THE OLYMPIAD CORNER

No. 220

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We begin with the problems of the two days of the Turkish Mathematical Olympiad 1998. Thanks go to Ed Barbeau for collecting these problems at the IMO in Romania.

VI TURKISH MATHEMATICAL OLYMPIAD

Second Round

First Day - December 11, 1998 (Time: 4.5 hours)

1. On the base of the isosceles triangle ABC ($|AB| = |AC|$) we choose a point D such that $|BD| : |DC| = 2 : 1$ and on $[AD]$ we choose a point P such that $m(\widehat{BAC}) = m(\widehat{BPD})$.

Prove that $m(\widehat{DPC}) = m(\widehat{BAC})/2$.

2. Prove that

$$(a+3b)(b+4c)(c+2a) \;\geq\; 60abc$$

for all real numbers $0 \leq a \leq b \leq c$.

3. The points of a circle are coloured by three colours. Prove that there exist infinitely many isosceles triangles with vertices on the circle and of the same colour.

Second Day - December 12, 1998 (Time: 4.5 hours)

4. Determine all positive integers x, n satisfying the equation $x^3 + 3367 = 2^n$.

5. Given the angle XOY, variable points M and N are considered on the arms $[OX]$ and $[OY]$, respectively, so that $|OM| + |ON|$ is constant. Determine the geometric locus of the mid-point of $[MN]$.

6. Some of the vertices of unit squares of an $n \times n$ chessboard are coloured so that any $k \times k$ square formed by these unit squares on the chess board has a coloured point on at least one of its sides. If $l(n)$ stands for the minimum number of coloured points required to satisfy this condition, prove that

$$\lim_{n\to\infty} \frac{l(n)}{n^2} \;=\; \frac{2}{7}\,.$$

As a second set for this issue we give the Turkish Team Selection Examination for the 40^{th} IMO, 1999. Thanks again to Ed Barbeau for collecting them at the IMO in Romania.

TURKISH TEAM SELECTION EXAMINATION FOR THE 40^{th} IMO
First Day - March 20, 1999
(Time: 4.5 hours)

1. Let $m \le n$ be positive integers and p be a prime. Let p–expansions of m and n be

$$\begin{aligned} m &= a_0 + a_1 p + \cdots + a_r p^r\,, \\ n &= b_0 + b_1 p + \cdots + b_s p^s\,, \end{aligned}$$

respectively, where a_r, $b_s \neq 0$, for all $i \in \{0, 1, \ldots, r\}$ and for all $j \in \{0, 1, \ldots, s\}$, we have $0 \le a_i, b_j \le p-1$.

If $a_i \le b_i$ for all $i \in \{0, 1, \ldots, r\}$, we write $m \prec_p n$. Prove that

$$p \not| \binom{n}{m} \quad \Longleftrightarrow \quad m \prec_p n\,.$$

2. Let L and N be the mid-points of the diagonals $[AC]$ and $[BD]$ of the cyclic quadrilateral $ABCD$, respectively. If BD is the bisector of the angle ANC, then prove that AC is the bisector of the angle BLD.

3. Determine all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that the set

$$\left\{ \frac{f(x)}{x} : x \neq 0 \quad \text{and} \quad x \in \mathbb{R} \right\}$$

is finite, and for all $x \in \mathbb{R}$

$$f(x - 1 - f(x)) \;=\; f(x) - x - 1\,.$$

Second Day - March 21, 1999
(Time: 4.5 hours)

4. Let the area and the perimeter of a cyclic quadrilateral C be A_C and P_C, respectively. If the area and the perimeter of the quadrilateral which is tangent to the circumcircle of C at the vertices of C are A_T and P_T, respectively, prove that $\frac{A_C}{A_T} \ge \left(\frac{P_C}{P_T}\right)^2$.

5. Each of A, B, C, D, E and F knows a piece of gossip. They communicate by telephone via a central switchboard, which can connect only two of them at a time. During a conversation, each side tells the other everything he or she knows at that point. Determine the minimum number of calls for everyone to know all six pieces of gossip.

6. Prove that the plane is not a union of the inner regions of finitely many parabolas. (The outer region of a parabola is the union of the lines not intersecting the parabola. The inner region of a parabola is the set of points of the plane that do not belong to the outer region of the parabola.)

As a final contest for this issue we give the Final Round of the Japanese Mathematical Olympiad 1999. Thanks go to Ed Barbeau for collecting this when he was Canadian Team Leader to the IMO in Romania.

JAPANESE MATHEMATICAL OLYMPIAD 1999

Final Round — February 11, 1999

Duration: 4 hours

1. You can place a stone at each of 1999×1999 squares on a grid pattern. Find the minimum number of stones to satisfy the following condition.

Condition: When an arbitrary blank square is selected, the total number of stones placed in the corresponding row and column shall be 1999 or more.

2. Let $f(x) = x^3 + 17$. Prove that for each natural number n, $n \geq 2$, there is a natural number x, for which $f(x)$ is divisible by 3^n but not by 3^{n+1}.

3. Let $2n + 1$ weights (n is a natural number, $n \geq 1$) satisfy the following condition.

Condition: If any one weight is excluded, then the remaining $2n$ weights can be divided into a pair of n weights that balance each other.

Prove that all the weights are equal in this case.

4. Prove that

$$f(x) = (x^2 + 1^2)(x^2 + 2^2)(x^2 + 3^2) \cdots (x^2 + n^2) + 1$$

cannot be expressed as a product of two integral-coefficient polynomials with degree greater than 1.

5. For the convex hexagon $ABCDEF$ having side lengths that are all 1, find the maximum value M and minimum value m of three diagonals AD, BE, and CF and their possible ranges.

Next, we turn to readers' comments and solutions to problems of the 13^{th} Iranian Mathematical Olympiad 1995, given [1999 : 456].

1. Find all real numbers $a_1 \leq a_2 \leq \cdots \leq a_n$ satisfying

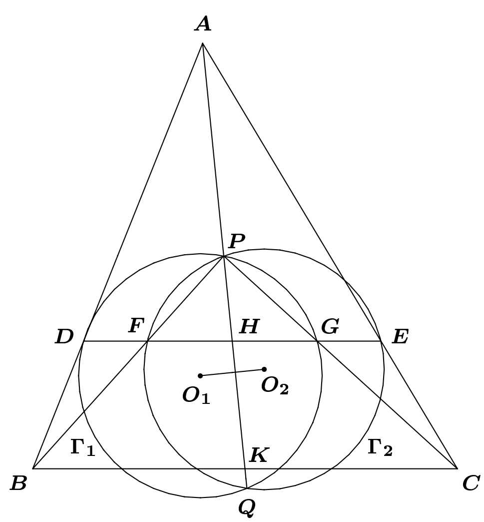
$$\sum_{i=1}^{n} a_i \;=\; 96\,, \quad \sum_{i=1}^{n} a_i^2 \;=\; 144\,, \quad \sum_{i=1}^{n} a_i^3 \;=\; 216\,.$$

Solutions by Moubinool Omarjee, Paris, France; and by George Evagelopoulos, Athens, Greece. Comments by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Pontoise, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Wang notes that this is the same as ***CRUX with MAYHEM*** problem #1982, which appeared [1994 : 250] with solution(s) [1995 : 256–257]. Bornsztein points out the similarity to ***CRUX with MAYHEM*** problem #1838, where a solution in positive integers is required.

2. Points D and E are situated on the sides AB and AC of triangle ABC in such a way that $DE \| BC$. Let P be an arbitrary point inside the triangle ABC. Lines PB and PC intersect DE at F and G, respectively. Let O_1 be the circumcentre of triangle PDG and let O_2 be that of PFE. Show that $AP \perp O_1O_2$.

Solution by Toshio Seimiya, Kawasaki, Japan.



Let Γ_1 and Γ_2 be the circumcircles of $\triangle PDG$ and $\triangle PFE$, respectively, so that O_1 and O_2 are centres of Γ_1 and Γ_2, respectively. Let Q be the intersection of Γ_1 and Γ_2 other than P. Then $O_1O_2 \perp PQ$.

Let H and K be the intersections of AP with DE and BC, respectively.

Since $DE \| BC$, we get

$$DH : BK = AH : AK = HE : KC \text{, so that}$$

$$DH : HE = BK : KC . \quad (1)$$

Since $FG \| BC$, we similarly get

$$FH : HG = BK : KC . \quad (2)$$

From (1) and (2), we have

$$DH : HE = FH : HG .$$

Thus,

$$DH \cdot HG = FH \cdot HE .$$

Note that $DH \cdot HG$ and $FH \cdot HE$ are the powers of H with respect to Γ_1 and Γ_2, respectively. Hence, H is a point on the radical axis of Γ_1 and Γ_2.

Since P and Q are intersections of Γ_1 and Γ_2, we have that PQ is the radical axis of Γ_1 and Γ_2. Therefore, H is a point on the line PQ, so that A, P, Q are collinear.

Since, $PQ \perp O_1O_2$, we have $AP \perp O_1O_2$.

3. Let $P(x)$ be a polynomial with rational coefficients such that $P^{-1}(\mathbb{Q}) \subseteq \mathbb{Q}$. Show that P is linear.

Solutions by Pierre Bornsztein, Pontoise, France; and by Moubinool Omarjee, Paris, France. We give Bornsztein's argument.

Let $P(x) \in \mathbb{Q}[x]$ such that $P^{-1}(\mathbb{Q}) \subset \mathbb{Q}$. It follows that

$$P(\mathbb{Q}) \subset \mathbb{Q} \text{ (since } P \in \mathbb{Q}[x]) \quad (1)$$

and

$$P(\mathbb{R} - \mathbb{Q}) \subset \mathbb{R} \backslash \mathbb{Q} \text{ (since } P^{-1}(\mathbb{Q}) \subset \mathbb{Q}) . \quad (2)$$

It is easy to see that P cannot be a constant polynomial.

Since $P \not\equiv 0$, multiply by the denominators of the coefficients of P. We obtain another polynomial with integer coefficients, satisfying (1) and (2).

Moreover, if c is the leading coefficient of this last polynomial, then, using $\left(x \mapsto \frac{x}{c}\right)$ and multiplying by c^{n-1} (where n is the degree of P), we obtain a monic polynomial, with (1) and (2).

Thus, with no loss of generality, we may suppose that $P \in \mathbb{Z}[x]$, where P has leading coefficient 1 (P is monic), and P satisfies (1) and (2).

The desired result follows immediately from this claim:

Claim: If $P \in \mathbb{Z}[x]$ is a monic polynomial, with degree greater than 1, then there exists an integer a such that $P(x) - a$ has a positive real irrational root.

Proof. Let p be a prime such that $p > P(1) - P(0)$ and greater than the largest real roots of $P(x) - P(0) - x$.

Let $a = p + P(0)$. Then $P(1) - a = P(1) - P(0) - p < 0$ and $P(p) - a = P(p) - P(0) - p > 0$.

From the Intermediate Value Theorem, it follows that $P(x) - a$ has a real root in $(1, p)$, say α.

Since $P(x) - a$ is a monic polynomial with integer coefficients, it is well known that, if α is a rational root of $P(x) - a$, then α divides $P(0) - a = -p$. Whence, $\alpha = 1$ or $\alpha = p$. Thus, α is irrational.

This ends the proof of the claim.

It follows from the claim that if $P \in \mathbb{Z}[x]$ is a monic polynomial satisfying (2), then P cannot have a degree greater than 1. That is, P is linear.

4. Let $S = \{x_1, x_2, \ldots, x_n\}$ be an n–element subset of the set $\{x \in R \mid x \geq 1\}$. Find the maximum number of elements of the form

$$\sum_{i=1}^{n} \varepsilon_i x_i\,, \quad \varepsilon_i = 0, 1$$

which belong to I, where I varies over all open intervals of length 1, and S over all n–element subsets.

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Pontoise, France; and by Moubinool Omarjee, Paris, France. We give Bornsztein's argument.

We will prove that the desired maximum, denoted by M, is

$$M = \binom{n}{\lfloor \frac{n}{2} \rfloor} \text{ where } \lfloor\cdot\rfloor \text{ is the integer part function.}$$

Let S be an n–element subset of $(1, +\infty)$. Denote by $s(A)$ the sum of the elements of the subset A of S, with $s(\emptyset) = 0$. Let I be an open interval of length one.

Claim: If A, B are two subsets of S with $A \subsetneq B$ then $s(A)$ or $s(B)$ does not belong to I.

Proof. Since $A \subsetneq B$ there exists $x \in B \backslash A$. Thus,

$$s(B) \;=\; x + s(B - \{x\}) \;\geq\; x + s(A) \;\geq\; 1 + s(A)\,.$$

Then, $s(B) - s(A) \geq 1$.

It follows that the numbers $s(B)$ and $s(A)$ cannot belong to a common open interval of length one.

From the claim, it follows immediately that the number of elements of the form $\sum_{i=1}^{n} \varepsilon_i x_i$, with $\varepsilon_i = 0, 1$, (that is the number of $s(A)$ where $A \subset S$) which belong to I is not greater than the maximum number of subsets A_1, $A_2, \ldots,$ of S which may be constructed such that none of the A_i's is included in another.

Such a family of subsets is a Sperner family. It is well known that a Sperner family has at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ elements (see [1]).

Since it is true for any I, S under the assumptions of the exercise, we deduce that

$$M \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}. \tag{1}$$

Conversely: Let p be an integer such that $\lfloor \frac{n}{2} \rfloor < p$.

For $i = 1, \ldots, n$, let $x_i = 1 + \frac{1}{p+i}$.

Let A be any $\left\lfloor \frac{n}{2} \right\rfloor$–subset of $S = \{x_1, \ldots, x_n\}$. Then,

$$\begin{aligned} \left\lfloor \frac{n}{2} \right\rfloor < s(A) &= \left\lfloor \frac{n}{2} \right\rfloor + \sum_{\substack{i \\ x_i \in A}} \frac{1}{p+i} \\ &< \left\lfloor \frac{n}{2} \right\rfloor + \sum_{\substack{i \\ x_i \in A}} \frac{1}{p} \\ &= \left\lfloor \frac{n}{2} \right\rfloor + \frac{1}{p} \left\lfloor \frac{n}{2} \right\rfloor . \end{aligned}$$

Thus,

$$s(A) \in I = \left(\left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor + 1 \right) .$$

I is clearly an open interval of length one, and I contains all $s(A)$ where $A \subset S$ with $\mathbf{Card}(A) = \lfloor \frac{n}{2} \rfloor$.

Then, at least $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ elements of the form $\sum \varepsilon_i x_i$, $\varepsilon_i = 0, 1$, belong to I.

It follows that

$$M \geq \binom{n}{\lfloor \frac{n}{2} \rfloor}. \tag{2}$$

From (1) and (2), we deduce that $M = \binom{n}{\lfloor \frac{n}{2} \rfloor}$, as claimed.

Reference.

[1] K. Engel, "Sperner theory", Cambridge University Press, p. 1–3.

5. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ that fulfils all of the following conditions:

(a) $f(1) = 1$

(b) there exists $M > 0$ such that $-M < f(x) < M$

(c) if $x \neq 0$ then

$$f\left(x + \frac{1}{x^2}\right) \;=\; f(x) + \left(f\left(\frac{1}{x}\right)\right)^2 \;?$$

Solution by Mohammed Aassila, Strasbourg, France.

Let n be the smallest integer for which $f(x) < n$ for all $x \neq 0$. Then, we can find $x \neq 0$ such that $f(x) \geq n - 1$. Then,

$$\left(f\left(\frac{1}{x}\right)\right)^2 \;=\; f\left(x + \frac{1}{x^2}\right) - f(x) \;<\; n - (n-1) \;=\; 1\,,$$

and thus, $f(\frac{1}{x}) > -1$. Now, substituting $\frac{1}{x}$ for x in the original equation, we have

$$(n-1)^2 \;\leq\; f(x)^2 \;=\; f\left(\frac{1}{x^2} + x\right) - f\left(\frac{1}{x}\right) \;<\; n + 1\,.$$

Thus, $(n-1)^2 < n+1$, and thus, $n \in \{1, 2\}$. But, putting $x = 1$ in the original equation, we get $f(2) = 2$, and therefore, $n > 2$, a contradiction.

Comment by Pierre Bornsztein, Pontoise, France. The problem and solution can be found in "*36*$^{\text{th}}$ *International Mathematical Olympiad*" published by the Canadian Mathematical Society, p. 124.

6. Let a, b, c be positive real numbers. Find all real numbers x, y, z such that

$$x + y + z \;=\; a + b + c$$
$$4xyz - (a^2x + b^2y + c^2z) \;=\; abc\,.$$

Solution by Mohammed Aassila, Strasbourg, France.

The second equation is equivalent to

$$4 \;=\; \frac{a^2}{yz} + \frac{b^2}{zx} + \frac{c^2}{xy} + \frac{abc}{xyz}$$

and also to

$$4 \;=\; x_1^2 + y_1^2 + z_1^2 + x_1y_1z_1\,,$$

where

$$0 \;<\; x_1 \;=\; \frac{a}{\sqrt{yz}} \;<\; 2\,, \qquad 0 \;<\; y_1 \;=\; \frac{b}{\sqrt{zx}} \;<\; 2\,,$$
$$0 \;<\; z_1 \;=\; \frac{c}{\sqrt{xy}} \;<\; 2\,.$$

Setting $x_1 = 2\sin u$, $0 < u < \frac{\pi}{2}$, and $y_1 = 2\sin v$, $0 < v < \frac{\pi}{2}$, we have

$$4 \;=\; 4\sin^2 u + 4\sin^2 v + z_1^2 + 4\sin u \cdot \sin v \cdot z_1 \,.$$

Hence,

$$z_1 + 2\sin u \cdot \sin v \;=\; 2\cos u \cdot \cos v \,,$$

and then,

$$z_1 \;=\; 2(\cos u \cdot \cos v - \sin u \cdot \sin v) \quad \big(=\; 2\cos(u+v)\big) \,.$$

Thus,

$$\begin{array}{rcl} a &=& 2\sqrt{yz}\sin u \,, \\ b &=& 2\sqrt{zx}\sin v \,, \\ c &=& 2\sqrt{xy}(\cos u \cdot \cos v - \sin u \cdot \sin v) \,. \end{array}$$

From $x + y + z = a + b + c$, we get

$$\left(\sqrt{x}\cos v - \sqrt{y}\cos u\right)^2 + \left(\sqrt{x}\sin v + \sqrt{y}\sin u - \sqrt{z}\right)^2 \;=\; 0$$

which implies

$$\sqrt{z} \;=\; \sqrt{x}\sin v + \sqrt{y}\sin u \;=\; \sqrt{x}\,\frac{y_1}{2} + \sqrt{y}\,\frac{x_1}{2} \,.$$

Therefore,

$$\sqrt{z} \;=\; \sqrt{x}\cdot\frac{b}{2\sqrt{zx}} + \sqrt{y}\cdot\frac{a}{2\sqrt{yz}} \,,$$

and thus, $z = \dfrac{a+b}{2}$. Similarly, $x = \dfrac{a+b}{2}$, $y = \dfrac{c+a}{2}$.

The triple

$$(x,y,z) \;=\; \left(\frac{b+c}{2}, \frac{c+a}{2}, \frac{a+b}{2}\right)$$

is the unique solution.

Comment by Pierre Bornsztein, Pontoise, France. The problem and a solution are in "*36th International Mathematical Olympiad*", published by the Canadian Mathematical Society, p. 122–123.

Next we turn to the February 2000 number of the *Corner* and solutions to problems of the Final (Selection) Round of the Estonian Mathematical Contests 1995-96 given [2000 : 6].

1. The numbers x, y and $\frac{x^2+y^2+6}{xy}$ are positive integers. Prove that $\frac{x^2+y^2+6}{xy}$ is a perfect cube.

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's write-up and comment.

Suppose $\frac{x^2+y^2+6}{xy} = k$ for some positive integers x, y and k. We prove that necessarily $k = 8$.

Consider

$$\frac{x^2+y^2+6}{xy} \;=\; k \qquad (1)$$

as a Diophantine equation in two variables x and y. Let (a, b) denote the solution of (1) with the *least* positive a value. We first show that $a = 1$. Due to symmetry, we may assume that $a \le b$. Note first that a and b must both be odd since they clearly must have the same parity and if they are both even, then modulo 4, $a^2 + b^2 + 6 \equiv 2$ while $kab \equiv 0$. If $a = b$, then from $k = 2 + \frac{6}{a^2}$ we deduce immediately that $a = 1$. If $a < b$ then $b \ge a + 1$.

Since

$$\frac{(ka-b)^2+a^2+6}{(ka-b)a} \;=\; \frac{k^2a^2-2kab+kab}{ka^2-ab} \;=\; k$$

and

$$ka - b \;=\; \frac{a^2+b^2+6}{b} - b \;=\; \frac{a^2+6}{b} \;>\; 0\,,$$

we see that $(ka - b, a)$ is also a solution of (1) in natural numbers. Note that $a^2+6-ab \le a^2+6-a(a+1) = 6-a < 0$ if $a > 6$. Thus, $ka-b = \frac{a^2+6}{b} < a$ if $a > 6$, contradicting the minimality of a. Hence, $a \le 6$. It remains to show that $a \ne 3$, 5. If $a = 3$, then we get $b^2 + 15 = 3kb$, which implies that 3 divides b and thus, $b^2 + 15 \equiv 6$, while $3kb \equiv 0 \pmod 9$. If $a = 5$, then we get $b^2 + 31 = 5kb$. Since b is an integer and 31 is a prime we deduce from the relation between roots and coefficients that $5k = 32$, which is impossible. Therefore, we conclude that $a = 1$, which implies $k = \frac{b^2+7}{b} = b + \frac{7}{b}$. Hence, $b = 7$ and $k = 8$, which is a cube.

Comment. This is a beautiful problem which parallels the following "infamous" problem of the 29^{th} IMO, supposedly the most difficult IMO problem ever (with an average score of 0.6 out of 7):

> "Suppose a and b are positive integers such that $ab + 1$ divides $a^2 + b^2$. Prove that $\frac{a^2+b^2}{ab+1}$ is a perfect square."

2. Let a, b, c be the sides of a triangle and α, β, γ the opposite angles of the sides, respectively. Prove that if the inradius of the triangle is r, then $a\sin\alpha + b\sin\beta + c\sin\gamma \geq 9r$.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea; by David Loeffler, student, Cotham School, Bristol, UK; by Toshio Seimiya, Kawasaki, Japan; and by Panos E. Tsaoussoglou, Athens, Greece. We first give Loeffler's solution.

First we substitute $\sin(\alpha) = \frac{a}{2R}$, etc., so that the required inequality is equivalent to

$$9r \;\leq\; \frac{a^2}{2R} + \frac{b^2}{2R} + \frac{c^2}{2R}\,, \qquad \text{or} \qquad 18Rr \;\leq\; a^2 + b^2 + c^2\,.$$

The product Rr can be neatly expressed by comparing two well-known formulae for the area of the triangle: $\frac{abc}{4R}$ and $rs = \frac{r(a+b+c)}{2}$. Equating these gives $Rr = \frac{abc}{2(a+b+c)}$.

Thus, the required inequality becomes:

$$\frac{9abc}{a+b+c} \;\leq\; a^2 + b^2 + c^2\,,$$

$$9abc \;\leq\; (a^2 + b^2 + c^2)(a+b+c)\,.$$

However, applying the AM-GM inequality, we see that

$$(abc)^{\frac{2}{3}} \;\leq\; \frac{a^2 + b^2 + c^2}{3}\,;$$

that is,

$$3(abc)^{\frac{2}{3}} \;\leq\; a^2 + b^2 + c^2\,,$$

and

$$(abc)^{\frac{1}{3}} \;\leq\; \frac{a+b+c}{3}\,, \qquad \text{or} \qquad 3(abc)^{\frac{1}{3}} \;\leq\; a+b+c\,.$$

Multiplying these together it follows that $9abc \leq (a+b+c)(a^2+b^2+c^2)$, which we showed to be equivalent to the required inequality.

Now we give Klamkin's generalization and comment.

Since $a = 2R\sin\alpha$, etc., where R is the circumradius, the inequality is now

$$a^2 + b^2 + c^2 \;\geq\; 18Rr\,.$$

We prove the stronger inequality

$$a^2 + b^2 + c^2 \;\geq\; xRr - (2x - 36)r^2 \;\geq\; 18Rr\,,$$

where $18 \leq x \leq 24$. The right-hand inequality reduces to $(x-18)(R-2r) \geq 0$

which is the well-known Chapple-Euler inequality. For the left-hand inequality, we use an identity [1; p. 52].

$$a^2 + b^2 + c^2 \;=\; 2(s^2 - 4Rr - r^2) \;\geq\; xRr - (2x - 36)r^2$$

where s is the semiperimeter. This reduces the inequality to

$$2s^2 \;\geq\; (x + 8)Rr + (38 - 2x)r^2\,.$$

Since it is known [1; p. 50] that $s^2 \geq 16Rr - 5r$, we must have here that

$$32Rr - 10r^2 \;\geq\; (x + 8)Rr + (38 - 2x)r^2$$

or that $(24 - x)Rr \geq (48 - 2x)r^2$ and which is valid if $x \leq 24$ as well.

Comment. As a known complementary inequality, we also have

$$9R^2 \;\geq\; a^2 + b^2 + c^2\,.$$

For one simple proof of this consider the expansion of $(A + B + C)^2 \;\geq\; 0$ where A, B, C are vectors from the circumcentre to the vertices A, B, C of the triangle ABC (note that $A^2 = R^2$, $2B \cdot C = 2R^2 - a^2$, etc.). Another proof follows from $OH^2 = 9R^2 - (a^2 + b^2 + c^2)$ where O and H are the circumcentre and orthocentre of ABC.

Reference:

[1] D.S. Mitrinovic, J.E. Pecaric, V. Volenic, **Recent Advances in Geometric Inequalities**, Kluwer, Dordrecht, 1989.

3. Prove that the polynomial $P_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$ has no zeros if n is even and has exactly one zero if n is odd.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give the write-up of Bataille (with his added supplementary remark).

We shall prove by induction the following property

$$(\pi_n)\begin{cases} P_{2n} & \text{has no (real) zeros} \\ P_{2n+1} & \text{has exactly one (real) zero.} \end{cases}$$

Clearly (π_0) is true since $P_0(x) = 1$ and $P_1(x) = 1 + x$.

Suppose now that (π_n) is true for an integer $n \geq 0$. We will denote by x_n the unique zero of P_{2n+1}. The continuous function P_{2n} has no zeros and is positive for $x \geq 0$. Hence, $P_{2n}(x) > 0$ for all x.

Since the derivative P'_{2n+1} of P_{2n+1} is P_{2n}, the function P_{2n+1} is increasing. Furthermore, $\lim\limits_{x\to-\infty} P_{2n+1}(x) = -\infty$ and $\lim\limits_{x\to+\infty} P_{2n+1}(x) = +\infty$, so that $P_{2n+1}(x) < 0$ for $x < x_n$ and $P_{2n+1}(x) > 0$ for $x > x_n$.

Now, using $P'_{2n+2} = P_{2n+1}$, we see that $P_{2n+2}(x)$ is a minimum when $x = x_n$. Hence, for all x

$$\begin{aligned} P_{2n+2}(x) \ \geq \ P_{2n+2}(x_n) &= P_{2n+1}(x_n) + \frac{x_n^{2n+2}}{(2n+2)!} \\ &= \frac{x_n^{2n+2}}{(2n+2)!} = \frac{(x_n^{n+1})^2}{(2n+2)!} > 0 . \end{aligned}$$

Thus, $P_{2n+2}(x)$ never takes the value 0.

Also, as above, P_{2n+3} is increasing, continuous, and $\lim\limits_{x\to-\infty} P_{2n+3}(x) = -\infty$, $\lim\limits_{x\to+\infty} P_{2n+3}(x) = +\infty$. Hence, P_{2n+2} has a unique zero. This completes the induction and shows that property (π_n) is true for all non-negative integer n.

Remark. As a supplement, we show that $\lim\limits_{n\to\infty} x_n = -\infty$.

Consider

$$P_{2n+1}(-2n-3) = \sum_{k=0}^{2n+1} \frac{(-2n-3)^k}{k!} = \sum_{p=0}^{n} (2n+3)^{2p} \left(\frac{1}{(2p)!} - \frac{2n+3}{(2p+1)!} \right) .$$

We have $P_{2n+1}(-2n-3) < 0$ (since $2n+3 > 2p+1$ for $p = 0, \ldots, n$) and thus, $x_n > -2n-3$. Hence,

$$P_{2n+3}(x_n) = P_{2n+3}(x_n) - P_{2n+1}(x_n) = \frac{x_n^{2n+2}}{(2n+2)!} \left(1 + \frac{x_n}{2n+3}\right) > 0 ,$$

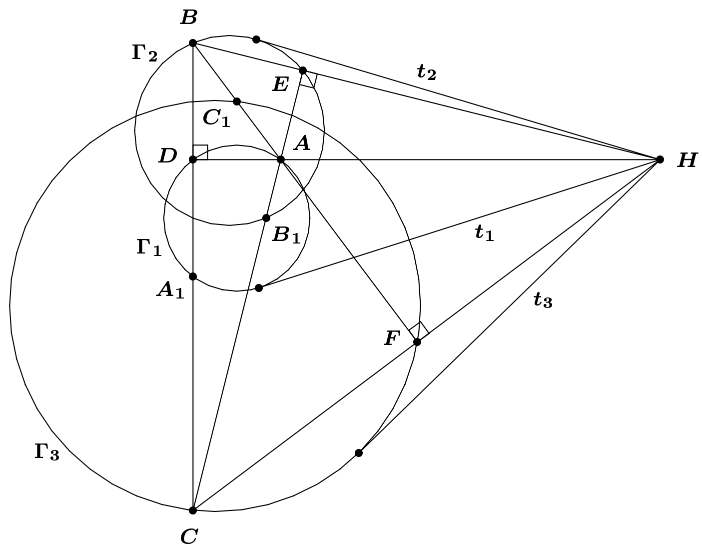
which implies $x_n > x_{n+1}$. Therefore, (x_n) is a decreasing sequence of real numbers and, as such, either $\lim_{n\to\infty} x_n = -\infty$ or (x_n) converges to a real number m. Assume that the latter does occur. Note that $m \leq x_n < 0$ for all n. Since for all $n \geq 0$ we have $P_{2n+1}(x) \leq e^x \leq P_{2n}(x)$ for all $x \leq 0$ [easy induction], we would have

$$\begin{aligned} 0 \ \leq \ e^{x_n} \ \leq \ P_{2n}(x_n) &= P_{2n+1}(x_n) - \frac{x_n^{2n+1}}{(2n+1)!} \\ &= -\frac{x_n^{2n+1}}{(2n+1)!} \leq -\frac{m^{2n+1}}{(2n+1)!} . \end{aligned}$$

But then, $\lim_{n\to\infty} e^{x_n} = 0$ (because $\lim\limits_{n\to\infty} \dfrac{m^{2n+1}}{(2n+1)!} = 0$) while, by the continuity of the exponential function, we must have $\lim\limits_{n\to\infty} e^{x_n} = e^m \neq 0$. This contradiction shows that $\lim\limits_{n\to\infty} x_n = -\infty$.

4. Let H be the orthocentre of an obtuse triangle ABC and A_1, B_1, C_1 arbitrary points taken on the sides BC, AC, AB, respectively. Prove that the tangents drawn from the point H to the circles with diameters AA_1, BB_1, CC_1 are equal.

Solutions by Michel Bataille, Rouen, France; and by Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution.



The lines HA, HB, HC meet BC, CA, AB at D, E, F, respectively.

Then, $AD \perp BC$, $BE \perp AC$ and $CF \perp AB$.

Let Γ_1, Γ_2, Γ_3 be circles with diameters AA_1, BB_1, CC_1, respectively.

Since $\angle ADA_1 = \angle BEB_1 = CFC_1 = 90°$, we have that Γ_1, Γ_2, Γ_3 pass through D, E, F, respectively.

Let t_1, t_2, t_3 be the tangent segments from H to Γ_1, Γ_2, Γ_3, respectively.

Since $\angle ADB = \angle AEB = 90°$, A, D, B, E are concyclic, so that

$$HA \cdot HD \;=\; HB \cdot HE\,.$$

Hence, we have

$$t_1^2 \;=\; HA \cdot HD \;=\; HB \cdot HE \;=\; t_2^2\,.$$

Thus, $t_1 = t_2$.

Similarly, we have $t_1 = t_3$. Therefore, $t_1 = t_2 = t_3$.

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions for all $x \in \mathbb{R}$.

(a) $f(x) = -f(-x)$;

(b) $f(x+1) = f(x) + 1$;

(c) $f\left(\frac{1}{x}\right) = \frac{1}{x^2} f(x)$, if $x \neq 0$.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Courdimanche, France. We give the write-up by Aassila.

Set $g(x) = f(x) - x$. Then g satisfies

(a') $g(x) = -g(-x)$;

(b') $g(x+1) = g(x)$;

(c') $g\left(\frac{1}{x}\right) = \frac{1}{x^2}g(x)$, if $x \neq 0$.

By (a')–(b') we get $g(0) = g(-1) = 0$, and for all $x \neq 0$, -1 we have

$$\begin{aligned} g(x) &= g(x+1) = (x+1)^2 g\left(\frac{1}{x+1}\right) = -(x+1)^2 g\left(-\frac{1}{x+1}\right) \\ &= -(x+1)^2 g\left(1 - \frac{1}{x+1}\right) = -(x+1)^2 g\left(\frac{x}{x+1}\right) \\ &= -(x+1)^2 \frac{x^2}{(x+1)^2} g\left(\frac{x+1}{x}\right) = -x^2 g\left(1 + \frac{1}{x}\right) = -x^2 g\left(\frac{1}{x}\right) \\ &= -g(x)\,. \end{aligned}$$

Hence, $g(x) \equiv 0$, and $f(x) = x$ for all $x \in \mathbb{R}$.

Now we turn to solutions from our readers to problems of the Japan Mathematical Olympiad, Final Round, 1996 given [2000 : 7].

2. For positive integers m, n with $\gcd(m, n) = 1$, determine the value $\gcd(5^m + 7^m, 5^n + 7^n)$.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by George Evagelopoulos, Athens, Greece. We give Bornsztein's solution.

We will use the following lemma (see [1] for a proof).

Lemma. If a and b are relatively prime integers with $a > b$, then for every pair of positive integers m and n we have

$$(a^m - b^m, a^n - b^n) = a^{(m,n)} - b^{(m,n)}$$

where (x, y) denotes the gcd of integers x and y.

Let m, n be positive integers, with $(m, n) = 1$.

Let $S_m = 5^m + 7^m$, $S_n = 5^n + 7^n$, $\mathcal{S} = (S_m, S_n)$ and $U_a = 7^a - 5^a$ for $a \in \mathbb{N}^*$.

If $m = n = 1$, we have $\mathcal{S} = S_m = S_n = 12$.

Since $(m, n) = 1$ and m, n are playing symmetric parts, we may suppose that $m > n$.

Let $m = a + n$. Then, $a \in \mathbb{N}^*$ and $(a, n) = 1$.

We have

$$S_m - 5^a S_n \ = \ 7^n U_a \, . \tag{1}$$

Since 5 and 7 are relatively prime, it is easy to see that $(\mathcal{S}, 5^a) \ = \ 1$ and $(\mathcal{S}, 7^n) = 1$.

Then, from (1), we deduce that $\mathcal{S} = (U_a, S_n)$.

Case 1. a is odd.

Let $S_n = \mathcal{S}l$, $U_a = \mathcal{S}k$, where k, $l \in \mathbb{N}^*$ and $(k, l) = 1$. Then, using the binomial expansion

$$7^{an} \ = \ (\mathcal{S}l - 5^n)^a \ = \ \mathcal{S}L - 5^{an}$$

and

$$7^{an} \ = \ (\mathcal{S}k + 5^a)^n \ = \ \mathcal{S}K + 5^{an}$$

where K, L are integers.

Thus,

$$\mathcal{S}(K - L) \ = \ 2 \cdot 5^{an} \, .$$

It follows that $\mathcal{S}$ divides $2 \cdot 5^{an}$. Since $(\mathcal{S}, 5) = 1$, we deduce (using Gauss' theorem) that $\mathcal{S}$ divides 2.

Moreover, S_m, S_n are even. Thus, 2 divides $\mathcal{S}$. Then, $\mathcal{S} = 2$.

Case 2. a is even.

Let $a = 2b$, where $b \in \mathbb{N}^*$ and $(b, n) = 1$.

Moreover, m and n are odd, since $m = 2b + n$ and $(m, n) = 1$. Then $\mathcal{S}$ divides $S_n U_n = U_{2n} = 7^{2n} - 5^{2n}$ and $\mathcal{S}$ divides $U_a = 7^{2b} - 5^{2b}$.

From the lemma, we deduce that $\mathcal{S}$ divides

$$7^2 - 5^2 = 24 \, . \tag{2}$$

Since m, n are odd, we have $7^m \equiv -1 \pmod 8$ and $5^m \equiv 5 \pmod 8$.

It follows that 4 divides S_m and S_n, but 8 does not divide either S_m or S_n.

In the same way, $S_m \equiv S_n \equiv 1^m + (-1)^m \equiv 0 \pmod 3$. Since 3 and 4 are relatively prime, it follows that

$$S_m \ \equiv \ S_n \ \equiv \ 0 \pmod{12}$$

but

$$S_m \ \not\equiv \ 0 \pmod{24} \ .$$

From (2), we may conclude that $\mathcal{S} = 12$. Then:

if m, n are odd, we have $(S_m, S_n) = 12$;

if m, n have opposite parities, we have $(S_m, S_n) = 2$.

Reference.

[1] *Math. Magazine*, Exercise no 1091, vol. 54, no. 2, March 1981, pp. 86–87.

3. Let x be a real number with $x > 1$ and such that x is not an integer. Let $a_n = \lfloor x^{n+1} \rfloor - x\lfloor x^n \rfloor$ $(n = 1, 2, 3, \ldots)$. Prove that the sequence of numbers $\{a_n\}$ is not periodic. (Here $\lfloor y \rfloor$ denotes, as usual, the largest integer $\leq y$.)

Solutions by Mohammed Aassila, Strasbourg, France; and by Michel Bataille, Rouen, France. We give Bataille's solution.

Suppose for purpose of contradiction, that there exists a positive integer p such that $a_{n+p} = a_n$ for all positive integers n. Defining the integer u_n by $u_n = \lfloor x^{n+p} \rfloor - \lfloor x^n \rfloor$, we would have $u_{n+1} = xu_n$ for all n. We now distinguish the following mutually exclusive cases:

Case 1. $u_1 = \lfloor x^{p+1} \rfloor - \lfloor x \rfloor \neq 0$.

Then, $x = \frac{u_2}{u_1}$ is a rational number that we can also write as $x = \frac{k}{l}$ where k, l are coprime integers. Note that $k > l > 1$ (since $x > 1$ and $x \notin \mathbb{N}$). For all n, $u_{n+1} = x^n u_1$ so that $l^n u_{n+1} = k^n u_1$ and, l^n being coprime with k^n, l^n divides u_1. This would mean that u_1 has an infinite number of divisors (since $l > 1$), which is clearly impossible.

Case 2. $u_1 = \lfloor x^{p+1} \rfloor - \lfloor x \rfloor = 0$.

Then, $u_n = 0$ for all n, and an easy induction shows that, for all positive integers m,

$$\left.\begin{array}{c} \lfloor x^{mp+1} \rfloor \;=\; \lfloor x \rfloor \,,\, \lfloor x^{mp+2} \rfloor \;=\; \lfloor x^2 \rfloor \,, \ldots \,,\, \lfloor x^{mp+p-1} \rfloor \;=\; \lfloor x^{p-1} \rfloor \\ \text{and} \quad \lfloor x^{mp} \rfloor \;=\; \lfloor x^p \rfloor \,. \end{array}\right\} \qquad (1)$$

However, from $x < x^2 < \cdots < x^p < x^{p+1}$, we get

$$\lfloor x \rfloor \leq \lfloor x^2 \rfloor \leq \cdots \leq \lfloor x^p \rfloor \leq \lfloor x^{p+1} \rfloor$$

and since $\lfloor x^{p+1} \rfloor = \lfloor x \rfloor$, we have $\lfloor x^r \rfloor = \lfloor x \rfloor$ for $r = 1, 2, \ldots, p+1$. Actually, we even have $\lfloor x^n \rfloor = \lfloor x \rfloor$ for all positive integers n (divide n by p and use (1)). Again this is impossible since $\lim\limits_{n\to\infty} x^n = +\infty$ and thus, $\lim\limits_{n\to\infty} \lfloor x^n \rfloor = +\infty$.

Thus, in both cases we are led to an impossibility, and $\{a_n\}$ cannot be periodic.

5. Let q be a real number such that $\frac{1+\sqrt{5}}{2} < q < 2$. When we represent a positive integer n in binary expansion as

$$n = 2^k + a_{k-1} \cdot 2^{k-1} + \cdots + a_1 \cdot 2 + a_0$$

(here $a_i = 0$ or 1), we define p_n by

$$p_n = q^k + a_{k-1}q^{k-1} + \cdots + a_1 q + a_0 \,.$$

Prove that there exist infinitely many positive integers k which satisfy the following condition: There exists no positive integer l such that $p_{2k} < p_l < p_{2k+1}$.

Solution by Mohammed Aassila, Strasbourg, France.

By induction on n, we can prove that $k = q_n$ satisfies the required condition, where q_n is defined by

$$\begin{cases} q_{2m} & = \sum\limits_{k=0}^{m} 2^{2k} \\ q_{2m+1} & = \sum\limits_{k=0}^{m} 2^{2k+1} \,. \end{cases}$$

That completes the *Corner* for this issue of ***CRUX with MAYHEM***. We are entering Olympiad season. Send me your nice solutions and generalizations as well as Olympiad Contests.

Professor Toshio Seimiya

Regular readers of this section will be aware of the many beautiful geometry problems that have been proposed by Professor Toshio Seimiya. We dedicated some to him in March 2001 [2001 : 114], in celebration of his 90$^{\text{th}}$ birthday. Unfortunately, we did not have a photograph available then. We do now! See page 100.