SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2605. [2001 : 49] *Proposed by K.R.S. Sastry, Bangalore, India.*

In triangle ABC, with median AD and internal angle bisector BE, we are given $AB = 7$, $BC = 18$ and $EA = ED$. Find AC.

I. *Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let M, N be feet of the perpendiculars from A, E to BC, respectively. Let J be on BN such that $BJ = BA = 7$. Then $\triangle ABE \cong \triangle JBE$ by SAS. Therefore, $EJ = EA = ED$. Furthermore, $JN = ND = 1$ and $NC = 10$. Also $AE/EC = AB/BC = 7/18$. Hence

$$MC \;=\; NC \times \frac{AC}{EC} \;=\; 10 \times \frac{25}{18} \;=\; \frac{125}{9}\,.$$

Thus

$$BM \;=\; 18 - \frac{125}{9} = \frac{37}{9}.$$

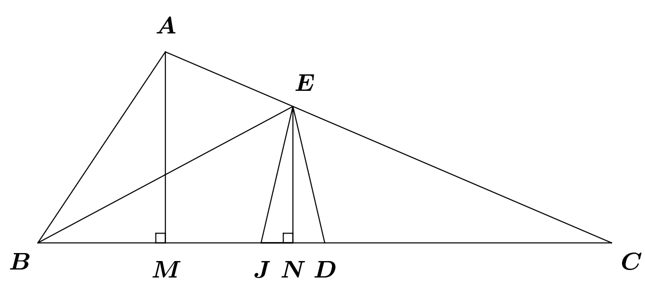
By the Theorem of Pythagoras:

$$AC^2 - AB^2 \;=\; (MC^2 + AM^2) - (MB^2 + AM^2) \;=\; MC^2 - MB^2\,.$$

Therefore,

$$\begin{aligned} AC^2 &= AB^2 + \left(\frac{125}{9}\right)^2 - \left(\frac{37}{9}\right)^2 \\ &= 49 + \frac{(125+37)(125-37)}{81} = 49 + \frac{162 \cdot 88}{81} \;=\; 225\,, \end{aligned}$$

from which we conclude that $AC = 15$.



II. *Solution by Henry Liu, student, University of Memphis, TN, USA.*

Let J lie on side BC with $BJ = 7$. Then (as in solution I above) $\triangle ABE \cong \triangle JBE$. Thus, $\angle BAE = \angle BJE$. Also, $ED = EA = EJ$, which implies that $\triangle JDE$ is isosceles, whence $\angle CDE = \angle BJE = \angle BAE$. Therefore, $\triangle CDE$ and $\triangle CAB$ are similar. Set $y = EA = ED$. Then

$$\begin{aligned} \frac{9}{y} &= \frac{AC}{7} &\Longrightarrow\quad & AC \cdot y \;=\; 63 \\ \text{and}\qquad \frac{AC-y}{9} &= \frac{18}{AC} &\Longrightarrow\quad & AC^2 - AC\cdot y \;=\; 162 \\ & &\Longrightarrow\quad & AC^2 \;=\; 162+63 \;=\; 225 \\ & &\Longrightarrow\quad & AC \;=\; 15\,. \end{aligned}$$

III. *Solution by Paul Jeffreys, student, Berkhamsted Collegiate School, UK.*

Since BE bisects $\angle ABC$, we have $7/AE = 18/EC$. Thus

$$\frac{AE}{EC} \;=\; \frac{7}{18} \qquad \text{and} \qquad EC \;=\; \frac{18}{25}AC. \tag{1}$$

Let $\alpha = \angle BAC$ and $\beta = \angle ABC$. By the Law of Sines on $\triangle ABE$, we have

$$\frac{AE}{\sin(\beta/2)} \;=\; \frac{BE}{\sin\alpha} \qquad \Longleftrightarrow \qquad AE\sin\alpha \;=\; BE\sin(\beta/2).$$

Applying the Law of Sines to $\triangle BDE$ yields:

$$\frac{DE}{\sin(\beta/2)} \;=\; \frac{BE}{\sin(\angle BDE)} \quad \Longleftrightarrow \quad DE\sin(\angle BDE) \;=\; BE\sin(\beta/2)\,.$$

Since $AE = ED$ we have $\sin(\angle BDE) = \sin\alpha$. If $\angle BDE = \alpha$, then $\triangle BAE$ is similar to $\triangle BDE$. Since $BD = 9 \neq 7 = BA$, we get a contradiction, whence we have $\angle BDE = 180^\circ - \alpha$. Thus $BDEA$ is a cyclic quadrilateral. Since $\angle BDE = 180^\circ - \alpha$, we also have $\angle CDE = \alpha$, which implies that $\triangle CDE$ is similar to $\triangle CAB$. Therefore, $9/AC = EC/18$. From (1), we have

$$\frac{9}{AC} \;=\; \frac{18}{25}\cdot\frac{AC}{18}\,, \quad \text{so that} \quad 9\cdot 25 \;=\; AC^2\,.$$

Thus, $AC = 15$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AUSTRIAN IMO TEAM 2001; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; SCOTT BROWN, Auburn University, Montgomery, AL; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES DIMINNIE and KARL HAVLAK, Angelo State University, San Angelo, TX; C. FESTRAETS-HAMOIR, Brussels, Belgium; VINAYAK GANESHAN, student, University of Waterloo, Waterloo, Ontario; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; KEE-WAI LAU, Hong Kong, China; GERRY LEVERSHA, St. Paul's School, London, England; DAVID LOEFFLER, *student, Cotham School, Bristol, UK; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; REVAÌ MATH CLUB, Győr, Hungary; JOEL SCHLOSBERG, student, New York University, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany (2 solutions); ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; KENNETH M. WILKE, Topeka, KS, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incorrect solution.*

Most solvers used a variation of solution III above. Kandall observes "It is easy to show that if $a = 2c$, then $EA = ED$ regardless of the length of AC", where a and c, of course, are the lengths of the sides opposite A and C, respectively.

Kandall, Konečný, and Seiffert all consider the more general problem with $BC = a$, $AB = c$ and $DE = EA$, and conclude that $AC = \sqrt{\frac{1}{2}a(a+c)}$.

2607. [2001 : 49] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

(a) Suppose that $q > p$ are odd primes such that $q = pn+1$, where n is an integer greater than 1. Let z be a complex number such that $z^q = 1$.

Prove that $\dfrac{z^p-1}{z^p+1} = \displaystyle\sum_{j=1}^{q-1} (-1)^{\lfloor \frac{j-1}{p} \rfloor} z^j$.

(b) Suppose that $q > 3$ is an odd prime such that $q = 3n+2$, where n is an integer greater than 1. Let z be a complex number such that $z^q = 1$.

Prove that $\dfrac{z^3-1}{z^3+1} = \displaystyle\sum_{j=1}^{q-1} (-1)^{\lfloor \frac{j-3}{3} \rfloor} z^j$.

Solution by Michel Bataille, Rouen, France.

(a) Observing that n is necessarily even, we may write

$$\begin{aligned}\sum_{j=1}^{q-1} (-1)^{\lfloor \frac{j-1}{p} \rfloor} z^j &= \sum_{j=1}^{pn} (-1)^{\lfloor \frac{j-1}{p} \rfloor} z^j \\ &= \sum_{j=1}^{p} z^j - \sum_{j=1}^{p} z^{j+p} + \sum_{j=1}^{p} z^{j+2p} - \cdots - \sum_{j=1}^{p} z^{j+(n-1)p}.\end{aligned}$$

If $z = 1$, this sum is $p - p + p - p + \cdots + p - p = 0$. Note that $\dfrac{z^p-1}{z^p+1} = 0$ as well.

If $z \neq 1$, the sum is

$$\begin{aligned} & z\frac{z^p-1}{z-1} - z^{p+1}\frac{z^p-1}{z-1} + \cdots - z^{(n-1)p+1}\frac{z^p-1}{z-1} \\ &\quad = \frac{z^p-1}{z-1}(z)\left(1 - z^p + z^{2p} - \cdots - z^{(n-1)p}\right) \\ &\quad = \frac{z^p-1}{z-1}(z)\frac{(-z^p)^n-1}{(-z)^p-1} = \frac{z^p-1}{z^p+1}\,\frac{(-z)(z^{q-1}-1)}{z-1} = \frac{z^p-1}{z^p+1}\end{aligned}$$

(the latter, using $z^{q-1} = \frac{1}{z}$). Thus, we have (a). Note that it is not necessary to suppose that p and q are primes — it is sufficient to require that p and q are odd.

(b) This time, n is necessarily odd. We have

$$\begin{aligned} S &= \sum_{j=1}^{q-1}(-1)^{\lfloor\frac{j-3}{3}\rfloor}z^j = \sum_{j=1}^{3n+1}(-1)^{\lfloor\frac{j-3}{3}\rfloor}z^j \\ &= -z - z^2 + \left(z^3+z^4+z^5\right) - \left(z^6+z^7+z^8\right) + \cdots \\ &\qquad - \left(z^{3(n-1)} + z^{3(n-1)+1} + z^{3(n-1)+2}\right) + z^{3n} + z^{3n+1} \\ &= -z - z^2 + z^{3n} + z^{3n+1} \\ &\qquad + z^3\left(1+z+z^2\right)\left(1 - z^3 + z^6 - \cdots - z^{3(n-2)}\right) . \end{aligned}$$

If $z=1$, we have $S = 0 = \frac{z^3-1}{z^3+1}$.

If $z \neq 1$, we have

$$S = -z - z^2 + z^{q-2} + z^{q-1} + \frac{z^3-1}{z-1}\,z^3\,\frac{1-(-z^3)^{n-1}}{1+z^3} .$$

Using $z^{q-2} = \frac{1}{z^2}$, $z^{q-1} = \frac{1}{z}$ and $z^{3n-3} = \frac{1}{z^5}$, we obtain

$$S = \frac{z^3-1}{z^3+1}\left(\frac{-(z+1)(z^3+1)}{z^2} + \frac{z^3}{z-1}\left(1 - \frac{1}{z^5}\right)\right) = \frac{z^3-1}{z^3+1} .$$

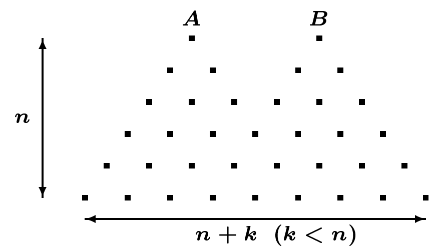
This completes the proof of (b). Similarly to (a), it is sufficent to require that q is odd.

Also solved by AUSTRIAN IMO–TEAM 2001; VINAYAK GANESHAN, student, University of Waterloo, Walerloo, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Cotham School, Bristol, UK; JOEL SCHLOSBERG, student, New York University, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer.

In fact, only Janous and Seiffert considered the case: $z = 1$. The other solvers ignored it. The editor was generous in not classing their solutions as incomplete.

2612★. [2001 : 49] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Two "Galton"–figures are given as follows:



(There are n levels in total; there are k levels such that there is no "intersection" between the levels emanating from A and B.)

Let two balls start at the same time from A and B. Each ball moves either $\swarrow$ or $\searrow$ with probability $\frac{1}{2}$.

Determine the probability $P(n,k)$ $(1 \leq k < n)$ such that the two balls reach the bottom level without colliding.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Denote by $c(n,k)$, the number of pairs of colliding paths. Then, $c(n,k) = 0$ if $k \geq n$, $c(n, n-1) = 1$, and $c(n,0) = 2^{n-1} \cdot 2^{n-1} = 2^{2n-2}$. Also, there are $c(n-1,k)$ pairs of colliding paths starting with $(\swarrow, \swarrow)$, $c(n-1,k+1)$ with $(\swarrow, \searrow)$, $c(n-1,k-1)$ with $(\searrow, \swarrow)$, and $c(n-1,k)$ with $(\searrow, \searrow)$. Therefore,

$$c(n,k) \;=\; c(n-1,k-1) + 2c(n-1,k) + c(n-1,k+1)\,.$$

We claim that $c(n,k) \;=\; \binom{2n-2}{n-k-1} + 2\sum_{i=1}^{n-k-2}\binom{2n-2}{i}$.

Indeed, using the identity $\binom{m}{j} \;=\; \binom{m-2}{j-2} + 2\binom{m-2}{j-1} + \binom{m-2}{j}$, we see easily that it satisfies the recursion. Moreover, since

$$\binom{2n-2}{n-1} + 2\sum_{i=0}^{n-2}\binom{2n-2}{i} \;=\; \sum_{i=0}^{2n}\binom{2n-2}{i} \;=\; 2^{2n-2}\,,$$

we see that all three boundary conditions are satisfied as well. Hence,

$$P(n,k) \;=\; 1 - \frac{c(n,k)}{2^{2n-2}} \;=\; \frac{2^{2n-2} - c(n,k)}{2^{2n-2}} \;=\; \frac{1}{2^{2n-2}}\sum_{i=n-k}^{n+k-1}\binom{2n-2}{i}\,.$$

Also solve by MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; KEITH EKBLAW, Walla Walla, WA, USA; RICHARD I. HESS, Rancho Palos

Verdes, CA, USA; ERIC POSTPISCHIL, Nashua, NH, USA; and JOEL SCHLOSBERG, student, New York University, NY, USA. There was one incomplete solution.

The submitted solutions varied in length from the above to one of ten pages. Postpischil noted that, in the CRC Concise Encyclopedia of Mathematics, *CRC Press, Boca Raton, FL, USA* (1999), *p. 138, Eric Weisstein shows how to write the sum in terms of the Beta Function* (*also known as the Eulerian Integral of the Second Kind*) *and the incomplete Beta Function.*

2613. [2001 : 136] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.* Dedicated to Toshio Seimiya, and based on his problem 2515.

In $\triangle ABC$, the three cevians AD, BE and CF through a non-exterior point P are such that $AF + BD + CE = s$ (the semi-perimeter). Characterize $\triangle ABC$ for each of the cases when P is (i) the orthocentre, and (ii) the Lemoine point.

[Ed. The Lemoine point is also known as the symmedian point. See, for example, James R. Smart, *Modern Geometries*, 4^{th} Edition, 1994, Brooks/Cole, California, USA. p. 161.]

Solution by Paragiou Theoklitos, Limassol, Cyprus, Greece.

In both cases $\triangle ABC$ is isosceles.

(i) When P is the orthocentre, AD, CF, and BE are the altitudes. We have therefore, $BD = c\cos B = \dfrac{c^2+a^2-b^2}{2a}$, and likewise, $CD = \dfrac{a^2+b^2-c^2}{2b}$, and $AF = \dfrac{b^2+c^2-a^2}{2c}$. Thus, the following statements are equivalent to the given condition:

$$AF + BD + CE \;=\; s\,,$$

$$\frac{b^2+c^2-a^2}{2c} + \frac{c^2+a^2-b^2}{2a} + \frac{a^2+b^2-c^2}{2b} \;=\; \frac{a+b+c}{2}\,,$$

$$ab(b^2-a^2) + bc(c^2-b^2) + ac(a^2-c^2) \;=\; 0\,,$$

$$(a-b)(b-c)(c-a)(a+b+c) \;=\; 0\,.$$

The last equation holds if and only if the given triangle is isosceles.

(ii) Standard references tell us that when P is the Lemoine point,

$$AF \;=\; \frac{cb^2}{a^2+b^2}\,,\quad BD \;=\; \frac{ac^2}{b^2+c^2}\,,\quad \text{and} \quad CE \;=\; \frac{ba^2}{c^2+a^2}\,.$$

Thus, the following statements are equivalent to the given condition:

$$AF + BD + CE \;=\; s\,,$$

$$\frac{cb^2}{a^2+b^2} + \frac{ac^2}{b^2+c^2} + \frac{ba^2}{c^2+a^2} \;=\; \frac{a+b+c}{2}\,,$$

$$\frac{(a-b)(b-c)(c-a)[a^3(b+c)+2a^2bc+a(b^3+2b^2c+2bc^2+c^3)+bc(b^2+c^2)]}{2(a^2+b^2)(b^2+c^2)(c^2+a^2)} = 0\,.$$

Once again, the last equation holds if and only if the given triangle is isosceles.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; JOEL SCHLOSBERG, student, New York University, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; and by the proposer.

Most solvers used essentially the same argument as in the featured solution, sometimes with more detail, sometimes with less. Together with problem 2515 [2000 : 114; 2001 : 144], *we now have established that*

$$AF + BD + CE \;=\; s \quad \text{if and only if} \quad \triangle ABC \text{ is isosceles}$$

when P is the incentre, the orthocentre, and the Lemoine point. David Loeffler continues the theme with the comment:

In fact, huge numbers of triangle centres may be dealt with in the same way, if you have a computer algebra system or incredible patience! The Mittenpunkt — Kimberling's X(9) [Clark Kimberling, Encyclopedia of Triangle Centers, `http://cedar.evansville.edu/~ck6/tcenters/`; or *Math. Mag.* **67**:3 (June 1994), 163–187] — *satisfies the condition only for isosceles triangles, although this is slightly more tricky to prove than the above. The same is true of the Spieker center X(10), all of the power points, and various others such as X(37), X(38), X(39), X(42), and X(43).*

Klamkin wondered whether the same is true of the circumcentre. However, before we get carried away with wild conjectures note that $AF + BD + CE = s$ for **all** *triangles when P is either the centroid or the Gergonne point.*

2614. [2001 : 136] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.* Dedicated to Toshio Seimiya, and suggested by his problem 2514.

In $\triangle ABC$, the two cevians through a non-exterior point P meet AC and AB at D and E, respectively. Suppose that $AE = BD$ and $AD = CE$. Characterize $\triangle ABC$ for the cases when P is (i) the orthocentre, (ii) the centroid, and (iii) the Lemoine point.

Solution by Henry Liu, student, University of Memphis, Memphis, TN, USA.

(i) When P is the orthocentre, BD and CE are the altitudes. By SAS and the given conditions, $\triangle ADB \cong \triangle CEA$, so that $AB = AC$ and consequently, $\angle ABC = \angle ACB$. Further, $\triangle BEC \cong \triangle CDB$ since they are similar right triangles that share their hypotenuse BC. Thus, $BD = CE = AD$. Therefore. $\triangle ABC$ is isosceles with $A = 45^\circ$ and $B = C = 67.5^\circ$.

(ii) When P is the centroid, E and D are mid-points of their respective sides. By SSS, $\triangle BEC \cong \triangle BDC$. Since these triangles have BC in common while the vertices D and E lie on the same side of the line BC, the two triangles must coincide (with E and D the same vertex). Since AB

contains E and AC contains D, the two sides of $\triangle ABC$ coincide, and the triangle is therefore degenerate (with $\angle A = 0^\circ$ and the vertices B and C coincident).

(iii) As in problem 2613, when P is the Lemoine point,

$$AE \;=\; \frac{cb^2}{a^2+b^2} \quad \text{and} \quad EB \;=\; \frac{ca^2}{a^2+b^2}\,.$$

By Stewart's Theorem we have $c(CE^2 + AE \cdot EB) = a^2AE + b^2EB$, so that $c\left(CE^2 + \left(\frac{abc}{a^2+b^2}\right)^2\right) = \frac{2a^2b^2c}{a^2+b^2}$ and therefore,

$$CE^2 \;=\; a^2b^2\left(\frac{2a^2+2b^2-c^2}{(a^2+b^2)^2}\right)\,.$$

Similarly, we have

$$AD \;=\; \frac{bc^2}{a^2+c^2} \quad \text{and} \quad BD^2 = a^2c^2\left(\frac{2a^2+2c^2-b^2}{(a^2+c^2)^2}\right)\,.$$

The given conditions imply $\left(\frac{AE}{AD}\right)^2 = \left(\frac{BD}{CE}\right)^2$, so that

$$\left(\frac{cb^2(a^2+c^2)}{bc^2(a^2+b^2)}\right)^2 \;=\; \frac{a^2c^2}{a^2b^2}\cdot\frac{(a^2+b^2)^2}{(a^2+c^2)^2}\cdot\frac{2a^2+2c^2-b^2}{2a^2+2b^2-c^2}\,. \qquad (1)$$

Set $x = a^2$, $y = b^2$, $z = c^2$, and (1) reduces to

$$y^2(2x+2y-z)(x+z)^4 - z^2(2x+2z-y)(x+y)^4 \;=\; 0\,.$$

Since $y = z$ satisfies the equation, we can factor out $(y - z)$ to get

$$(y-z)(y^4z^2 - y^3z^3 + y^2z^4 + P(x,y,z)) \;=\; 0\,,$$

where $P(x,y,z)$ is a polynomial in x, y, and z having all its terms positive. Note that the first three terms in the factor on the right satisfy

$$y^4z^2 - y^3z^3 + y^2z^4 \;=\; y^2z^2((y-z)^2 + yz) > 0\,.$$

Since $y - z = (b-c)(b+c)$, we see that (1) is equivalent to

$$(b-c)Q(a,b,c) \;=\; 0\,,$$

where Q is a polynomial that is positive for all positive values of a, b, and c. We conclude that $b = c$, so that $\triangle ABC$ is isosceles.

It remains to determine the shape of $\triangle ABC$. [Here the editor has replaced Liu's argument with a shorter calculation.]

Instead of (1), use $\left(\frac{AE}{CE}\right)^2 = \left(\frac{BD}{AD}\right)^2$, and set $b = c$ to get

$$b^4 \;=\; a^2(2a^2+b^2)\,.$$

Since $a, b > 0$, this equation implies that $b = \sqrt{2}a$. Therefore, $\triangle ABC$ must satisfy $b = c = \sqrt{2}a$ with $\angle A = \cos^{-1}\left(\frac{3}{4}\right)$. One easily checks that these conditions are also sufficient.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; JOEL SCHLOSBERG, student, New York University, NY, USA ((i) *and* (ii) *only*)*; PETER Y. WOO, Biola University, La Mirada, CA, USA* ((i) *and* (ii) *only*)*; and the proposer.*

2615. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

Suppose that $x_1, x_2, \ldots, x_n$, are non-negative numbers such that

$$\sum x_1^2 + \sum (x_1x_2)^2 = \frac{n(n+1)}{2},$$

where the sums here and subsequently are symmetric over the subscripts $1, 2, \ldots, n$.

(a) Determine the maximum of $\sum x_1$.

(b)⋆ Prove or disprove that the minimum of $\sum x_1$ is $\sqrt{\frac{n(n+1)}{2}}$.

Solution by Heinz-Jürgen Seiffert, Berlin, Germany.

(a) The given equation,

$$\sum_{k=1}^{n} x_k^2 + \sum_{1\le j<k\le n} (x_jx_k)^2 = \frac{n(n+1)}{2}, \qquad (1)$$

is satisfied when $x_1 = x_2 = \ldots = x_n = 1$, so that the maximum of $\sum x_1$ is greater than or equal to n. Using the trivial inequality $2t - t^2 \le 1$, $t \in R$, we obtain

$$\begin{aligned}
\left(\sum x_1\right)^2 &= \left(\sum_{k=1}^{n} x_k\right)^2 = \sum_{k=1}^{n} x_k^2 + 2\sum_{1\le j<k\le n} x_jx_k \\
&= \sum_{k=1}^{n} x_k^2 + 2\sum_{1\le j<k\le n} x_jx_k + \sum_{1\le j\ <\ k\le n} (x_jx_k)^2 - (x_jx_k)^2 \\
&= \frac{n(n+1)}{2} + \sum_{1\le j<k\le n} 2x_jx_k - (x_jx_k)^2 \\
&\le \frac{n(n+1)}{2} + \sum_{1\le j<k\le n} 1 = n^2,
\end{aligned}$$

or $\sum x_1 \le n$. Hence, the maximum of $\sum x_1$ is n.

(b) Clearly, if $n = 1$, then the minimum of $\sum x_1$ is 1.

Let $n = 2$. From the condition $x_1^2 + x_2^2 + (x_1x_2)^2 = 3$, we have $x_1x_2 \le \sqrt{3} < 2$, so that $(x_1 + x_2)^2 = 3 + x_1x_2(2 - x_1x_2) \ge 3$, or $x_1 + x_2 \ge \sqrt{3}$. Since an equality is attained when $x_1 = \sqrt{3}$ and $x_2 = 0$, it follows that the minimum of $\sum x_1$ is $\sqrt{3}$.

Let $n = 3$. First, suppose that $\max\{x_1x_2, x_1x_3, x_2x_3\} > 2$. If $x_1x_2 > 2$, then $x_1 + x_2 + x_3 \ge x_1 + x_2 \ge 2\sqrt{x_1x_2} > 2\sqrt{2} > \sqrt{6}$. Otherwise, we have $(x_1+x_2+x_3)^2 = 6+x_1x_2(2-x_1x_2)+x_1x_3(2-x_1x_3)+x_2x_3(2-x_2x_3) \ge 6$, or $x_1 + x_2 + x_3 \ge \sqrt{6}$. Since an equality is attained when $x_1 = \sqrt{6}$ and $x_2 = x_3 = 0$, we conclude that the minimum of $\sum x_1$ is $\sqrt{6}$.

We choose to disprove for $n \ge 4$. The equation (1) is satisfied when $x_1 = x_2 = \sqrt{\frac{1}{2}\left(\sqrt{2n(n+1)+4} - 2\right)}$ and $x_3 = x_4 = \cdots = x_n = 0$. However, $\sum x_1 = \sqrt{2\left(\sqrt{2n(n+1)+4} - 2\right)} < \sqrt{\frac{1}{2}n(n+1)}$, because the last inequality easily reduces to $n(n+1) > 16$, which is true for $n \ge 4$.

Also solved by PETER Y. WOO, Biola University, La Mirada, CA, USA. Part (a) *only was solved by the proposer. One solver sent an incomplete solution. Another solver misinterpreted the condition and solved a different problem.*

2617. [2001 : 137] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

A problem in one book was to prove that each edge of an isosceles tetrahedron is equally inclined to its opposite edge. A problem in another book was to prove that the three angles formed by the opposite edges of a tetrahedron cannot be equal unless they are at right angles.

1. Show that only the second result is valid.
2. Show that a tetrahedron which is both isosceles and orthocentric must be regular.

Solution by Joel Schlosberg, student, New York University, NY, USA.

If the three edges that come out of a single vertex of the tetrahedron are labelled as the three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, then the six sides of the tetrahedron are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{c} - \mathbf{b}$, $\mathbf{b} - \mathbf{a}$, $\mathbf{a} - \mathbf{c}$.

If X, Y and Z are the three angles formed by the opposite edges, then

$$\cos X = \pm\frac{\mathbf{a}\cdot(\mathbf{c}-\mathbf{b})}{|\mathbf{a}||\mathbf{c}-\mathbf{b}|} = \pm\frac{\mathbf{a}\cdot\mathbf{c} - \mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{c}-\mathbf{b}|},$$

$$\cos Y = \pm\frac{\mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}||\mathbf{a}-\mathbf{c}|}, \qquad \cos Z = \pm\frac{\mathbf{b}\cdot\mathbf{c} - \mathbf{a}\cdot\mathbf{c}}{|\mathbf{c}||\mathbf{b}-\mathbf{a}|}.$$

(The "angle between the opposite segments" is a slightly ambiguous phrase which could refer to either of the two supplementary angles. However, this does not affect the problem, since the [absolute values of the] cosines of two supplementary angles are the same.)

1. A counterexample to the first result is the tetrahedron with vertices $(0,0,0)$, $(0,2u,0)$, (u,u,v), $(-u,u,v)$, where $2u^2 \neq v^2$, so that $\mathrm{a} = (0,2u,0)$, $\mathrm{b} = (u,u,v)$ and $\mathrm{c} = (-u,u,v)$. A simple calculation shows that

$$\begin{aligned} |\mathrm{c}-\mathrm{b}| &= |(-2u,0,0)| = |(0,2u,0)| = |\mathrm{a}|, \\ |\mathrm{a}-\mathrm{c}| &= |(u,u,-v)| = |(u,u,v)| = |\mathrm{b}|, \\ |\mathrm{b}-\mathrm{a}| &= |(u,-u,v)| = |(-u,u,v)| = |\mathrm{c}|, \end{aligned}$$

so that the tetrahedron is isosceles. However,

$$\cos X = \pm\frac{\mathrm{a}\cdot\mathrm{c}-\mathrm{a}\cdot\mathrm{b}}{|\mathrm{a}||\mathrm{c}-\mathrm{b}|} = \pm\frac{2u^2-2u^2}{|\mathrm{a}|^2} = 0,$$

but

$$\cos Y = \pm\frac{\mathrm{a}\cdot\mathrm{b}-\mathrm{b}\cdot\mathrm{c}}{|\mathrm{b}||\mathrm{a}-\mathrm{c}|} = \pm\frac{2u^2-v^2}{|\mathrm{b}|^2} \neq 0.$$

Therefore $\cos X \neq \pm\cos Y$, and these angles are not equal.

To prove the second result, suppose that the three angles formed by opposite sides are equal. Without loss of generality, assume that $\mathrm{a}\cdot\mathrm{b} \leq \mathrm{a}\cdot\mathrm{c} \leq \mathrm{b}\cdot\mathrm{c}$. Then

$$|\cos X| = |\cos Y| = |\cos Z|,$$

and thus,

$$\frac{\mathrm{a}\cdot\mathrm{c}-\mathrm{a}\cdot\mathrm{b}}{|\mathrm{a}||\mathrm{c}-\mathrm{b}|} = \frac{\mathrm{b}\cdot\mathrm{c}-\mathrm{a}\cdot\mathrm{b}}{|\mathrm{b}||\mathrm{a}-\mathrm{c}|} = \frac{\mathrm{b}\cdot\mathrm{c}-\mathrm{a}\cdot\mathrm{c}}{|\mathrm{c}||\mathrm{b}-\mathrm{a}|}.$$

However, any equation of the form $D(x-y) = E(z-y) = F(z-x)$ has the unique solution $x = y = z$. (The equivalent homogeneous linear system $Dx + (E-D)y - Ez = 0$, $Fx - Ey + (E-F)z = 0$, has the solution set $x = y = z$; these are the only solutions, since any solution to a given n–equation, $(n+1)$–variable system of homogeneous linear equations is a scalar multiple of any other solution)

Therefore $\mathrm{a}\cdot\mathrm{b} = \mathrm{a}\cdot\mathrm{c} = \mathrm{b}\cdot\mathrm{c}$, so that $\mathrm{a}\cdot(\mathrm{c}-\mathrm{b}) = 0$, $\mathrm{b}\cdot(\mathrm{a}-\mathrm{c}) = 0$ and $\mathrm{c}\cdot(\mathrm{b}-\mathrm{a}) = 0$. Hence $\mathrm{a} \perp (\mathrm{c}-\mathrm{b})$, $\mathrm{b} \perp (\mathrm{a}-\mathrm{c})$ and $\mathrm{c} \perp (\mathrm{b}-\mathrm{a})$, implying that the opposite angles are right angles.

2. If a tetrahedron with sides a, b, c, $\mathrm{c}-\mathrm{b}$, $\mathrm{b}-\mathrm{a}$, $\mathrm{a}-\mathrm{c}$ is isosceles, then $|\mathrm{a}| = |\mathrm{c}-\mathrm{b}|$, $|\mathrm{b}| = |\mathrm{a}-\mathrm{c}|$ and $|\mathrm{c}| = |\mathrm{b}-\mathrm{a}|$. Therefore,

$$|\mathrm{a}|^2 = |\mathrm{c}-\mathrm{b}|^2 \quad \text{or} \quad \mathrm{a}\cdot\mathrm{a} = (\mathrm{c}-\mathrm{b})\cdot(\mathrm{c}-\mathrm{b}) = \mathrm{c}\cdot\mathrm{c}+\mathrm{b}\cdot\mathrm{b}-2\mathrm{b}\cdot\mathrm{c},$$

and thus,

$$-\mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{c} \;=\; 2\mathbf{b}\cdot\mathbf{c}\,.$$

Similarly

$$\mathbf{a}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{c} \;=\; 2\mathbf{a}\cdot\mathbf{c} \qquad \text{and} \qquad \mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}-\mathbf{c}\cdot\mathbf{c} = 2\mathbf{a}\cdot\mathbf{b}\,.$$

If the tetrahedron is orthocentric, then $\mathbf{a}\cdot(\mathbf{c}-\mathbf{b}) = \mathbf{b}\cdot(\mathbf{a}-\mathbf{c}) = \mathbf{c}\cdot(\mathbf{b}-\mathbf{a}) = 0$, and further, we have $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c} = \mathbf{b}\cdot\mathbf{c}$. Combining this with the previous equations gives us

$$-\mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{c} \;=\; \mathbf{a}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{c} \;=\; \mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}-\mathbf{c}\cdot\mathbf{c}\,.$$

Combining these equations pairwise allows us to deduce that $\mathbf{a}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{b} = \mathbf{c}\cdot\mathbf{c}$, and hence that $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$. Combining this with the equations $|\mathbf{a}| = |\mathbf{c}-\mathbf{b}|$, $|\mathbf{b}| = |\mathbf{a}-\mathbf{c}|$ and $|\mathbf{c}| = |\mathbf{b}-\mathbf{a}|$ shows us that all six sides are equal in length, so that the tetrahedron is regular.

Also solved by JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2618 *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

Determine a geometric problem whose solution is given by the positive solution of the equation

$$3x^2\left(\frac{1}{\sqrt{4R^2+x^2-a^2}}+\frac{1}{\sqrt{4R^2+x^2-b^2}}+\frac{1}{\sqrt{4R^2+x^2-c^2}}\right)$$
$$= \left(\sqrt{4R^2+x^2-a^2}+\sqrt{4R^2+x^2-b^2}+\sqrt{4R^2+x^2-c^2}+a+b+c\right),$$

where a, b, c and R are the sides and circumradius of a given triangle ABC.

Solution by the proposer.

We show that x is the altitude to the face ABC of an orthocentric tetrahedron of maximum isoperimetric quotient, $\dfrac{V}{E^3}$, where V and E are the volume and total edge length of the tetrahedron, respectively.

If $PABC$ is an orthocentric tetrahedron, then P must lie on a line through H, the orthocentre of ABC, and perpendicular to the plane of ABC. Then

$$PA \;=\; \sqrt{4R^2\cos^2 A+x^2} \;=\; \sqrt{4R^2+x^2-a^2}\,, \quad \text{etc.},$$

and $3V = x[ABC]$. The given equation corresponds to $\dfrac{d}{dx}\left(\dfrac{V}{E^3}\right) = 0$. That the maximum is unique follows by dividing both sides of the given equation by x, and then noting that the left hand side is an increasing function of x, whereas the right hand side is a decreasing one.

No other solutions were received.

2620. *Proposed by Bill Sands, University of Calgary, Calgary, Alberta, dedicated to Murray S. Klamkin, on his 80th birthday.*

Three cards are handed to you. On each card are three non-negative real numbers, written one below the other, so that the sum of the numbers on each card is 1. You are permitted to put the three cards in any order you like, then write down the first number from the first card, the second number from the second card, and the third number from the third card. You add these three numbers together.

Prove that you can always arrange the three cards so that your sum lies in the interval $\left[\frac{1}{2}, \frac{3}{2}\right]$. (Corrected)

Solution by Michel Bataille, Rouen, France.

Denote by (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3) the triples of numbers respectively written on each of the cards. By hypothesis, these numbers are non-negative and

$$a_1 + a_2 + a_3 \;=\; b_1 + b_2 + b_3 \;=\; c_1 + c_2 + c_3 \;=\; 1\,.$$

Following the process described in the statement of the problem, we can form six sums, namely

$$\begin{aligned} s_1 &= a_1 + b_2 + c_3\,, & s_2 &= a_1 + c_2 + b_3\,, \\ s_3 &= b_1 + a_2 + c_3\,, & s_4 &= b_1 + c_2 + a_3\,, \\ s_5 &= c_1 + a_2 + b_3\,, & s_6 &= c_1 + b_2 + a_3\,. \end{aligned}$$

For the purpose of contradiction we will suppose that $s_i \notin \left[\frac{1}{2}, \frac{3}{2}\right]$ for $i = 1$, 2, …, 6. Since $s_2 + s_3 + s_6 = 3$, one of the numbers s_2, s_3, s_6 must be at least 1, and one must be at most 1. Say, $s_2 \le 1$ and $s_6 \ge 1$. By supposition, we even have $s_2 < \frac{1}{2}$ and $s_6 > \frac{3}{2}$. The latter implies that we cannot have $s_3 > \frac{3}{2}$ (otherwise $s_3 + s_6 > 3$); hence $s_3 < \frac{1}{2}$. Now, $s_2 + s_3 < 1$, which means that we actually have $s_6 > 2$, or

$$c_1 + b_2 + a_3 \;>\; 2\,. \tag{1}$$

Similarly, since $s_1 + s_4 + s_5 = 3$, two of the numbers s_1, s_4, s_5 are less than $\frac{1}{2}$ (and the third is greater than 2).

- If $s_1 < \frac{1}{2}$ and $s_4 < \frac{1}{2}$, then $b_2 < \frac{1}{2}$ and $a_3 < \frac{1}{2}$, which implies $b_2 + a_3 < 1$. From (1) we have $c_1 > 1$, contradicting $c_1 \le c_1 + c_2 + c_3 = 1$.

- If $s_1 < \frac{1}{2}$ and $s_5 < \frac{1}{2}$, then $b_2 < \frac{1}{2}$ and $c_1 < \frac{1}{2}$, which implies $b_2 + c_1 < 1$. From (1) we have the contradiction $a_3 > 1$.

- If $s_4 < \frac{1}{2}$ and $s_5 < \frac{1}{2}$, then $a_3 < \frac{1}{2}$ and $c_1 < \frac{1}{2}$, which implies $a_3 + c_1 < 1$. From (1) we have the contradiction $b_2 > 1$.

In the same way it is readily checked that we are also led to a contradiction when the condition (1) is replaced by $s_2 > 2$ or $s_3 > 2$. The conclusion follows.

[*Editor's comment*: When this problem was originally printed, the interval that appeared was $\left[\frac{1}{3}, \frac{2}{3}\right]$. (This problem is impossible since the interval is too restricted.) A subsequent issue, [2001 : 213], (incorrectly) corrected the interval to $\left[\frac{1}{3}, \frac{3}{2}\right]$. This new problem can now be proven, but is not as sharp as it could be, since we can improve the lower bound on the interval. In the next issue, [2001 : 267], the problem was further corrected to the (correct) interval listed in the problem statement above. Unfortunately, some solvers missed this last correction, and solved the previously stated (weaker) problem. As a result, we have split the solvers into two groups: those who have solved the intended problem, and those who solved the weaker problem.]

Also solved by CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KEE-WAI LAU, Hong Kong; JOEL SCHLOSBERG, student, New York University, NY, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. The weaker version of the problem was correctly solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ERIC POSTPISCHIL, Nashua, NH; and CHRIS WILDHAGEN, Rotterdam, the Netherlands. There was one incorrect solution.

Both Dimminie and the proposer show that these bounds are the best possible:

- $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, $(0, 0, 1)$ *can give a sum of* $\frac{1}{2}$ *or* 2, *and*
- $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, $(0, 0, 1)$, $(0, 0, 1)$ *can give a sum of* 0 *or* $\frac{3}{2}$.

The proposer also asks about generalizing the problem to n cards of n numbers each. He conjectures that the best intervals for achievable sums appear to be

$$\left[1 - \frac{2}{n}, 1 + \frac{2}{n}\right] \quad \text{for } n \text{ even}, \qquad \left[1 - \frac{2}{n+1}, 1 + \frac{2}{n+1}\right] \quad \text{for } n \text{ odd},$$

but he has no proof. Perhaps our readers can check these bounds and supply proofs!