# Problem of the Month

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**Problem**. (a) Given positive numbers $a_1$, $a_2$, $a_3$, $\cdots$, $a_n$ and the quadratic function $f(x) = \sum_{i=1}^{n} (x - a_i)^2$, show that $f(x)$ attains its minimum value at $\frac{1}{n}\sum_{i=1}^{n} a_i$, and prove that $\sum_{i=1}^{n} a_i^2 \geq \frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2$.

(b) The sum of sixteen positive numbers is $100$ and the sum of their squares is $1000$. Prove that none of the sixteen numbers is greater than $25$.

(1996 Canadian Open, Problem B3)

**Solution.** (a) The quadratic function $f(x)$ is a parabola, and the graph $y = f(x)$ opens upward. (The $x^2$–coefficient is positive.) Hence the vertex of the graph is a minimum point; that is, there is a unique value of $x$ that minimizes $y = f(x)$, and the point $(x, y)$ is the vertex.

It is known that the $x$–coordinate of the vertex of the function $f(x) = ax^2 + bx + c$ is $-b/2a$. Our given function, after expanding, is $f(x) = nx^2 - 2x\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} a_i^2$, and the $x$-coordinate of the vertex is $\frac{1}{n}\sum_{i=1}^{n} a_i$. Hence, for this value of $x$, the value of $f(x)$ is minimized.

We also know that $f(x)$ is greater than or equal to $0$, since it is the sum of non-negative squares. Hence, the discriminant must be less than or equal to $0$. (This condition corresponds to $f(x)$ having one or no roots.)

If we let the discriminant be $D$, then $D/4 = \left(\sum_{i=1}^{n} a_i\right)^2 - n \cdot \sum_{i=1}^{n} a_i^2 \leq 0$. Rearranging this inequality gives us the desired result, $\sum_{i=1}^{n} a_i^2 \geq \frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2$.

(b) Let the largest value of the $a_i$'s be $b$. Consider the $15$ $a_i$'s, excluding $b$. Then, apply the result in $(a)$ to these $15$ numbers. We have $\sum a_i^2 - \frac{1}{15}\left(\sum a_i\right)^2 \geq 0$, where both summations are taken over the $15$ $a_i$'s excluding $b$.

$$\begin{aligned} \sum a_i^2 - \frac{1}{15}\left(\sum a_i\right)^2 &= \frac{1}{15} \cdot \left\{15\left(1000 - b^2\right) - (100 - b)^2\right\} \\ = \frac{1}{15} \cdot \left(-16b^2 + 200b + 5000\right) &= -\frac{8}{15} \cdot (b - 25)(2b + 25) \geq 0\,. \end{aligned}$$

Since $b$ is a positive value, the inequality holds true if and only if $b \leq 25$. In other words, the largest of the $a_i$'s must not exceed $25$, QED.