MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 977 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z7 (NEW!)**. The electronic address is

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MAYHEM PROBLEMS

Envoyez vos propositions et solutions à MATHEMATICAL MAYHEM, Faculté de mathématiques, Université de Waterloo, 200 University Avenue West, Waterloo, ON, N2L 3G1, ou par courriel à

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N'oubliez pas d'inclure à toute correspondance votre nom, votre année scolaire, le nom de votre école, ainsi que votre ville, province ou état et pays. Nous sommes surtout intéressés par les solutions d'étudiants du secondaire. Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le *1er septembre 2002*. Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

Pour être admissibles au DÉFI MAYHEM de ce mois-ci, les solutions doivent avoir été postées *avant le* 1er juillet 2002, cachet de la poste faisant foi.

M34. *Proposé par l'équipe de Mayhem.*

Les nombres 1 à 2002 sont écrits au tableau noir et l'on décide de jouer au jeu suivant :

On lance une pièce de monnaie et on efface deux nombres x et y du tableau. Si l'on tombe sur pile, on écrit $x + y$ au tableau, sinon on écrit $|x - y|$; on continue le processus jusqu'à ce qu'il ne reste plus qu'un nombre. Montrer que ce dernier nombre est impair.

The numbers 1 to 2002 are written on a blackboard so you decide to play a fun game. You flip a coin, then erase two numbers, x and y, from the board. If the coin was heads you write the number $x+y$ on the board, if the coin was tails you write the number $|x-y|$. You continue this process until only one number remains. Prove that the last number is odd.

M35. *Proposé par l'équipe de Mayhem.*

On définit deux suites par $x_1 = 4732$, $y_1 = 847$, $x_{n+1} = \frac{x_n+y_n}{2}$ and $y_{n+1} = \frac{2x_ny_n}{x_n+y_n}$. Trouver

$$\lim_{n\to\infty} x_n \qquad \text{et} \qquad \lim_{n\to\infty} y_n .$$

. . . .

Two sequences are defined by: $x_1 = 4732$, $y_1 = 847$, $x_{n+1} = \frac{x_n+y_n}{2}$ and $y_{n+1} = \frac{2x_ny_n}{x_n+y_n}$. Find

$$\lim_{n\to\infty} x_n \qquad \text{and} \qquad \lim_{n\to\infty} y_n .$$

M36. *Proposé par l'équipe de Mayhem.*

Dans un triangle ABC, soit AM la médiane issue du sommet A. Montrer que $AM \leq \frac{AB+AC}{2}$.

. . . .

In $\triangle ABC$, AM is the median from A. Prove $AM \leq \frac{AB+AC}{2}$.

M37. *Proposé par J. Walter Lynch, Athens, GA, USA.*

Trouver deux entiers positifs différents, plus petits que 100, et tels que la somme des chiffres des deux entiers soit égale au plus grand et que produit de leurs chiffres soit égal au plus petit.

. . . .

Find two (different) positive integers less than 100 such that the sum of the digits in both integers is the larger integer and the product of the digits in both integers is the smaller integer.

M38. *Proposé par l'équipe de Mayhem.*

Trouver toutes les valeurs de n telles que $1! + 2! + 3! + \cdots + n!$ soit un carré parfait.

. . . .

Find all values of n such that $1! + 2! + 3! + \cdots + n!$ is a perfect square.

Challenge Board Solutions

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

In this issue we present some of the solutions to the Konhauser Problemfest presented in the April 2001 issue [2001 : 204].

1. Last season, the Minnesota Timberwolves won 5 times as many games as they lost, in games in which they scored 100 or more points. On the other hand, in games in which their opponents scored 100 or more points, the Timberwolves lost 50% more games than they won. Given that there were exactly 34 games in which either the Timberwolves or their opponents scored 100 or more points, what was the Timberwolves' win-loss record in games in which *both* they and their opponents scored 100 or more points?

Solution: Let the desired win-loss record be $a-b$, so that of the games when both the Timberwolves and their opponents scored at least 100 points, the Timberwolves won a and their opponents won b. Also, let c be the number of games in which only the Timberwolves scored at least 100 points and d be the number of games in which only their opponents scored at least 100 points. Thus, we have

$$a+c = 5b\,, \qquad (1)$$
$$b+d = 1.5a\,, \qquad (2)$$
$$a+b+c+d = 34\,. \qquad (3)$$

Thus, we get $c=5b-a$, and $d=1.5a-b$ from (1) and (2), so that (3) yields $3a+10b=68$.

Now a and b are non-negative integers, and in particular $b \le 6$ and $68-10b$ is divisible by 3, which is true only for $b=2$ and $b=5$. If $b=2$, then $a=16$, $c=5b-1=-6<0$, which is a contradiction. Thus, we must have $b=5$ and $a=6$. The Timberwolves were $6-5$ in games in which both they and their opponents scored at least 100 points.

2. Three circles are drawn in chalk on the ground. To begin with, there is a heap of n pebbles inside one of the circles, and there are "empty heaps" (containing no pebbles) in the other two circles. Your goal is to move the entire heap of n pebbles to a different circle, using a series of moves of the following type. For any non-negative integer k, you may move exactly 2^k pebbles from one heap (call it heap **A**) to another (heap **B**), provided that heap **B** begins with fewer than 2^k pebbles, and that after the move, heap **A** ends up with fewer than 2^k pebbles. Naturally, you want to reach your goal in as few moves as possible. For what values of $n \le 100$ would you need the largest number of moves?

Solution: This is a close relative of the classic "Towers of Hanoi" problem. In that problem, a tapering stack of d discs is to be moved from one peg to one of the other two pegs, which are originally empty. No disc is ever allowed to rest on a larger disc, and the discs must be moved one at a time. If b_d is the number of moves required for d discs, one shows that $b_{d+1} = 2b_d + 1$ (that is, to move the bottom disc, first move the other discs onto a single peg, then after the bottom disc is moved, the others are moved back onto it). However, all we need for our present purpose is that $b_{d+1} > b_d$.

The connection between the given problem and the "Towers of Hanoi" problem can be seen by writing n as a sum of distinct powers of 2. Then, n can be written as such a sum in exactly one way.

The distinct powers of 2 correspond to the discs in the "Towers of Hanoi" problem, but *larger powers of* 2 *correspond to smaller discs*. The condition for moving 2^k pebbles corresponds to the fact that the top disc must be moved before any of the discs underneath.

Note that d corresponds to the number of distinct powers of 2 that add up to n (or the number of 1's in the binary expansion of n). So our problem boils down to: what numbers $n \leq 100$ have the most digits 1 in their binary expansion?

The powers of 2 that can be used in the binary expansion of numbers $n \leq 100$ are 2^6, 2^5, 2^4, 2^3, 2^2, 2^1, 2^0. However, they cannot *all* be used since their sum is $127 > 100$. It is possible to use all but one of them in just two ways:

$$\begin{aligned} 2^6 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 &= 95\,, \\ 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 &= 63\,. \end{aligned}$$

Hence, 63 *and* 95 are the desired values of n.

3. (a) Begin with a string of 10 A's, B's, and C's, for example

```
A B C C B A B C B A
```

and underneath, form a new row, of length 1 shorter, as follows: between two consecutive letters that are different, you write the third letter, and between two letters that are the same, you write that same letter again. Repeat this process until you have only one letter in the new row. For example, for the string above, you will now have:

```
A B C C B A B C B A
 C A C A C C A A C
  B B B B C B A B
   B B B A A C C
    B B C A B C
     B A B C A
      C C A B
       C B C
        A A
         A
```

Prove that the letters at the corners of the resulting triangle are always either all the same or all different.

Solution: Think of the letters A, B, C as representing the numbers 0, 1, 2, respectively, $\pmod 3$. Then, if in some row we have $\cdots \quad x \quad y \quad \cdots$ where x and y are integers $\pmod 3$, we get $\cdots -x-y \cdots$ in the next row, where $-x-y$ is computed $\pmod 3$. If the original row corresponds to the integers $x_1 \quad x_2 \quad \cdots \quad x_{10} \pmod 3$, then the next rows are:

$$\begin{array}{ccccccc} -x_1-x_2 & & -x_2-x_3 & & -x_3-x_4 & & \cdots \\ & x_1+2x_2+x_3 & & x_2+2x_3+x_4 & & \cdots & \\ & & -x_1-3x_2-3x_3-x_4 & & \cdots & & \end{array}$$

and we see the pattern (which can be proved by induction): apart from the $\pm$ signs, the coefficients are those of Pascal's triangle, the binomial coefficients. In particular, the tenth (bottom) row will consist of the single entry

$$-x_1 - \binom{9}{1}x_2 - \binom{9}{2}x_3 - \cdots - \binom{9}{8}x_9 - x_{10} \pmod 3 .$$

But since 9 is a power of the prime 3, all binomial coefficients $\binom{9}{1}$, $\binom{9}{2}$, $\ldots$, $\binom{9}{8}$ are divisible by 3, so that the entry in the bottom row is equal to $-x_1 - x_{10} \pmod 3$. Thus, the three corner numbers $(x_1, x_{10}, -x_1 - x_{10}) \pmod 3$ that add to $0 \pmod 3$ and it follows that they are either all the same or all different.

(b) For which positive integers n (besides 10) is the result from part (a) true for all strings of n A's, B's, and C's?

Solution: For $n = 1$, and for $n = 3^k + 1$, $k \geq 0$. For such an n, the n^{th} row will have the form:

$$-x_1 - \binom{3^k}{1}x_2 - \binom{3^k}{2}x_3 - \cdots - \binom{3^k}{3^k-1}x_{n-1} - x_n \equiv -x_1 - x_n \pmod 3 ,$$

because the binomial coefficients are divisible by 3.

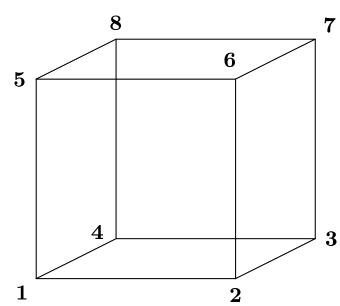
4. When Mark climbs a staircase, he ascends either 1, 2, or 3 stairsteps with each stride, but in no particular pattern from one foot to the next. In how many ways can Mark climb a staircase of 10 steps? (Note that he must finish on the top step. Two ways are considered the same if the number of steps for each stride are the same; that is, it does not matter whether he puts his best or his worst foot forward first.) Suppose that a spill has occurred on the 6^{th} step and Mark wants to avoid it. In how many ways can he climb the staircase without stepping on the 6^{th} step?

Solution: Let a_n be the number of ways for Mark to climb a staircase of n steps. Then $a_1 = 1$, $a_2 = 2$ and $a_3 = 4$. For $n > 3$, consider Mark's last stride. If his last stride was 1–step, then before that he climbed an $n-1$–step staircase, and there are a_{n-1} ways in which that can be done. Using a similar argument for 2– and 3–step last steps, we get $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n > 3$. Using this we can calculate $a_{10} = 274$. Thus there are 274 ways that Mark can climb the stairs before the spill has occurred.

Once the spill has occurred, we can work through the same way. If b_n is the number of ways to get to step n and not step on step 6, then we have: $b_n = a_n$ for $n \leq 5$, $b_6 = 0$. Thus, using the recurrence relation for a_n we get $b_7 = 0 + 13 + 7 = 20$, $b_8 = 20 + 0 + 13 = 33$, $b_9 = 0 + 20 + 33 = 53$ and $b_{10} = 20 + 33 + 53 = 106$. Thus, there are 106 ways to go up the stairs and avoid the spill.

5. Number the vertices of a cube from 1 to 8. Let A be the 8×8 matrix whose (i, j) entry is 1 if the cube has an edge between vertices i and j, and is 0 otherwise. Find the eigenvalues of A, and describe the corresponding eigenspaces.

Solution:Let the vertices be numbered as shown.



Then we have

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

We can find a few eigenvectors by inspection; for example $(1\ 1\ 1\ 1\ 1\ 1\ 1\ 1)^T$ is an eigenvector for $\lambda = 3$ and $(1\ \ 1\ \ 1\ \ 1\ \ -1\ \ -1\ \ -1\ \ -1)^T$ is an eigenvector for $\lambda = -1$. From these a pattern starts to emerge. Note that the entries 1 occur in positions 1, 2, 3, 4 and the entries -1 occur in positions 5, 6, 7, 8. Now, $\{1, 2, 3, 4\} \cup \{5, 6, 7, 8\} = S_1 \cup S_2$ is a decomposition of the vertex set $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ into two disjoint parts S_1, S_2 with the property that each vertex in S_i is connected to two vertices in S_i and one vertex in S_j for $i,\ j \in \{1,2\}$, $i \neq j$. We now have the observation, whose proof follows from the definition of A given in the problem:

Let $S = S_1 \cup S_2$ be any decomposition of $S = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8\}$ into disjoint subsets S_1, S_2 such that there exist integers k and l with $k + l = 3$ such that each vertex in S_i is connected to k vertices in S_i and l vertices in S_j for $i,\ j \in \{1,\ 2\}$, $i \neq j$. Then the vector with entries 1 in the positions corresponding to S_1 and -1 in the positions corresponding to S_2 is an eigenvector of A for the eigenvalue $k - l$.

Note that while we could interchange the subsets S_1 and S_2, that would just change the eigenvector to its opposite. Except for this, there are eight different decompositions of S of the desired type, as shown in the table below.

S_1	S_2	k	l	λ	Eigenvector
S	ϕ	3	0	3	$(1,1,1,1,1,1,1,1)^T$
$\{1,\ 2,\ 3,\ 4\}$	$\{5,\ 6,\ 7,\ 8\}$	2	1	1	$(1,1,1,1,-1,-1,-1,-1)^T$
$\{1,\ 2,\ 5,\ 6\}$	$\{3,\ 4,\ 7,\ 8\}$	2	1	1	$(1,1,-1,-1,1,1,-1,-1)^T$
$\{1,\ 4,\ 5,\ 8\}$	$\{2,\ 3,\ 6,\ 7\}$	2	1	1	$(1,-1,-1,1,1,-1,-1,1)^T$
$\{1,\ 2,\ 7,\ 8\}$	$\{3,\ 4,\ 5,\ 6\}$	1	2	-1	$(1,1,-1,-1,-1,-1,1,1)^T$
$\{1,\ 4,\ 6,\ 7\}$	$\{2,\ 3,\ 5,\ 8\}$	1	2	-1	$(1,-1,-1,1,-1,1,1,-1)^T$
$\{1,\ 3,\ 5,\ 7\}$	$\{2,\ 4,\ 6,\ 8\}$	1	2	-1	$(1,-1,1,-1,1,-1,1,-1)^T$
$\{1,\ 3,\ 6,\ 8\}$	$\{2,\ 4,\ 5,\ 7\}$	0	3	-3	$(1,-1,1,-1,-1,1,-1,1)^T$

(Geometrically, the eigenvectors for $\lambda = 1$ in this table can be thought of as corresponding to pairs of opposite faces of the cube; the eigenvectors for $\lambda = -1$ can be thought of as corresponding to pairs of diagonal planes through the cube.) It is easily checked that the eigenvectors for $\lambda = 1$ (similarly for $\lambda = -1$) listed above are linearly independent. Therefore, for matrix A we have eigenvalues $\lambda = 3$, $\lambda = 1$ (with multiplicity 3), $\lambda = -1$ (with multiplicity 3) and $\lambda = -3$, and the eigenspaces are spanned by the vectors in the table.

6. Let $f(x)$ be a twice-differentiable function on the open interval $(0,1)$ such that

$$\lim_{x\to 0^+} f(x) = -\infty\,, \quad \text{and} \quad \lim_{x\to 1^-} f(x) = +\infty\,.$$

Show that $f''(x)$ takes on both negative and positive values.

Solution: Take an arbitrary number x_0 in the interval $(0,1)$ and let $f'(x_0) = m$. Suppose that $f''(x) \geq 0$ for all x in $(0,1)$. Then, in particular, for all x with $0 < x < x_0$ we must have $f(x) \geq f(x_0) + m(x - x_0)$.

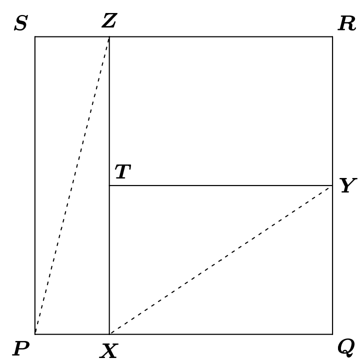
But $f(x) \geq f(x_0) + m(x - x_0)$ shows that, as $x \to 0^+$, we have $f(x) \geq f(x_0) - mx_0$, contradicting the given $\lim_{x\to 0^+} f(x) = -\infty$.

Similarly, if $f''(x) \leq 0$ for all x in $(0,1)$ then for all x with $x_0 < x < 1$ we must have $f(x) \leq f(x_0) + m(x - x_0)$, and this contradicts $\lim_{x\to 1^-} f(x) = +\infty$.

Thus, we must have $f''(x) > 0$ for some x in $(0, 1)$ as well as $f''(x) < 0$ for some x in $(0, 1)$, and we are done.

7. Three stationary sentries are guarding an important public square which is, in fact, square, with each side measuring 8 rods (recall that one rod equals 5.5 yards). (If any of the sentries see trouble brewing at any location on the square, the sentry closest to the trouble spot will immediately cease to be stationary and dispatch to that location. And like all good sentries, these three are continually looking in all directions for trouble to occur.) Find the maximum value of D, so that no matter how the sentries are placed, there is always some spot in the square that is at least D rods from the closest sentry.

Solution: Divide the square into rectangles as shown, in such a way that $QY = YR = 4$ and $PZ = XY$. We claim once this has been done $D_0 = \frac{1}{2}XY$ is the maximum value of D.



Proof: One way to place the sentries is at the centres of the rectangles $PXZS$, $XQYT$, $TYRZ$; for this placement, every point on the square is at most D_0 from the sentry at the centre of the rectangle it belongs to. Thus, a value $D > D_0$ is not possible. To show that $D = D_0$ actually works, we have to show that there is no placement of the sentries for which every point of the square has distance less than D_0 to the closest sentry. Suppose we did have such a placement. Since there are four corners P, Q, R, S of the square and only three sentries, at least two corners would be guarded by the same sentry. Without loss of generality, let P and S be two corners guarded by the same sentry. Now note that X and Y cannot be guarded by a single sentry, because by our assumption that sentry would have distance less than D_0 to both X and Y, so that $XY < 2D_0 = XY$, a contradiction. Similarly, no other two opposite corners of the three smaller rectangles that we constructed can be guarded by a single sentry. Thus, S, X, Y are all guarded by different sentries, say 1, 2, 3, in that order, and since S and P are guarded by the same sentry P, Z, Y are also guarded by 1, 2, 3 in that order. Thus, Q and R must be guarded by 3. But, we will show that the mid-point O of XQ is too far from Z and R to be guarded by 2 and 3, respectively, which will provide a contradiction.

Let $PX = a$, $XQ = 8 - a$. Then

$$PZ = XY \iff PX^2 + XZ^2 = XQ^2 + QY^2 ,$$

which gives $a = 1$, so that $PX = 1$, $QX = 7$, $XO = \frac{7}{2}$ and $ZO > ZP = 2D_0$, showing that O is indeed too far from Z. Thus, the answer is $D_0 = \frac{1}{2}\sqrt{65}$.

8. The Union Atlantic Railway is planning a massive project: a railroad track joining Cambridge, Massachusetts and Northfield, Minnesota. However, the funding for the project comes from the will of Orson Randolph Kane, the eccentric founder of the U.A.R., who has specified some strange conditions on the railway; thus the skeptical builders are unsure whether or not it is possible to build a railway subject to his unusual requirements.

Kane's will insists that there must be exactly 100 stops (each named after one of his great-grandchildren) between the termini, and he even dictates precisely what the distance along the track between each of these stops must be. (Unfortunately, the tables in the will *do not* list the order in which the stops are to appear along the railway.) Luckily, it is clear that Kane has put some thought into these distances; for any three distinct stops, the largest of the three distances among them is equal to the sum of the smaller two, which is an obvious necessary condition for the railway to be possible. (Also, all the given distances are shorter than the distance along a practical route from Cambridge to Northfield!)

U.A.R.'s engineers have pored over the numbers and noticed that for any four of Kane's stops, it would be possible to build a railway with these four stops and the distances between them as Kane specifies. Prove that, in fact, it is possible to complete the entire project to Kane's specifications.

Solution: Let $d(i, j)$ be the required distance from stop i to stop j along the track. Note that if four stops are arranged successfully along the track, there is a unique greatest distance between two of those four stops: the one that ends up closest to Cambridge, and the one that ends up closest to Northfield.

Now for the proof.

The case $n = 4$ is trivial. Suppose the statement is true for N stops ($N \geq 4$) and we have a table of distances $d(i, j)$ for $N + 1$ stops satisfying the given conditions. Find the largest of these distances. This largest distance can only occur *once*, because if $d(i, j) = d(k, l)$ with $\{i, j\} \neq \{k, l\}$, it would be impossible to arrange stops i, j, k, l (or if two of them were equal, say $j = l$, then stops i, j, k, m with $m \neq i$, j, k) successfully along the track. Without loss of generality, assume that the greatest distance in the table is $d(1, N + 1)$. Let a, b be any two distinct stops with $2 \leq a$, $b \leq N$. Then, since stops 1, a, b, $N+1$ could be arranged successfully, and $d(1, N+1)$ is the largest distance, we must have either $d(1, a) = d(1, b) + d(a, b)$ or $d(1, b) = d(1, a) + d(a, b)$ (but we cannot have $d(a, b) = d(1, a) + d(1, b)$). Now, ignore stop $N + 1$ for the moment and arrange stops $1, 2, \ldots, N$ successfully along the track, which is possible by the induction hypothesis. Since we cannot have $d(a, b) = d(1, a) + d(1, b)$ for any a, $b \in \{2, 3, \ldots, N\}$, we have that 1 cannot be between any two other stops in this successful arrangement; that is, all stops $2, 3, \ldots, N$ must be the same side of stop 1. Since they all have the proper distances to stop 1 and $d(1, N + 1)$ is larger than all those distances, we can place stop $N + 1$ on the track in such a way that all stops $2, 3, \ldots, N$ are between stop 1 and stop $N + 1$, that stops 1 and $N + 1$ are the proper distance apart, and that all stops $1, 2, \ldots, N$ are still in the same places.

All we now need to do is to check that stop $N+1$ is the correct distance $d(a, N + 1)$ to each of the stops a, $2 \le a \le N$. Then, since $d(1, N + 1)$ is less than the distance from Cambridge to Northfield, we can arrange for the stops to follow each other in this order between Cambridge and Northfield, and we will be done.

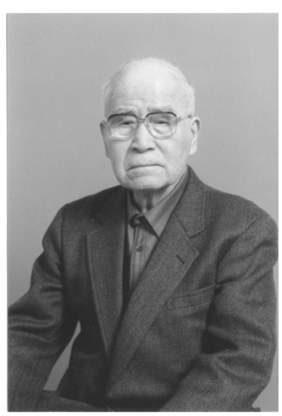
Consider any stop a, $2 \le a \le N$. Since it has the correct distance $d(a, 1)$ to stop 1, in the above placement its distance to stop $N + 1$ is given by $d(1, N + 1) - d(a, 1)$. But since $d(1, N + 1)$ is the largest of the required distances between stops 1, a, $N + 1$, we have

$$d(1, N + 1) \;=\; d(a, 1) + d(a, N + 1)\,,$$

so that

$$d(1, N + 1) - d(a, 1) \;=\; d(a, N + 1)\,,$$

and we are done.



Professor Toshio Seimiya

Problem of the Month

Jimmy Chui, student, University of Toronto

Problem. (a) Given positive numbers a_1, a_2, a_3, $\cdots$, a_n and the quadratic function $f(x) = \sum_{i=1}^{n} (x - a_i)^2$, show that $f(x)$ attains its minimum value at $\frac{1}{n}\sum_{i=1}^{n} a_i$, and prove that $\sum_{i=1}^{n} a_i^2 \geq \frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2$.

(b) The sum of sixteen positive numbers is 100 and the sum of their squares is 1000. Prove that none of the sixteen numbers is greater than 25.

(1996 Canadian Open, Problem B3)

Solution. (a) The quadratic function $f(x)$ is a parabola, and the graph $y = f(x)$ opens upward. (The x^2–coefficient is positive.) Hence the vertex of the graph is a minimum point; that is, there is a unique value of x that minimizes $y = f(x)$, and the point (x, y) is the vertex.

It is known that the x–coordinate of the vertex of the function $f(x) = ax^2 + bx + c$ is $-b/2a$. Our given function, after expanding, is $f(x) = nx^2 - 2x\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} a_i^2$, and the x-coordinate of the vertex is $\frac{1}{n}\sum_{i=1}^{n} a_i$. Hence, for this value of x, the value of $f(x)$ is minimized.

We also know that $f(x)$ is greater than or equal to 0, since it is the sum of non-negative squares. Hence, the discriminant must be less than or equal to 0. (This condition corresponds to $f(x)$ having one or no roots.)

If we let the discriminant be D, then $D/4 = \left(\sum_{i=1}^{n} a_i\right)^2 - n\cdot\sum_{i=1}^{n} a_i^2 \leq 0$. Rearranging this inequality gives us the desired result, $\sum_{i=1}^{n} a_i^2 \geq \frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2$.

(b) Let the largest value of the a_i's be b. Consider the 15 a_i's, excluding b. Then, apply the result in (a) to these 15 numbers. We have $\sum a_i^2 - \frac{1}{15}\left(\sum a_i\right)^2 \geq 0$, where both summations are taken over the 15 a_i's excluding b.

$$\begin{aligned}\sum a_i^2 - \frac{1}{15}\left(\sum a_i\right)^2 &= \frac{1}{15}\cdot\left\{15\left(1000 - b^2\right) - (100 - b)^2\right\} \\ = \frac{1}{15}\cdot\left(-16b^2 + 200b + 5000\right) &= -\frac{8}{15}\cdot(b - 25)(2b + 25) \geq 0\,.\end{aligned}$$

Since b is a positive value, the inequality holds true if and only if $b \leq 25$. In other words, the largest of the a_i's must not exceed 25, QED.

SKOLIAD No. 60

Shawn Godin

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mayhem-editors@cms.math.ca

Please include on any correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 July 2002*. A copy of ***MATHEMATICAL MAYHEM*** **Vol. 2** will be presented to the pre-university reader(s) who send in the best set of solutions before the deadline. The decision of the editor is final.

This issue's item comes to us from Manitoba. My thanks go to Diane Dowling at St.Paul's College in Winnipeg for forwarding the material to me.

MANITOBA MATHEMATICAL CONTEST, 2001

For Students in Senior 4

9:00 a.m. – 11:00 a.m. Wednesday, February 21, 2001

Sponsored by

The Actuaries' Club of Winnipeg, The Manitoba Association of Mathematics Teachers, The Canadian Mathematical Society, and The University of Manitoba

Answer as much as possible. You are not expected to complete the paper. See both sides of this sheet. Hand calculators are not permitted. Numerical answers only, without explanation, will not be given full credit.

1. (a) Solve the equation $\frac{6}{x} - \frac{1}{x^2} = 9$.

(b) Find real numbers a and b such that $(2^a)(4^b) = 8$ and $a+7b = 4$.

2. (a) Find all real numbers x such that $25|x| = x^2 + 144$.

(b) If x is a real number and $\frac{5}{\sqrt{7+3\sqrt{x}}} = \sqrt{7-3\sqrt{x}}$ what is the value of x?

3. (a) If two of the roots of $x^3+px^2+qx+r=0$ are equal in absolute value but opposite in sign, prove that $pq = r$. (p, q and r are real numbers.)

(b) If a, b and c are real numbers such that $a+b+c = 14$, $c^2 = a^2+b^2$ and $ab = 14$, find the numerical value of c.

4. (a) If $0^\circ < \theta < 180^\circ$ and $2\sin^2\theta + 3\sin\theta \geq 2$ what is the largest possible value of θ?

(b) Let O be the origin, P the point whose coordinates are $(2, 3)$ and F a point on the line whose equation is $y = \frac{x}{2}$. If PF is perpendicular to OF find the coordinates of F.

5. How many consecutive zeros are there at the end of the product of all the integers from 16 to 100 inclusive?

6. In triangle ABC, $\angle ACB = 135^\circ$, $CA = 6$ and $BC = \sqrt{2}$. If M is the mid-point of the side AB, find the length of CM.

7. A circle of radius 2 has its centre in the first quadrant and has both coordinate axes as tangents. Another smaller circle also has both coordinate axes as tangents and has exactly one point in common with the larger circle. Find the radius of the smaller circle.

8. A parallelogram has an area of 36 and diagonals whose lengths are 10 and 12. Find the lengths of its sides.

9. a, b, c, d are distinct integers such that $(x-a)(x-b)(x-c)(x-d)=4$ has an integral root r. Prove that $a + b + c + d = 4r$.

10. If x, y and z are positive real numbers, prove that

$$(x + y - z)(x - y)^2 + z(x - z)(y - z) \;\geq\; 0\,.$$

Next we present the official solutions to the Mandelbrot competitions from the October issue [2001 : 385].

The Mandelbrot Competition
Division B Round Two Individual Test
December 1997

1. If a group of positive integers has a sum of 8, what is the greatest product the group can have? **(1 point)**

Solution.

Clearly we do not want any 1's in our group, since they contribute to the sum but not the product. Any 5 in the group can be replaced by a 2 and a 3, which have the same sum but a greater product. Similarly, any 6 can be replaced by two 3's, and so on, since every number contributes more to the product if broken down into 2's and 3's. (A 4 can be replaced with two 2's, with no effect on the sum or product.) Hence, our optimal group should consist of 2's and 3's only. The only such sets of numbers summing to 8 are $2+2+2+2$ or $2+3+3$. The products in these cases are 16 and 18. Hence, $\boxed{18}$ is optimal.

2. There is one two-digit number such that if we add 1 to the number and reverse the digits of the result, we obtain a divisor of the number. What is the number? **(1 point)**

Solution.

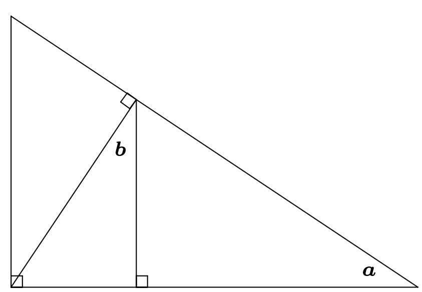
Let the two-digit number n be $\boxed{\boxed{t \mid u}}$. If u is not 9, then the number obtained by adding one and reversing the digits is $\boxed{\boxed{u+1 \mid t}}$. Since this potential divisor cannot equal n, it must be half of n or less, so that $2(u+1) \leq t$ and $2\left((u+1) \times 10 + t\right) \leq t \times 10 + u$. This restricts us to the numbers $n = 30$, 40, 50, 51, 60, 61, 70, 71, 80, 81, 82, 90, 91, or 92. A quick check shows that none of these numbers works. Hence, u must be 9. If $n = \boxed{\boxed{t \mid 9}}$, then $n + 1 = \boxed{\boxed{t+1 \mid 0}}$ (unless $t = 9$), so that the divisor would be $t + 1$. But n equals $10(t+1) - 1$, which cannot be a multiple of $t + 1$. Thus t must also be 9, yielding $n = \boxed{99}$, making the divisor equal to 001, or 1.

3. Ten slips of paper, numbered 1 through 10, are placed in a hat. Three numbers are drawn out, one after another. What is the probability that the three numbers are drawn in increasing order? **(2 points)**

Solution.

Let the numbers chosen be A, B, and C. There are six orders in which the slips can be chosen: ABC, ACB, BAC, BCA, CAB, CBA. Of these six, only one is in the increasing order we desire. Hence, the probability is $\boxed{\frac{1}{6}}$.

4. The three marked angles are right angles. If $\angle a = 20°$, then what is $\angle b$? **(2 points)**



Solution.

Note that $\angle b$ is complementary to an angle which is complementary to $\angle a$; hence, $\angle b = \angle a$, so that $\angle b = \boxed{20°}$.

5. Vicky asks Charlene to identify all non-congruent triangles $\triangle ABC$ given:

(a) the value of $\angle A$

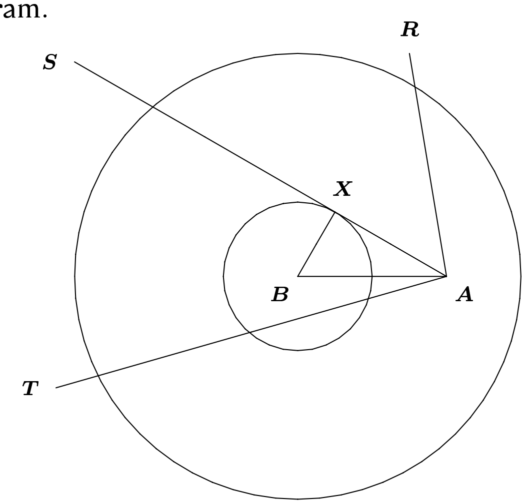
(b) $AB = 10$, and

(c) length BC equals either 5 or 15.

Charlene responds that there are only two triangles meeting the given conditions. What is the value of $\angle A$? **(2 points)**

Solution.

Consider $\overline{AB}$ to be a fixed segment of length 10. Since BC is either 5 or 15, C must lie on one of the two circles with center B and radii 5 and 15, as in the diagram.



There are many possible values for $\angle A$; the possible positions for point C are the points where $\angle A$ intersects the two circles. For some angles, as with $\angle A = \angle BAR$ in the diagram, there is only one possible point C; for others, as with $\angle A = \angle BAT$, there are three. The only value which gives exactly two points C is that which makes the angle tangent to the inner circle, as with $\angle A = \angle BAS$. Call the point of tangency X. Since $AB = 10$, $BX = 5$, and $\angle BXA$ is a right angle, we conclude that triangle ABX is a $30°$-$60°$-$90°$ triangle. Hence, $\angle A = \boxed{30°}$.

6. Five pirates find a cache of five gold coins. They decide that the shortest pirate will become bursar and distribute the coins — if half or more of the pirates (including the bursar) agree to the distribution, it will be accepted; otherwise, the bursar will walk the plank and the next shortest pirate will become bursar. This process will continue until a distribution of coins is agreed upon. If each pirate always acts so as to stay aboard if possible and maximize his wealth, and would rather see another pirate walk the plank than not (all else being equal), then how many coins will the shortest pirate keep for himself? **(3 points)**

Solution.

Call the pirates $p_1, p_2, \dots, p_5$, with p_1 the shortest and p_5 the tallest. Consider what would happen if only p_4 and p_5 remained. Whatever division strategy p_4 suggested would hold, since p_4's vote alone would constitute half the total vote. Thus, p_4 would simply allot himself all the gold. Next consider the situation where three pirates remained. Whatever distribution p_3 chose, p_5 would have to agree, as long as he got one or more coins. His only alternative would be to go to the two-pirate situation, in which p_5 gets nothing at all. Hence, p_3 would simply take 4 coins for himself and allot 1 coin to p_5, getting a majority vote from himself and p_5. Similarly, with four pirates remaining, p_2, the bursar, would take 4 coins for himself and allot 1 coin to p_4. Again, since p_4 would have otherwise gotten nothing, he would have to support the plan. With five pirates, the bursar, p_1, would allot himself $\mathbf{3}$ coins and give one coin each to p_3 and p_5. Since each of p_3 and p_5 gets more than they would get by vetoing the plan, they must support it. The shortest pirate gets $\boxed{3}$ coins.

7. The twelve positive integers $a_1 \le a_2 \le \cdots \le a_{12}$ have the property that no three of them can be the side lengths of a non-degenerate triangle. Find the smallest possible value of $\frac{a_{12}}{a_1}$. **(3 points)**

Solution.

If a, b and c can be sides of a non-degenerate triangle with $a \le b \le c$, we always must have $c < a + b$. Hence, if no three of our integers can form a non-degenerate triangle, we must have $a_i \ge a_j + a_k$ for any three with $i > j, k$. Since the numbers are increasing, it suffices to show that

$$a_1 + a_2 \le a_3\,, \quad a_2 + a_3 \le a_4\,, \quad a_3 + a_4 \le a_5$$

and so on. Substituting the first inequality into the second, we have

$$a_1 + 2a_2 \le a_4\,.$$

Substituting this and the second inequality into the third, we get

$$2a_1 + 3a_2 \le a_5\,.$$

Substituting into the fifth inequality:

$$3a_1 + 5a_2 \le a_6\,.$$

We continue in this manner. At each step, we simply increase the coefficients of a_1 and a_2; they must increase as Fibonacci numbers, since at each step we get the new coefficient by adding the previous two. Hence, the final inequality will read

$$55a_1 + 89a_2 \le a_{12}\,.$$

To obtain the smallest possible ratio we choose $a_1 = a_2$. This simply yields $144a_1 \le a_{12}$, or $a_{12}/a_1 \ge 144$. Since this limit can be attained, with $a_1 = a_2 = 1$, $a_3 = 2$, $a_4 = 3$, $a_5 = 5$, $\ldots$, $a_{12} = 144$, we find that $\boxed{144}$ is the least value.

The Mandelbrot Competition
Division B Round Two Team Test
December 1997

FACTS: A polynomial $p(x)$ of degree n or less is determined by its value at $n+1$ x–coordinates. For $n=1$ this is a familiar statement; a line (degree one polynomial) is determined by two points. Moreover, the value of $p(x)$ at any other x–value can be computed in a particularly nice way using Lagrange interpolation, as outlined in the essay *An Interpretation of Interpolation*.

We will also need a result from linear algebra which states that a system of n "different" linear equations in n variables has exactly one solution. For example, there is only one choice for x, y, and z which satisfies the equations $x+y+z=1$, $x+2y+3z=4$, and $x+4y+9z=16$.

SETUP: Let $p(x)$ be a degree three polynomial for which we know the values of $p(1)$, $p(2)$, $p(4)$, and $p(8)$. By the facts section there is exactly one such polynomial. According to Lagrange interpolation the number $p(16)$ can be deduced; it equals

$$p(16) \;=\; A_0p(1) + A_1p(2) + A_2p(4) + A_3p(8)\,,$$

for some constants A_0 through A_3. The goal of this team test will be to compute the A_i and use them to find information about $p(16)$ *without ever finding an explicit formula for* $p(x)$.

Problems:

Part i: (4 points) We claim that the A_i can be found by subtracting

$$x^4-(x-1)(x-2)(x-4)(x-8) \;=\; A_3x^3+A_2x^2+A_1x+A_0\,. \tag{1}$$

Implement this claim to compute A_0 through A_3.

Solution.

Apparently we have been handed a magic formula which generates the constants A_i needed for computing $p(16)$. It is referred to frequently in this solution, so we reproduce it here:

$$x^4-(x-1)(x-2)(x-4)(x-8) = A_3x^3+A_2x^2+A_1x+A_0\,. \tag{2}$$

Rather than marvel at this stroke of good fortune so early on, we set about computing the values A_0 through A_3. Carefully multiplying out the left-hand side yields

$$\begin{aligned}
&x^4-(x-1)(x-2)(x-4)(x-8)\\
&\quad= x^4-(x^2-3x+2)(x^2-12x+32)\\
&\quad= x^4-(x^4-15x^3+70x^2-120x+64)\\
&\quad= 15x^3-70x^2+120x-64\\
&\quad= A_3x^3+A_2x^2+A_1x+A_0\,.
\end{aligned}$$

Therefore we should use $A_3 = 15$, $A_2 = -70$, $A_1 = 120$, and $A_0 = -64$.

Part ii: (4 points) To show that these A_i are in fact the correct numbers we must show that they correctly predict $p(16)$ for four "different" polynomials. We begin with the case $p(x) = x$. Show that the value of $p(16)$ agrees with the prediction $A_0p(1) + A_1p(2) + A_2p(4) + A_3p(8)$. (HINT: try $x = 2$ in (1).)

Solution.

According to Lagrange Interpolation, if $p(x)$ is a polynomial of degree three or less, then we should be able to predict $p(16)$ based on the values of $p(x)$ at $x = 1, 2, 4$ and 8. We will now show that the constants A_i just computed do the job by showing that $p(16)$ always equals the sum $A_3p(8) + A_2p(4) + A_1p(2) + A_0p(1)$. According to the facts section we need only verify that the A_i work in four different cases to know that they will always work. In checking these four cases we employ a somewhat clever method that never actually uses the numbers calculated in part i, just the equation (2) that produced them.

First suppose that $p(x) = x$. Then clearly $p(1) = 1$, $p(2) = 2$, $p(4) = 4$, and $p(8) = 8$. Let us check whether or not $A_3p(8) + A_2p(4) + A_1p(2) + A_0p(1)$, which is the same as $8A_3 + 4A_2 + 2A_1 + A_0$, correctly predicts $p(16)$. Substituting $x = 2$ into equation (2) gives us

$$2^4 - (2-1)(2-2)(2-4)(2-8) \ = \ 8A_3 + 4A_2 + 2A_1 + A_0 \,.$$

The left-hand side equals 16 since the $(2 - 2)$ factor causes the second term to vanish. However, the right hand side is our prediction for $p(16)$. Sure enough, we get $p(16) = 16$, just as we should for the function $p(x) = x$.

Part iii: (5 points) Continuing the previous part, show that the A_i correctly predict $p(16)$ for the three other polynomials $p(x) = 1$, $p(x) = x^2$ and $p(x) = x^3$.

Solution.

In all cases the prediction for $p(16)$ is $A_3p(8) + A_2p(4) + A_1p(2) + A_0p(1)$. Continuing our work from part ii we try $p(x) = 1$, so our prediction becomes $A_3 + A_2 + A_1 + A_0$. Using $x = 1$ in (2) yields

$$1^4 - (1-1)(1-2)(1-4)(1-8) \ = \ A_3 + A_2 + A_1 + A_0 \,.$$

The left-hand side reduces to 1, so that the prediction is $p(16) = 1$, which again is correct. When $p(x) = x^2$ the prediction for $p(16)$ becomes $64A_3 + 16A_2 + 4A_1 + A_0$. This can be found quickly by substituting $x = 4$ into equation (2):

$$4^4 - (4-1)(4-2)(4-4)(4-8) \ = \ 64A_3 + 16A_2 + 4A_1 + A_0 \,.$$

The now familiar cancellation occurs on the left hand side, leaving us with a prediction of 4^4 for $p(16)$. Since $p(x) = x^2$ we expect to have $p(16) = 16^2$, and indeed $16^2 = (4^2)^2 = 4^4$. The case of $p(x) = x^3$ works in exactly the same manner, substituting $x = 8$ into equation (2), so we encourage the reader to try it as practice. (Naturally teams were expected to show the details for this case as well in their solutions!)

Notice that we were able to do all of our checking without ever using the numerical values of A_0 through A_3. The other more obvious method is to plug in the values for the A_i and do the arithmetic. However, the slick technique can be generalized, while the more routine method cannot.

Part iv: (4 points) Suppose that $p(x)$ is a third degree polynomial with $p(1) = 0$, $p(2) = 1$, and $p(4) = 3$. What value should $p(8)$ have to guarantee that $p(x)$ has a root at $x = 16$?

Solution.

We have now verified that $A_3 = 15$, $A_2 = -70$, $A_1 = 120$, and $A_0 = -64$ are the correct values needed to interpolate $p(16)$. We are also told in this problem that $p(1) = 0$, $p(2) = 1$, and $p(4) = 3$. Furthermore, we want $p(16) = 0$ so that $p(x)$ has a root at $x = 16$. Substituting all of these values into our interpolation formula produces

$$\begin{aligned} p(16) &= A_3p(8) + A_2p(4) + A_1p(2) + A_0p(1) \\ \Longrightarrow \quad 0 &= 15p(8) - 70\cdot 3 + 120\cdot 1 - 64\cdot 0 \\ \Longrightarrow \quad 90 &= 15p(8) \\ \Longrightarrow \quad 6 &= p(8)\,. \end{aligned}$$

Thus we need $p(8) = 6$ to ensure that $p(16) = 0$ so that there is a root at $x = 16$.

Part v: (4 points) Let $p(x)$ be a degree three polynomial with $p(1) = 1$, $p(2) = 3$, $p(4) = 9$, and $p(8) = 27$. Calculate $p(16)$. How close does it come to the natural guess of 81?

Solution.

There are two ways to do this problem—a long way and a short way. The long way involves substituting all of the values for A_0 through A_3 and $p(1)$ through $p(8)$ into the interpolation formula shown above and cranking out the answer. The short way, hinted at by our above work, involves plugging $x = 3$ into equation (2). We opt for the short way, obtaining

$$3^4 - (3-1)(3-2)(3-4)(3-8) = 27A_3 + 9A_2 + 3A_1 + A_0\,.$$

The left-hand side reduces to $81 - (2)(1)(-1)(-5) = 71$, while the expression on the right is exactly the interpolation formula for $p(16)$. Therefore we have $p(16) = 71$, ten less than the intuitive guess of 81.