THE OLYMPIAD CORNER

No. 223

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

As a problem set in this issue we continue the selection of problems from the St. Petersburg Contests 1965–1984 which we began last issue. These problems were selected (and translated) by two school students and forwarded to me by Andy Liu, University of Alberta, Edmonton, Alberta. I hope you enjoy the choice by Oleg Ivrii and by Robert Barrington Leigh.

ST. PETERSBURG CONTESTS 1965–1984
Problems from Various Contests (continued)

24. An infinite sequence of light bulbs and an infinite sequence of switches are both numbered by the positive integers. Each switch has a finite number of positions. Whether a bulb is on or off depends only on the positions of a finite number of switches. In any setting of the switches, at least one bulb is on. Prove that there exists a finite set of bulbs such that for any setting of the switches, at least one of them is on.

25. The sum of two continuous periodic functions is a nonconstant continuous periodic function. Prove that the periods of these two functions are integral multiples of the period of their sum.

26. Four squares on a 25×25 chessboard are called a quartet if their centres form a rectangle with sides parallel to the sides of the board. What is the maximum number of quartets which do not have any common squares?

27. The intersection of 20 circles consists of more than one point. Prove that the boundary of this intersection is a union of at most 38 circular arcs.

28. An irreducible fraction $\frac{x}{y}$ is called a good approximation of a number c if $\left|c - \frac{x}{y}\right| < \frac{1}{y^{100}}$. Prove that in any interval, there is a number with infinitely many good approximations.

29. From each of k points on a plane, a few rays are drawn. No two rays intersect. Prove that one can choose $k - 1$ of the segments connecting these points such that they are disjoint from one another and from any of the rays, except possibly at those k points.

30. There are 25 magnetic tapes on reels and 1 empty reel. One can rewind a tape from a full reel to an empty one, thus, reversing its direction. Can one reverse the direction of every tape while leaving each on its original reel?

31. There are $p-1$ integers none of which is divisible by p, where p is an odd prime. Prove that one can replace some of these numbers with their additive inverses to get $p-1$ numbers whose sum is divisible by p.

32. There are 9 points in a 2×2 square. Prove that the distance between some 2 of these points is not greater than 1.

33. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let $f^{(1)}(x) = f(x)$ and let $f^{(n)}(x) = f^{(n-1)}(f(x))$ for any integer $n > 1$. For any polynomial $P(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_0$, define

$$P(f) = a_n f^{(n)}(x) + a_{n-1} f^{(n-1)}(x) + \cdots + a_1 f(x) + a_0 x\,.$$

(a) Let $P(t) = t^2 - t + 1$. If $P(f) = 0$, prove that $f^{(7)} = f$.

(b) Let P, Q and R be polynomials such that $Q = PR$. If $P(f) = 0$, prove that $Q(f) = 0$.

(c) Let $P(t) = t^2 - t + 1$. Prove that there exists a function f such that $P(f) = 0$.

(d) Is (c) true if P is an arbitrary polynomial?

(e) Let Q and P be polynomials such that $\{f : P(f) = 0\} \subset \{f : Q(f) = 0\}$. Prove that there exists a polynomial R such that $Q = PR$. In particular, prove this result for $Q(t) = t^n - 1$.

34. A straight line passes through the centre of a regular $2n$-gon. Prove that the sum of distances to this line from the vertices on one side of this line equals the sum of the distances from the remaining vertices.

35. The *luckies* form a finite subset of the positive integers. Let a_k be the largest number of luckies which can be written if we have k copies of each digit. The same lucky may be written several times. Prove that among the numbers $\frac{a_k}{k}$, there exists a maximum.

36. A set X in the coordinate plane intersects any unit square whose vertices have integral coordinates in two parallel segments joining the midpoints of the sides of the square. X is invariant if shifted 25 units horizontally or vertically. Prove that X contains polygonal lines of infinite length.

37. Two points (x_1, y_1) and (x_2, y_2) are said to be *dependable* if $(x_1 - x_2)^2 = (y_1 - y_2)(x_1y_2 - x_2y_1)$. Prove that if any two of four points are dependable, then they are collinear.

38. Red, blue and green arcs are used to join pairs of $2n$ points such that there is exactly one arc of each colour at each point. Let a, b and c be the numbers of red-blue, red-green and blue-green cycles. Prove that $n + a \geq b + c$.

39. Each of n lines on a plane is cut by the others into 2 rays and $n - 2$ equal segments. Prove that $n = 3$.

40. A strictly increasing sequence $\{a_n\}$ of positive integers is such that $a_2 = 2$ and $a_{mn} = a_m a_n$ if m and n are relatively prime. Prove that $a_n = n$ for all n.

41. A real number is placed in each square of an infinite chessboard. Numbers in the same row or column at a distance 1982 apart are equal. Each number is the average either of its two horizontal neighbours or of its two vertical neighbours. Prove that either all the numbers in each column are equal or all the numbers in each row are equal.

42. A strait is 10 kilometres wide. The speed of the patrol boat is 7 times the speed of the smugglers' barge. The patrol boat will discover the barge if the distance between them is 1 kilometre or less. Can the patrol boat always discover any barge coming down the strait?

43. H is a given point inside a circle. Prove that a fixed circle passes through the mid-points of the sides of any triangle inscribed in the circle and having H as its orthocentre.

44. A strictly increasing sequence $\{x_n\}$ of positive integers is such that for all $n > 1982$,

$$x_1^3 + x_2^3 + \cdots + x_n^3 \ = \ (x_1 + x_2 + \cdots + x_n)^2 \, .$$

Prove that $x_n = n$ for all n.

45. Let $P(z)$ and $Q(z)$ be complex polynomials, one of which is not constant. Every root of $P(z)$ is also a root of $Q(z)$ and vice versa. Every root of $P(z) - 1$ is also a root of $Q(z) - 1$ and vice versa. Prove that $P = Q$.

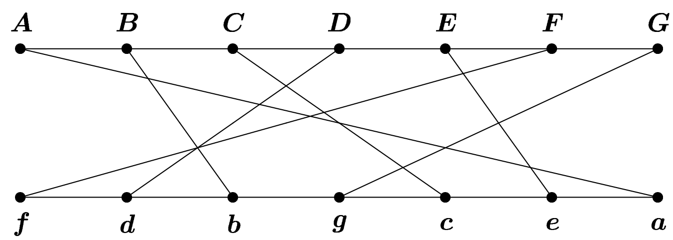
46. A calculator may be used to add two numbers, subtract one number from another, divide a number by any positive integer, and raise any number to the tenth power. Prove that it may be used to find the product of any ten numbers.

47. In a graph, it is possible to go from any vertex to any other passing through at most $n - 1$ vertices in between. The shortest cycle in this graph has length of $2n + 1$. Prove that all the vertices have the same degree.

Next we give a reader's response to a question we posed when giving solutions to problem 5 of the Georg Mohr Konkurrencen I Matematik 1996, [2001 : 239–240; 1999 : 261–262]. The solution we gave provided a negative answer for $n \equiv 2, 3 \pmod 4$.

5. In a ballroom 7 gentlemen, A, B, C, D, E, F and G are sitting opposite 7 ladies a, b, c, d, e, f and g in arbitrary order. When the gentlemen walk across the dance floor to ask each of their ladies for a dance, one observes that at least two gentlemen walk distances of equal length. Is that always the case?

The figure shows an example. In this example $Bb = Ee$ and $Dd = Cc$.



Editor's question. If n is congruent to either 0 or 1 modulo 4, is it always possible to arrange the n gentlemen and the n ladies in a way such that the distances are all different?

Solution by Peter de Caux, Eatonton, GA, USA.

We answer the Editor's two questions in the affirmative by constructing functions which specify for each "gentleman" a "lady" to whom he should walk. It will be convenient to call a function f *dispersive* provided:

(a) f is one-to-one and both its range and domain are the same initial segment I of the positive integers and

(b) the set $D = \{|m - f(m)| \mid m \in I\}$ is conumerous with I.

A simple consequence of this definition is that $D = \{m - 1 \mid m \in I\}$ and that for no two m and m' in I is $|m - f(m)| = |m' - f(m')|$.

Suppose now that n is a positive integer.

We build a dispersive function f with domain I consisting of the first $4n + 1$ positive integers. We do this in two steps. First we define a sequence S of $4n$ terms whose range is I with $3n + 1$ removed as follows:

$$\begin{array}{lrcccclcl} \text{for} & 1 & \leq & i & \leq & 2n & : & S(2i-1) & = & i\,, \\ \text{for} & 1 & \leq & i & \leq & n & : & S(2i) & = & 4n+2-i\,, \\ \text{for} & n+1 & \leq & i & \leq & 2n & : & S(2i) & = & 4n+1-i\,. \end{array}$$

Example 1. The consecutive terms of such a sequence if n were 5:

$$1\ \ 21\ \ 2\ \ 20\ \ 3\ \ 19\ \ 4\ \ 18\ \ 5\ \ 17\ \ 6\ \ 15\ \ 7\ \ 14\ \ 8\ \ 13\ \ 9\ \ 12\ \ 10\ \ 11\,.$$

Note that S is one-to-one and that the absolute values of the difference of consecutive terms of S are in descending order the first n positive integers with $2n$ removed.

Step 2 defines f to be the following collection of ordered pairs:

$$\{(S(k),\, S(k+1)) \mid 1 \le k \le 4n-1\} \;\cup\; \{(2n+1, 1),\, (3n+1, 3n+1)\}\,.$$

It is quickly checked that f is dispersive.

For $n = 5$ the values of f would be

$$\begin{array}{lclllclllclllcl}
f(1) & = & 21\,, & & f(21) & = & 2\,, & & f(6) & = & 20\,, & & f(20) & = & 3\,, \\
f(3) & = & 19\,, & & f(19) & = & 4\,, & & f(4) & = & 18\,, & & f(18) & = & 5\,, \\
f(5) & = & 17\,, & & f(17) & = & 6\,, & & f(6) & = & 15\,, & & f(15) & = & 7\,, \\
f(7) & = & 14\,, & & f(14) & = & 8\,, & & f(8) & = & 13\,, & & f(13) & = & 9\,, \\
f(9) & = & 12\,, & & f(12) & = & 10\,, & & f(10) & = & 11\,, & & f(11) & = & 1\,, \\
f(16) & = & 16\,. & & & & & & & & & & & &
\end{array}$$

If 21 gentlemen are evenly spaced in a straight line along one side of a rectangular dance floor and they are numbered consecutively 1, 2, $\ldots$, 21 and if each of the 21 ladies is evenly spaced on the side of the dance floor opposite the men and each is numbered as the gentleman opposite her is numbered, then, if gentleman m walks directly to lady $f(m)$ and another gentleman m' walks directly to lady $f(m')$, then m and m' walk different distances.

There is another dispersive function $\overline{f}$ with domain the positive integers less than or equal to $4n+1$, I, which can be constructed from a sequence $\overline{S}$ whose range is I with $n+1$ removed. Since the inverse of a dispersive function is again dispersive, we have four distinct dispersive functions with domains I : f, f^{-1}, $\overline{f}$, $\overline{f}^{-1}$.

Question: Are there more than four dispersive functions with domain I?

In an entirely analogous way we construct a dispersive function g with domain I consisting of the positive integers less than or equal to $4n$. We use a sequence of $4n-1$ terms, call it T, which has range I with $3n$ removed. The terms of T, $T(1)$, $T(2)$, $\ldots$, $T(4n-1)$ are

$$1,\, 4n,\, 2,\, 4n-1,\, \ldots,\, n,\, 3n+1,\, n+1,\, 3n-1,\, n+2,\, \ldots,$$
$$2n-2,\, 2n+2,\, 2n-1,\, 2n+1,\, 2n.$$

Using T, the ordered pairs in g are

$$(1, 4n),\, (4n, 2),\, \ldots,\, (n, 3n+1),\, (3n+1, n+1),\, (n+1, 3n-1),\, \ldots,$$
$$(2n-1, 2n+1),\, (2n+1, 2n),\, (2n, 1),\, (3n, 3n).$$

g is dispersive and together with f settle the editor's two questions.

Again a dispersive function $\overline{g}$ may be built from a sequence $\overline{T}$ whose terms are the first $4n$ positive integers with $n+1$ removed.

Question: Are there more than the four dispersive functions g, g^{-1}, $\overline{g}$, and $\overline{g}^{-1}$ which have domain the positive integers less than or equal to $4n$?

Next we fill in a missing solution when we gave responses to the Second Round of the 13th Iranian Mathematical Olympiad 1996 [1999 : 454–455; 2002 : 11–15].

6. In tetrahedron $ABCD$ let A', B', C', and D' be the circumcentres of faces BCD, ACD, ABD and ABC. We mean by $S(X,YZ)$, the plane perpendicular from point X to the line YZ. Prove that the planes $S(A,C'D')$, $S(B,D'A')$, $S(C,A'B')$, and $S(D,B'C')$ are concurrent.

Solution by Michel Bataille, Rouen, France.

Let R_3 and R_4 be the circumradii of $\triangle ABD$ and $\triangle ABC$, respectively. Then $AC' = R_3$ and $AD' = R_4$ so that $A \in \{M : MC'^2 - MD'^2 = R_3^2 - R_4^2\}$. This set of points, which is known to be a plane orthogonal to $C'D'$, is thus, $S(A,C'D')$.

Now let Σ be the circumsphere of the tetrahedron $ABCD$ (centre O, radius R) and Σ' be the circumsphere of the tetrahedron $A'B'C'D'$ (centre O', radius R'). The circumcircle of $\triangle ABC$ is the intersection of Σ with the plane (ABC). Hence, D' is the orthogonal projection of O onto (ABC) and $OD'^2 = AO^2 - AD'^2 = R^2 - R_4^2$. Similarly, $OC'^2 = R^2 - R_3^2$ and therefore, $OC'^2 - OD'^2 = R_4^2 - R_3^2$. Now, let Ω be the reflection of O in O'. Then, using $O'C' = O'D' = R'$,

$$\begin{aligned}
&\Omega C'^2 - \Omega D'^2 \\
&\quad = \Omega O'^2 + O'C'^2 + 2\,\overrightarrow{\Omega O'}\cdot\overrightarrow{O'C'} - \Omega O'^2 - O'D'^2 - 2\,\overrightarrow{\Omega O'}\cdot\overrightarrow{O'D'} \\
&\quad = 2\,\overrightarrow{OO'}\cdot\overrightarrow{O'D'} - 2\,\overrightarrow{OO'}\cdot\overrightarrow{O'C'} \\
&\quad = OD'^2 - OO'^2 - O'D'^2 - (OC'^2 - OO'^2 - O'C'^2) \\
&\quad = R_3^2 - R_4^2\,.
\end{aligned}$$

It follows that $\Omega \in S(A,C'D')$. Similarly, Ω belongs to the planes $S(B,D'A')$, $S(C,A'B')$, and $S(D,B'C')$, and we are done.

We next turn to solutions by readers to problems of the XXXIII Spanish Mathematical Olympiad 1996–97 given in the April 2000 number of the *Corner* [2000 : 196–197].

1. Show that any complex number $z \neq 0$ can be expressed as a sum of two complex numbers such that their difference and their quotient are purely imaginary (that is, with real part zero).

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; by Christopher J. Bradley, Clifton College, Bristol, UK; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the write-up by Bornsztein.

Let $z \in \mathbb{C}^*$, $z = a + ib$ with a, $b \in \mathbb{R}$. We are looking for α, $\beta \in \mathbb{C}$ such that

$$\begin{aligned} z &= \alpha + \beta && (1) \\ \alpha - \beta &= \mu i\,, \quad \text{where } \mu \in \mathbb{R} && (2) \\ \frac{\alpha}{\beta} &= \lambda i\,, \quad \text{where } \lambda \in \mathbb{R}\,. && (3) \end{aligned}$$

From (1) and (2), we deduce that $\mathfrak{Re}(\alpha) = \mathfrak{Re}(\beta) = \frac{1}{2}\mathfrak{Re}(z) = \frac{a}{2}$. Thus, we suppose that

$$\alpha = \frac{a}{2} + ix\,, \quad \beta = \frac{a}{2} + iy\,, \quad \text{where } x, y \text{ are reals.}$$

Then $x + y = b$.

Since we are looking for some possible values of x and y, we will omit for a time all the cases $a = 0$, $b = 0$, and so on ....

From (3), we have $a + 2ix = -2\lambda y + a\lambda i$. By identifying real parts, and imaginary parts, we may eliminate λ to get:

$$xy = -\frac{a^2}{4}\,.$$

Thus, we obtain

$$x + y = b\,, \quad xy = -\frac{a^2}{4}\,.$$

Since the quadratic $X^2 - bX - \frac{a^2}{4} = 0$ has a discriminant $\Delta = b^2 + a^2 > 0$ (since $z \neq 0$), it gives two real distinct roots:

$$x = \frac{b + \sqrt{b^2 + a^2}}{2} \quad \text{and} \quad y = \frac{b - \sqrt{b^2 + a^2}}{2}\,.$$

Conversely: We have $z = a + ib = \alpha + \beta$ where

$$\alpha = \frac{a}{2} + i\left(\frac{b + \sqrt{b^2 + a^2}}{2}\right) \quad \text{and} \quad \beta = \frac{a}{2} + i\left(\frac{b - \sqrt{b^2 + a^2}}{2}\right)\,.$$

It is clear that $\alpha - \beta$ is purely imaginary. Since $z \neq 0$, we must have

$$b + \sqrt{b^2 + a^2} \neq 0 \quad \text{or} \quad b - \sqrt{b^2 + a^2} \neq 0$$

then $\alpha \neq 0$ or $\beta \neq 0$.

Let us suppose $\beta \neq 0$ (the case $\alpha \neq 0$ is similar). Thus,

$$\frac{\alpha}{\beta} = \frac{a + i(b + \sqrt{b^2 + a^2})}{a + i(b - \sqrt{b^2 + a^2})} = \frac{2a\sqrt{b^2 + a^2}}{a^2 + (b - \sqrt{b^2 + a^2})^2}i$$

is purely imaginary.

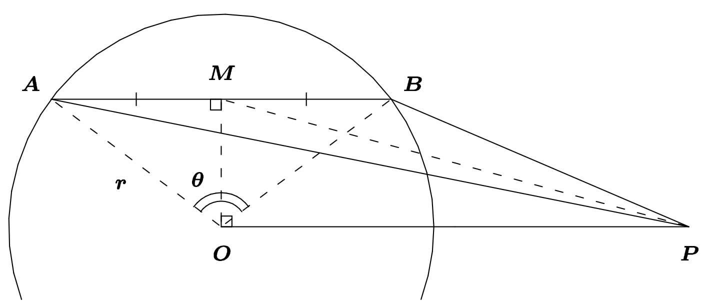
(Note the convention that 0 is both purely real and purely imaginary.)

2. Consider a circle of centre O, radius r, and let P be an external point. We draw a chord AB parallel to OP.

(a) Show that $PA^2 + PB^2$ is constant.

(b) Find the length of the chord AB which maximizes the area of the $\triangle ABP$.

Solutions by Michel Bataille, Rouen, France; by Christopher J. Bradley, Clifton College, Bristol, UK; by Toshio Seimiya, Kawasaki, Japan; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Seimiya's write-up.



(a) Let M be the mid-point of AB, then $OM \perp AB$. Since $AB \parallel OP$, we have $OM \perp OP$.

Since M is the mid-point of AB, we get

$$\begin{aligned} PA^2 + PB^2 &= 2(PM^2 + AM^2) \\ &= 2\{(OM^2 + OP^2) + (OA^2 - OM^2)\} \\ &= 2(OP^2 + OA^2) \\ &= 2(OP^2 + r^2) \;=\; \text{constant}. \end{aligned}$$

(b) Since $AB \parallel OP$ we have $[ABP] = [ABO]$, where $[XYZ]$ denotes the area of triangle XYZ.

We put $\angle AOB = \theta$, so

$$[ABO] \;=\; \frac{1}{2}OA \cdot OB \sin\theta \;=\; \frac{1}{2}r^2 \sin\theta \;\le\; \frac{1}{2}r^2 :$$

equality holds when $\theta = 90^\circ$, and $AB = \sqrt{2}\,r$. Therefore, $[ABP]$ is a maximum when $AB = \sqrt{2}\,r$.

3. Six musicians participate in a music festival. At each concert, some of them play music, and the others listen. What is the minimal number of concerts so that each musician listens to all the others?

Solutions by Christopher J. Bradley, Clifton College, Bristol, UK; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bradley's solution.

Call the musicians A, B, C, D, E, F. In all there have to be at least 30 acts of playing and listening since B, C, D, E, F have all to hear A, etc.

Now at one concert the maximum number of acts of playing and listening is 9, when 3 are playing and 3 listening (splits of $2/4$ give only 8 acts). Therefore, 4 concerts are necessary at a minimum. That it can be done in 4 concerts, see the table below

	Playing	Listening
Concert 1	ABC	DEF
2	AEF	BCD
3	BDF	CAE
4	CDE	ABF

4. The sum of two of the roots of the equation

$$x^3 - 503x^2 + (a+4)x - a \;=\; 0$$

is equal to 4. Determine the value of a.

Solutions by Pierre Bornsztein, Pontoise, France; by Christopher J. Bradley, Clifton College, Bristol, UK; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bornsztein's write-up.

We have $a = 1996$ — what a surprise!

Let α, β, γ denote the roots of $x^3 - 503x^2 + (a+4)x - a = 0$ with $\alpha + \beta = 4$.

Since $\alpha + \beta + \gamma = 503$, we deduce that $\gamma = 499$. Then

$$499^3 - 503 \cdot 499^2 + (a+4)499 - a \;=\; 0\,.$$

Thus, $a = 1996$ as claimed.

5. If a, b, c are positive real numbers, prove the inequality

$$a^2 + b^2 + c^2 - ab - bc - ca \;\geq\; 3(b-c)(a-b)\,.$$

When is the "$=$" sign valid?

Solutions by Pierre Bornsztein, Pontoise, France; by Christopher J. Bradley, Clifton College, Bristol, UK; by Toshio Seimiya, Kawasaki, Japan; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Seimiya's write-up.

We put $X = a - b$ and $Y = b - c$, so that $X + Y = a - c$. Since

$$\begin{aligned} & 2(a^2 + b^2 + c^2 - ab - bc - ca) - 6(b-c)(a-b) \\ = \;& (a-b)^2 + (b-c)^2 + (c-a)^2 - 6(b-c)(a-b) \\ = \;& X^2 + Y^2 + (X+Y)^2 - 6XY \\ = \;& 2(X^2 + Y^2 - 2XY) \\ = \;& 2(X-Y)^2 \geq 0 \quad \text{(equality holds when } X = Y)\,. \end{aligned}$$

Thus, we have

$$2(a^2+b^2+c^2-ab-bc-ca) \;\geq\; 6(b-c)(a-b)\,.$$

Further, we obtain

$$a^2+b^2+c^2-ab-bc-ca \;\geq\; 3(b-c)(a-b)\,.$$

Equality holds when $X = Y$; that is $a-b = b-c$. This implies that $a+c=2b$.

6. Find, with reasons, all the natural numbers n such that n^2 has only odd digits.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; by Christopher J. Bradley, Clifton College, Bristol, UK; and by Stewart Metchette, Gardena, CA, USA. We give Metchette's solution.

If n^2 is to have only odd digits, then it must terminate in an odd integer that is an odd quadratic residue of 10: 1, 5 or 9. Hence, for $n=1$ or 3, $n^2=1$ or 9 and contains only odd digits.

Further, all odd squares >9 must terminate in one of the 22 quadratic residues of 100, of which 11 are odd:

$$01\;\;09\;\;21\;\;25\;\;29\;\;41\;\;49\;\;61\;\;69\;\;81\;\;89\,.$$

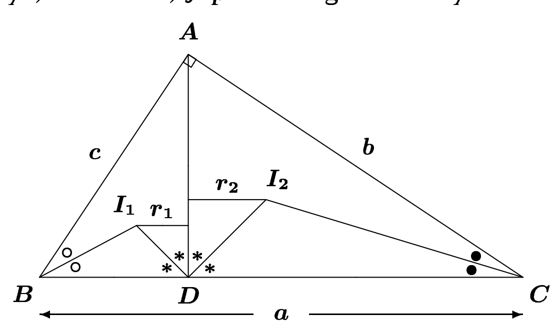
But then the ten's-digit of n^2 is always even and there is no $n^2>9$ that contains only odd digits.

Consequently, only for $n=1$ or 3 does $n^2=1$ or 9 have only odd digits.

7. The triangle ABC has $\widehat{A}=90^\circ$, and AD is the altitude from A. The bisectors of the angles $\widehat{ABD}$ and $\widehat{ADB}$ intersect at I_1; the bisectors of the angles $\widehat{ACD}$ and $\widehat{ADC}$ intersect at I_2.

Find the acute angles of $\triangle ABC$, given that the sum of distances from I_1 and I_2 to AD is $BC/4$.

Solutions by Christopher J. Bradley, Clifton College, Bristol, UK; and by Toshio Seimiya, Kawasaki, Japan. We give Seimiya's write-up.



We denote the distances from I_1 and I_2 to AD by r_1 and r_2, respectively.

Since I_1 and I_2 are incentres of $\triangle ABD$ and $\triangle ACD$, respectively, then r_1 and r_2 are inradii of $\triangle ABD$ and $\triangle ACD$, respectively.

Since $\angle ADB = \angle ADC = 90°$, we have

$$2r_1 \;=\; AD + BD - AB \quad \text{and} \quad 2r_2 \;=\; AD + DC - AC$$

so that

$$2(r_1 + r_2) \;=\; 2AD + BC - AB - AC\,. \qquad (1)$$

We put $BC = a$, $CA = b$ and $AB = c$. Since $\angle BAC = 90°$ and $AD \perp BC$ we get

$$AD \cdot BC \;=\; AB \cdot AC\,.$$

Thus, we have $AD = \frac{bc}{a}$. Since $r_1 + r_2 = \frac{BC}{4}$, we obtain from (1)

$$\frac{a}{2} \;=\; \frac{2bc}{a} + a - b - c\,.$$

Multiplying both sides by $2a$, we get

$$a^2 \;=\; 4bc + 2a^2 - 2a(b+c)\,.$$

Thus, $a^2 - 2a(b+c) + 4bc \;=\; 0$, so that

$$(a-2b)(a-2c) \;=\; 0\,.$$

Therefore, either $a - 2b = 0$ or $a - 2c = 0$. If $a - 2b = 0$, then $a = 2b$, since $\angle A = 90°$ so that we get $\angle B = 30°$ and $\angle C = 60°$. If $a - 2c = 0$, similarly, we get $\angle C = 30°$ and $\angle B = 60°$. Therefore, the acute angles of $\triangle ABC$ are $60°$ and $30°$.

8. For each real number x, we denote by $\lfloor x \rfloor$ the biggest integer which is less than or equal to x. We define

$$q(n) \;=\; \left\lfloor \frac{n}{\lfloor \sqrt{n} \rfloor} \right\rfloor, \quad n = 1, 2, 3, \ldots\,.$$

(a) Forming a table with the values of $q(n)$ for $1 \le n \le 25$, make a conjecture about the numbers n for which $q(n) > q(n+1)$.

(b) Determine, with reasons, all the positive integer n such that

$$q(n) \;>\; q(n+1)\,.$$

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; by Christopher J. Bradley, Clifton College, Bristol, UK; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bradley's write-up.

(a) The conjecture must be that $q(n) > q(n+1)$ if and only if $n = m^2 - 1$, where m is an integer ($m > 1$).

(b) If $n = m^2 - 1$, then

$$q(n) = \left\lfloor \frac{m^2-1}{\lfloor\sqrt{m^2-1}\rfloor} \right\rfloor = \left\lfloor \frac{m^2-1}{m-1} \right\rfloor = m+1\,,$$

whereas

$$q(n+1) = \left\lfloor \frac{m^2}{\lfloor\sqrt{m^2}\rfloor} \right\rfloor = \left\lfloor \frac{m^2}{m} \right\rfloor = m < q(n)\,.$$

Apart from these occasional decreases in the value of $q(n)$ when n is a perfect square, it is the case that $q(n+1) \geq q(n)$. To prove this, it is sufficient to show

$$q(m^2+k) \geq q(m^2+k-1) \quad \text{for} \quad 1 \leq k \leq 2m\,.$$

This is in fact trivial, since $m^2 + k > m^2 + k - 1$ and $\left\lfloor\sqrt{m^2+k}\right\rfloor = \left\lfloor\sqrt{m^2+k-1}\right\rfloor = m$ for such values of k.

Next we turn to solutions to some of the problems of the 20th Austrian-Polish Mathematical Competition 1997 given [2000 : 197–199].

1. P is the common point of straight lines l_1 and l_2. Two circles S_1 and S_2 are externally tangent at P and l_1 is their common tangent line. Similarly, two circles T_1 and T_2 are externally tangent at P and l_2 is their common tangent line. The circles S_1 and T_1 have common points P and A, the circles S_1 and T_2 have common points P and B, the circles S_2 and T_2 have common points P and C, and the circles S_2 and T_1 have common points P and D. Prove that the points A, B, C, D lie on a circle if and only if the lines l_1 and l_2 are perpendicular.

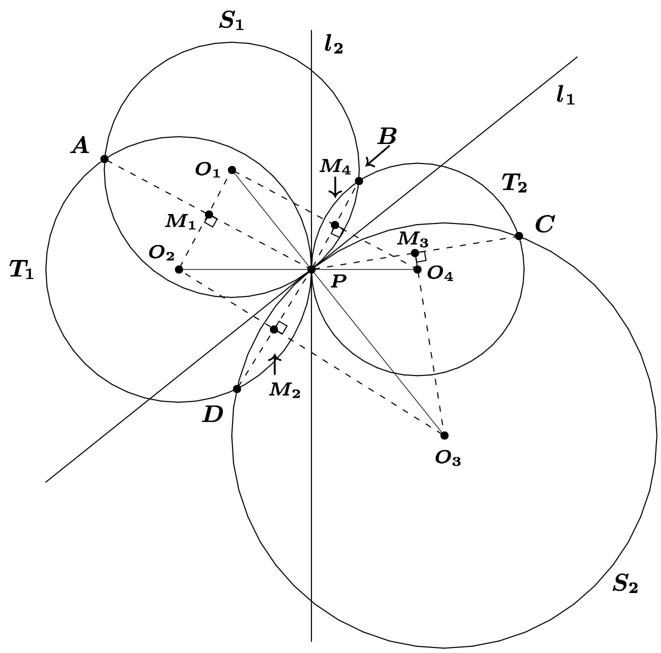
Solution by Toshio Seimiya, Kawasaki, Japan.

See figures on page 301.

Let O_1, O_2, O_3 and O_4 be centres of S_1, T_1, S_2 and T_2, respectively. Since S_1 and S_2 touch l_1 at P, then $O_1P \perp l_1$ and $O_3P \perp l_1$. Thus, O_1, P, and O_3 are collinear and $O_1O_3 \perp l_1$.

Similarly, O_2, P, and O_4 are collinear and $O_2O_4 \perp l_2$. Let PA meet O_1O_2 at M_1. Since S_1, T_1 intersect at P and A, M_1 is the mid-point of PA, and $PM_1 \perp O_1O_2$.

Let M_2, M_3 and M_4 be the intersections of PD, O_2O_3, PC, O_3O_4 and PB, O_4O_1, respectively. Then M_2, M_3 and M_4 are mid-points of O_2O_3,

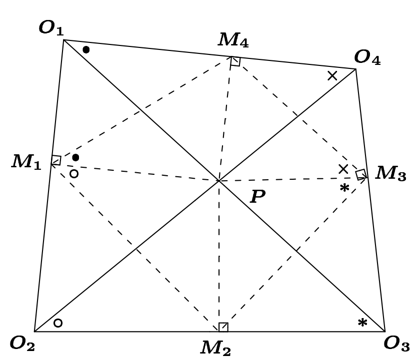


O_3O_4 and O_4O_1, respectively, and $PM_2 \perp O_2O_3$, $PM_3 \perp O_3O_4$, and $PM_4 \perp O_4O_1$.

Since M_1, M_2, M_3, and M_4 are mid-points of PA, PD, PC and PB, respectively, we have

$$\begin{aligned} A, B, C, D \text{ are concyclic} &\iff M_1, M_2, M_3, M_4 \text{ are concyclic.} \\ &\iff \angle M_2M_1M_4 + \angle M_2M_3M_4 = 180^\circ . \end{aligned}$$

Since $\angle PM_1O_1 = \angle PM_4O_1 = 90^\circ$, it follows that O_1, M_1, P, M_4 are concyclic, so that $\angle PM_1M_4 = \angle PO_1M_4 = \angle PO_1O_4$. Similarly, we have $\angle PM_1M_2 = \angle PO_2O_3$, $\angle PM_3M_2 = \angle PO_3O_2$ and $\angle PM_3M_4 = \angle PO_4O_1$.



Thus,

$$\begin{aligned}
&\angle M_2M_1M_4 + \angle M_2M_3M_4 \\
&\quad = \angle PM_1M_4 + \angle PM_1M_2 + \angle PM_3M_2 + \angle PM_3M_1 \\
&\quad = \angle PO_1O_4 + \angle PO_2O_3 + \angle PO_3O_2 + \angle PO_4O_1 \\
&\quad = (\angle PO_1O_4 + \angle PO_4O_1) + (\angle PO_2O_3 + \angle PO_3O_2) \\
&\quad = \angle O_1PO_2 + \angle O_1PO_2 \;=\; 2\angle O_1PO_2\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
\angle M_2M_1M_4 + \angle M_2M_3M_4 \;=\; 180^\circ \;&\Longleftrightarrow\; 2\angle O_1PO_2 \;=\; 180^\circ \\
&\Longleftrightarrow\; \angle O_1PO_2 \;=\; 90^\circ\,.
\end{aligned}$$

Since $O_1P \perp l_1$ and $O_2P \perp l_2$, we have that l_1 and l_2 coincide with O_2P and O_1P, respectively.

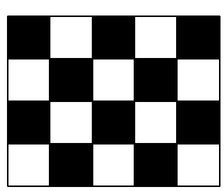
Thus, $\angle O_1PO_2 = 90^\circ$ if and only if $l_1 \perp l_2$. Therefore, we have that A, B, C, D are concyclic if and only if $l_1 \perp l_2$.

2. Let m, n, p, q be positive integers. We have a rectangular chess-board of dimensions $m \times n$ divided into $m \cdot n$ equal squares. The squares are referred to by their coordinates (x, y), where $1 \le x \le m$, $1 \le y \le n$. There is a piece on each square. A piece can be moved from the square (x, y), to the square (x', y') if and only if $|x - x'| = p$ and $|y - y'| = q$. We want to move each piece simultaneously so as to get again one piece on each square. Find the number of ways in which such a multiple move can be made.

Solution by Mohammed Aassila, Strasbourg, France.

If m is odd, then there are more black squares in the top row than in the second one (see the figure below). Hence, the move is impossible (for $p = q = 1$). Similarly, if n is odd, the move is impossible. If m and n are even, then the move is possible in only one way.

If p and q are general positive integers. We divide the $m \times n$ squares into $p \times q$ classes based on their x–coordinate $\pmod p$ and y–coordinate $\pmod q$. Hence, the problem is reduced to the case when $p = q = 1$. Thus, if $2p \mid m$ and $2q \mid n$, there is only one way in which such a move can be made, and it is impossible otherwise.



3. On the blackboard there are written the numbers 48, 24, 16, $\dots$, $\frac{48}{97}$; that is, rational numbers $\frac{48}{k}$ with $k = 1$, 2, 3, $\dots$, 97. In each step two arbitrarily chosen numbers a and b are cancelled and the number $2ab - a - b + 1$ is written on the blackboard. After 96 steps there is only one number on the blackboard. Determine the set of numbers which are possible outcomes of the procedure.

Solution by Pierre Bornsztein, Pontoise, France.

The only possible outcome is $\frac{1}{2}$.

Let E_i be the set of all numbers written on the blackboard after the i^{th} step. First note that $\frac{1}{2} = \frac{48}{96} \in E_0$.

Let $i \geq 1$ be a fixed integer. Suppose that $\frac{1}{2} \in E_{i-1}$. At the i^{th} step, we choose a and b.

- if neither a nor b is equal to $\frac{1}{2}$, then $\frac{1}{2} \in E_i$.
- if $a = \frac{1}{2}$, for example, then $2ab - a - b + 1 = \frac{1}{2}$.

Thus, $\frac{1}{2} \in E_i$. Then, in all cases $\frac{1}{2} \in E_i$.

We easily deduce by induction that $\frac{1}{2} \in E_i$ for $i = 0, 1, \ldots, 96$. Since E_{96} contains only one number, we must have $E_{96} = \left\{\frac{1}{2}\right\}$, as claimed.

4. In a convex quadrilateral $ABCD$ the sides AB and CD are parallel, the diagonals AC and BD intersect at point E and points F and G are the orthocentres of the triangles EBC and EAD, respectively. Prove that the midpoint of the segment GF lies on the line k perpendicular to AB such that $E \in k$.

Solutions by Toshio Seimiya, Kawasaki, Japan; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give Smeenk's solution.

We denote

$$\angle DAB = \alpha, \quad \angle ABC = \beta, \quad BC = b, \quad AD = d\,.$$

$$AB \parallel CD \implies d \sin\alpha = b \sin\beta \tag{1}$$

Write $\angle AED = \angle BEC = \varepsilon$ and let G', E', F' be the projections onto AB of G, E, and F, respectively. Then $\angle EGG' = \alpha$, $\angle EFF' = \beta$.

Denote by R_1 the circumradius of $\triangle AED$, and by R_2 that of $\triangle BEC$.

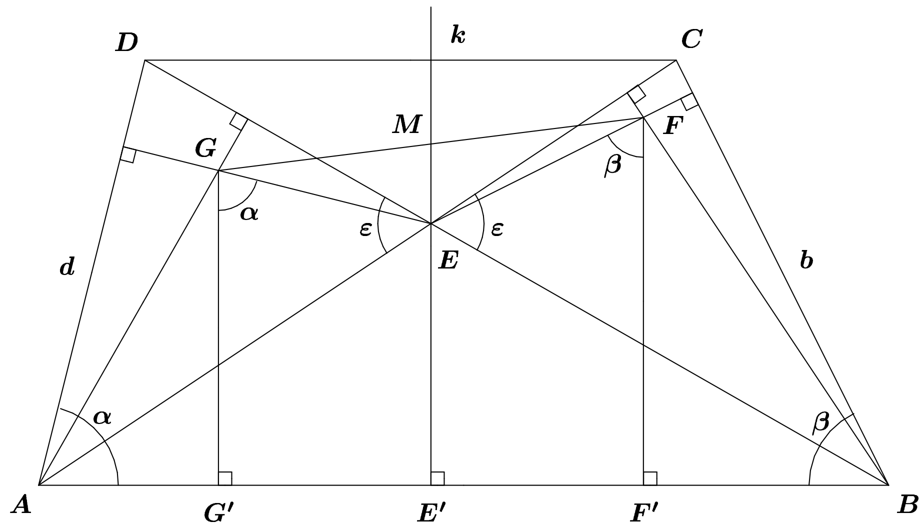
$$d = 2R_1 \sin\varepsilon \implies R_1 = \frac{d}{2\sin\varepsilon}\,;\; EG = 2R_1\cos\varepsilon = d\cot\varepsilon$$
$$\implies E'G' = EG\sin\alpha = d\cot\varepsilon\sin\alpha\,. \tag{2}$$

In the same way:

$$E'F' = b\cot\varepsilon\sin\beta \tag{3}$$

$$\text{(1), (2) and (3)} \implies E'G' = E'F'\,. \tag{4}$$

(4) $\implies M$, the mid-point of GF, lies on EE' $(= k)$. (See figure on page 304.)



5. Let p_1, p_2, p_3, and p_4 be four distinct prime numbers. Prove that there does not exist a cubic polynomial $Q(x) = ax^3 + bx^2 + cx + d$ with integer coefficients such that

$$|Q(p_1)| = |Q(p_2)| = |Q(p_3)| = |Q(p_4)| = 3 .$$

Solution by Pierre Bornsztein, Pontoise, France.

Let p_1, p_2, p_3, p_4 be four distinct prime numbers.

Suppose, for a contradiction, that there exists $Q(x) = ax^3+bx^2+cx+d$ with $a, b, c, d \in \mathbb{Z}$ and $a \neq 0$, such that $|Q(p_i)| = 3$ for $i = 1, 2, 3, 4$. With no loss of generality, we may suppose that at least two of the numbers $Q(p_i)$ are equal to 3 (otherwise we use $-Q$). Define $R(x) = Q(x) - 3$. Then $R \in \mathbb{Z}[x]$ and R has degree 3.

If $Q(p_i) = 3$ for each i: then the polynomial R has four distinct roots, which is impossible, since the degree of R is 3.

If exactly three of the $Q(p_i)$ are equal to 3: with no loss of generality, we may suppose that $Q(p_1) = Q(p_2) = Q(p_3) = 3$ and $Q(p_4) = -3$. Then p_1, p_2, p_3 are the roots of $R(n)$, and we have

$$R(x) = a(x - p_1)(x - p_2)(x - p_3) .$$

Thus, $|R(p_4)| = |a|\,|p_4 - p_1|\,|p_4 - p_2|\,|p_4 - p_3| = 6 = 2 \times 3$, with $|a|$, $|p_4 - p_1|$, $|p_4 - p_2|$, $|p_4 - p_3| \in \mathbb{N}^*$. It follows that at least two of these four numbers are equal to 1.

But if $|p_i - p_j| = 1$, then the integers p_i and p_j are consecutive. And, if they are primes, they have to be 2 and 3. Since p_1, p_2, p_3, p_4 are distinct such a situation can occur at most one time. It follows that:

$|a| = 1$ and (by symmetry) $|p_4 - p_1| = 1$, $|p_4 - p_2| = 2$, $|p_4 - p_3| = 3$.

But the difference between two primes is odd if and only if one of these primes is 2. Then $p_4 = 2$. It follows that $p_2 \in \{0; 4\}$, which is impossible.

If exactly two of the $Q(p_i)$ are equal to 3, we may suppose that

$$Q(p_1) = Q(p_2) = 3 \quad \text{and} \quad Q(p_3) = Q(p_4) = -3.$$

Let α be the third root of $R(x)$. We then have $p_1 + p_2 + \alpha = -\frac{b}{a}$. Thus, $a\alpha$ is an integer.

Moreover:

$$\begin{aligned} |R(p_3)| &= |p_3 - p_1|\,|p_3 - p_2|\,|ap_3 - a\alpha| = 6 \\ |R(p_4)| &= |p_4 - p_1|\,|p_4 - p_2|\,|ap_4 - a\alpha| = 6 \end{aligned}$$

with $|p_3 - p_1|$, $|p_3 - p_2|$, $|ap_3 - a\alpha|$, $|p_4 - p_1|$, $|p_4 - p_2|$, $|ap_4 - a\alpha| \in \mathbb{N}^*$.

As above, we deduce that:

(a) at least one of the integers $|p_4 - p_1|$ and $|p_4 - p_2|$ is equal to 1 or 3, and therefore, is odd.

(b) At least one of the integers $|p_3 - p_1|$ and $|p_3 - p_2|$ is equal to 1 or 3, and therefore, is odd.

Then:

• if $p_4 = 2$, we have $p_2 \neq 2$ and $p_1 \neq 2$. From (b), we must have $p_3 = 2 = p_4$. A contradiction.

• If $p_4 \neq 2$, then we may suppose that $p_1 = 2$ (from (a)). From (a) we deduce that $p_4 \in \{3, 5\}$, and from (b) that $p_3 \in \{3, 5\}$. We may suppose that $p_3 = 3$ and $p_4 = 5$. Then $p_2 \geq 7$ and $|p_2 - p_3| \geq 4$. Since $|p_2 - p_3|$ divides 6, we must have $p_2 - p_3 = 6$. Thus, $p_2 = 9$, which is not a prime, a contradiction.

And the proof is complete.

That completes the *Corner* for this issue. Send me your nice solutions and Olympiad materials.