On a "Problem of the Month"

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In the problem of the month [1999 : 106], one was to prove that

$$\sqrt{a+b-c}+\sqrt{b+c-a}+\sqrt{c+a-b} \;\leq\; \sqrt{a}+\sqrt{b}+\sqrt{c}\,,$$

where a, b, c are sides of a triangle.

It is to be noted that this inequality will follow immediately from the Majorization Inequality [1]. Here, if $\mathbf{A}$ and $\mathbf{B}$ are vectors $(a_1, a_2, \ldots, a_n)$, $(b_1, b_2, \ldots, b_n)$ where $a_1 \geq a_2 \geq \cdots \geq a_n$, $b_1 \geq b_2 \geq \cdots \geq b_n$, and $a_1 \geq b_1$, $a_1+a_2 \geq b_1+b_2$, $\ldots$, $a_1+a_2+\cdots+a_{n-1} \geq b_1+b_2+\cdots+b_{n-1}$, $a_1+a_2+\cdots+a_n = b_1+b_2+\cdots+b_n$, we say that $\mathbf{A}$ majorizes $\mathbf{B}$ and write it as $\mathbf{A} \succ \mathbf{B}$. Then, if F is a convex function,

$$F(a_1)+F(a_2)+\cdots+F(a_n) \;\geq\; F(b_1)+F(b_2)+\cdots+F(b_n)\,.$$

If F is concave, the inequality is reversed.

For the triangle inequality, we can assume without loss of generality that $a \geq b \geq c$. Then $a+b-c \geq a$, $(a+b-c)+(a+c-b) \geq a+b$, and $(a+b-c)+(a+c-b)+(b+c-a) = a+b+c$. Therefore, if F is concave,

$$F(a+b-c)+F(b+c-a)+F(c+a-b) \;\leq\; F(a)+F(b)+F(c)$$

(for the given inequality, $F = \sqrt{x}$ is concave).

As to the substitution $a = y+z$, $b = z+x$, $c = x+y$ which was used in the referred to solution and was called the Ravi Substitution, this transformation was known and used before he was born. Geometrically, x, y, z are the lengths which the sides are divided into by the points of tangency of the incircle. Thus, we have the following implications for any triangle inequality or identity:

$$\begin{aligned} F(a,b,c) \;\geq\; 0 \quad &\Longleftrightarrow \quad F(y+z, z+x, x+y) \;\geq\; 0\,, \\ F(x,y,z) \;\geq\; 0 \quad &\Longleftrightarrow \quad F((s-a),(s-b),(s-c)) \;\geq\; 0 \end{aligned}$$

(here s is the semiperimeter). This transformation eliminates the troublesome triangle constraints and lets one use all the machinery for a set of three non-negative numbers.

Another big plus for the Majorization Inequality is that we can obtain both upper and lower bounds subject to other kinds of constraints. Here are two examples:

(1) Consider the bounds on $\sin a_1 + \sin a_2 + \cdots + \sin a_n$ where $n \geq 4$, $\frac{\pi}{2} \geq a_i \geq 0$ and $\sum a_i = S \leq 2\pi$. Since

$$\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0, 0, \ldots, 0\right) \succ (a_1, a_2, \ldots, a_n) \succ \left(\frac{S}{n}, \frac{S}{n}, \ldots, \frac{S}{n}\right),$$

we have

$$4 \leq \sin a_1 + \sin a_2 + \cdots + \sin a_n \leq n \sin\left(\frac{S}{n}\right).$$

(2) Consider the bounds on $a_1^2 + a_2^2 + \cdots + a_n^2$ where $\sum a_i = S$ ($\geq n$) and the a_i's are positive integers. Since

$$(S - n + 1, 1, 1, \cdots, 1) \succ (a_1, a_2, \cdots, a_n) \succ \left(\frac{S}{n}, \frac{S}{n}, \cdots, \frac{S}{n}\right),$$

we have

$$(S - n + 1)^2 + n - 1 \geq a_1^2 + a_2^2 + \cdots + a_n^2 \geq n\left(\frac{S}{n}\right)^2.$$

For many other applications, see [1].

Reference

1. A.W. Marshall, I. Olkin, Inequalities: Theory of Majorization and Its Applications, Academic Press, NY, 1979.

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Substitutions, Inequalities, and History

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The Non-Ravi Substitution

A solution to the inequality (1996 Asian Pacific Mathematical Competition)

$$\sqrt{a+b-c}+\sqrt{a-b+c}+\sqrt{-a+b+c} \;\leq\; \sqrt{a}+\sqrt{b}+\sqrt{c} \qquad (1)$$

where a, b, c are the sides of a triangle, appeared in Crux [1999 : 106]. It starts with the substitution $a = x + y$, $b = y + z$, $c = z + x$ (x, y, $z > 0$). This substitution is referred to as the "Ravi Substitution" and reported to be known by this name, at least in Canadian IMO circles.

It seems that this awkward credit for the substitution diffused to wider circles. The same inequality (1) appears in a French problem solving book from 1999 [9, p. 146]. Although the solution proposed in [9] is different, it starts with the same substitution which, amazingly, is called there too the "Ravi Substitution". Further, [9] includes several other mentions and applications of the "Ravi Substitution" [9, pp. 130, 146, 147, 155, 237].

In response to Crux [1999 : 106], Klamkin [6] comments telegraphically that the substitution $a = x+y$, $b = y+z$, $c = z+x$ is a classical technique that was known and used before he (Ravi) was born. It is time to set things straight.

Although Klamkin gives no reference in his piece, he is right; this substitution is one of the first strategies taught in Olympiad training classes, at least — *à la Crux* — in Israeli circles. But since even folklore should be supported by some written evidence, can we find such support? The answer is positive. One example appears in Engel's book (an English translation of previous German versions) [2, p. 178]: the substitution $a = x+y$, $b = y+z$, $c = z + x$ with x, y, $z > 0$ is advice number 7 in Engel's list of 18 possible strategies for proving inequalities. It is explained in [2, p. 164] that this substitution is merely another manifestation of the triangle inequality. The same explanation appears in more detail in [8, Chapter II, p. 26], a book devoted to geometric inequalities. It provides some references to papers where such equivalent forms of the triangle inequality are mentioned (see pp. 35–36). One of these references, namely [5], is a paper from 1971 — written by Klamkin.

Consequently, even without digging into earlier references (which are probably easy to find) Klamkin's remark is evidently correct.

Karamata and the Majorization Inequality

We continue with a historical mood. Klamkin [6] proposes to solve (1) by using the Majorization Inequality. This inequality relates to two sequences $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ and states that: $a_1+\cdots+a_i \leq b_1+\cdots+b_i$ for $1 \leq i \leq n-1$ and $a_1+\cdots+a_n = b_1+\cdots+b_n$ if and only if $f(a_1)+\cdots+f(a_n) \geq f(b_1)+\cdots+f(b_n)$ for any convex function $f(x)$. Clearly, this inequality includes Jensen's Inequality as a special case. The proof of one direction is easy, and the more intricate part can be proved by applying the Abel Summation Formula

$$\sum_{i=1}^{n} a_i b_i = \sum_{j=1}^{n} \left(\sum_{i=1}^{j} a_i \right) (b_j - b_{j+1}) \,,$$

setting $b_{n+1} = 0$.

The Majorization Inequality is well known, but unfortunately, this generic name does not reveal its source: this inequality is due to Karamata, 1932 [4], and should therefore be called the Karamata Inequality, as in [1, pp. 31-32]. It turns out to be a strong tool with various applications, some of which can be found in [7] and [8, Chapter VIII]). We also note that before [6], the inequality (1) appeared in [3] (a paper in Hebrew) as an example (probably well known even before) of a case where the Karamata Inequality is a useful approach.

Substitution and the Karamata Inequality

We conclude with an example in the spirit of [6], where a substitution followed by the application of the Karamata Inequality leads to a solution. This is IMO 2000 Problem 2 (as solved by one of the Israeli contestants, Eran Assaf; the problem has some half dozen solutions): Let a, b, c be positive numbers with $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \;\leq\; 1 \,. \tag{2}$$

The substitution $a = x/y$, $b = y/z$, $c = z/x$ (x, y, $z > 0$) converts (2) to

$$(x - y + z)(y - z + x)(z - x + y) \;\leq\; xyz \,. \tag{3}$$

Without loss of generality, assume that $x \leq y \leq z$, so that $(y + z - x)$, $(z + x - y) > 0$. If $x + y - z \leq 0$ (3) follows immediately, so that we may assume that $x + y > z$ as well.

Since $x \geq x + y - z$, $x + y \geq (x + y - z) + (z + x - y)$, and $x + y + z = (x + y - z) + (z + x - y) + (y + z - x)$, we can apply the Karamata Inequality to the triplets (x, y, z) and $(x+y-z, x+z-y, z+y-x)$, and obtain (3) by writing

$$f(x) + f(y) + f(z) \;\leq\; f(x + y - z) + f(z + x - y) + f(y + z - x)$$

for the convex function $f(x) = -\ln x$.

Returning to history, it turns out that inequality (3) to which the IMO 2000 problem is reduced/equivalent, is not really a new one: it is due to A. Padoa, in 1925 (Period. Mat. (4)5 : 80–85). Moreover, (3) is equivalent to $a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c) \leq 3abc$, which is IMO 1964 Problem 2. Could you guess what substitution is helpful for proving these inequalities easily?

References

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