On a "Problem of the Month"

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In the problem of the month [1999 : 106], one was to prove that

$$\sqrt{a+b-c}+\sqrt{b+c-a}+\sqrt{c+a-b} \;\leq\; \sqrt{a}+\sqrt{b}+\sqrt{c}\,,$$

where a, b, c are sides of a triangle.

It is to be noted that this inequality will follow immediately from the Majorization Inequality [1]. Here, if $\mathbf{A}$ and $\mathbf{B}$ are vectors $(a_1, a_2, \ldots, a_n)$, $(b_1, b_2, \ldots, b_n)$ where $a_1 \geq a_2 \geq \cdots \geq a_n$, $b_1 \geq b_2 \geq \cdots \geq b_n$, and $a_1 \geq b_1$, $a_1+a_2 \geq b_1+b_2$, $\ldots$, $a_1+a_2+\cdots+a_{n-1} \geq b_1+b_2+\cdots+b_{n-1}$, $a_1+a_2+\cdots+a_n = b_1+b_2+\cdots+b_n$, we say that $\mathbf{A}$ majorizes $\mathbf{B}$ and write it as $\mathbf{A} \succ \mathbf{B}$. Then, if F is a convex function,

$$F(a_1)+F(a_2)+\cdots+F(a_n) \;\geq\; F(b_1)+F(b_2)+\cdots+F(b_n)\,.$$

If F is concave, the inequality is reversed.

For the triangle inequality, we can assume without loss of generality that $a \geq b \geq c$. Then $a+b-c \geq a$, $(a+b-c)+(a+c-b) \geq a+b$, and $(a+b-c)+(a+c-b)+(b+c-a) = a+b+c$. Therefore, if F is concave,

$$F(a+b-c)+F(b+c-a)+F(c+a-b) \;\leq\; F(a)+F(b)+F(c)$$

(for the given inequality, $F = \sqrt{x}$ is concave).

As to the substitution $a = y+z$, $b = z+x$, $c = x+y$ which was used in the referred to solution and was called the Ravi Substitution, this transformation was known and used before he was born. Geometrically, x, y, z are the lengths which the sides are divided into by the points of tangency of the incircle. Thus, we have the following implications for any triangle inequality or identity:

$$\begin{array}{lcl} F(a,b,c) \;\geq\; 0 & \Longleftrightarrow & F(y+z, z+x, x+y) \;\geq\; 0\,, \\ F(x,y,z) \;\geq\; 0 & \Longleftrightarrow & F((s-a),(s-b),(s-c)) \;\geq\; 0 \end{array}$$

(here s is the semiperimeter). This transformation eliminates the troublesome triangle constraints and lets one use all the machinery for a set of three non-negative numbers.

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Another big plus for the Majorization Inequality is that we can obtain both upper and lower bounds subject to other kinds of constraints. Here are two examples:

(1) Consider the bounds on $\sin a_1 + \sin a_2 + \cdots + \sin a_n$ where $n \geq 4$, $\frac{\pi}{2} \geq a_i \geq 0$ and $\sum a_i = S \leq 2\pi$. Since

$$\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0, 0, \ldots, 0\right) \succ (a_1, a_2, \ldots, a_n) \succ \left(\frac{S}{n}, \frac{S}{n}, \ldots, \frac{S}{n}\right),$$

we have

$$4 \leq \sin a_1 + \sin a_2 + \cdots + \sin a_n \leq n \sin\left(\frac{S}{n}\right).$$

(2) Consider the bounds on $a_1^2 + a_2^2 + \cdots + a_n^2$ where $\sum a_i = S\ (\geq n)$ and the a_i's are positive integers. Since

$$(S - n + 1, 1, 1, \cdots, 1) \succ (a_1, a_2, \cdots, a_n) \succ \left(\frac{S}{n}, \frac{S}{n}, \cdots, \frac{S}{n}\right),$$

we have

$$(S - n + 1)^2 + n - 1 \geq a_1^2 + a_2^2 + \cdots + a_n^2 \geq n\left(\frac{S}{n}\right)^2.$$

For many other applications, see [1].

Reference

1. A.W. Marshall, I. Olkin, Inequalities: Theory of Majorization and Its Applications, Academic Press, NY, 1979.

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