MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 975 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z5**, or to **Mathematical Mayhem, c/o Faculty of Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, Ontario. N2L 3G1**. The electronic address is

`mayhem-editors@cms.math.ca`

The Assistant Mayhem Editor is Chris Cappadocia (University of Waterloo). The rest of the staff consists of Adrian Chan (Harvard University), Jimmy Chui (University of Toronto), Donny Cheung (University of Waterloo), and David Savitt (Harvard University).

Editorial

Shawn Godin

Another year has come to an end and with it another volume of Crux with Mayhem. It is hard to believe that I have been at this job for a year; it feels like ten! (I have earned enough new grey hairs to count for ten years). Over the past year we have started changing the face of Mayhem and we hope that, as we iron out the wrinkles of our new format, Crux with Mayhem will become accessible to a broader audience. The current changes should be finished by early in 2002, and we should be in our new format for the better part of the next volume.

As our format changed, there were a few solutions that slipped though the cracks. The following solutions were received after the solutions were published due to being sent to the old problem editor and then forwarded to the interim problem editor (yours truly): JOSE LUIS DIAZ (H273), ROBERT BILINSKI (H273 and H274) and PAUL JEFFREYS (H281 and H282). My apologies to these, and any other readers who had their solutions not acknowledged.

At this point I should thank some people who have made my first year go a bit smoother. First and foremost a big thank you to BRUCE SHAWYER for all his help, advice and guidance in this year of a steep learning curve. He always was quick to answer the email, or phone and calm my shattered nerves, as well as not being too harsh when deadlines came and went (like this one!).

Another big asset was NAOKI SATO, the previous Mayhem editor. Naoki has been very helpful, meeting with me, sending me material and giving me advice and help when I needed it. I hope that he will continue to contribute to Mayhem over the coming years.

At CMS headquarters in Ottawa is GRAHAM WRIGHT, who I am sure has put in a new phone line just for my calls. Graham helps with all of my problems from getting stationery and back issues of the journal to helping secure some funds for prizes in our 2002 year of prizes.

The Mayhem staff: ADRIAN CHAN, DONNY CHEUNG, JIMMY CHUI and DAVID SAVITT have made my job that much easier. Their hard work and dedication have really helped shape Mayhem into the great journal that it is. Since we were planning on changing the format, Adrian, Donny and David decided that it was time for them to move on to other things. We will miss your work, and appreciate all the time you have put in over the years. We wish you the best, and hope you'll send us some material from time to time. The new Mayhem assistant editor, CHRIS CAPPADOCIA, has been there when I needed him, and is taking over a larger role in the upcoming year looking after the problem section.

Some other people that have been very helpful over the year that have to be mentioned for their continuous help are: ARLENE ANGEL, DAVID BRIGGS, BILL CLARKE, ELIZABETH ELTON, EDWARD WANG, RICHARD HOSHINO, and CYRUS HSIA.

The year 2002 will be a year of prizes at Mayhem. The Endowment Fund of the CMS has graciously provided us with funds to provide prizes for solutions for the upcoming year. Look for books, past volumes of Crux and Mayhem as well as subscriptions in the next volume. Prizes will be for individuals as well as schools. Look for a complete description in the next issue.

Mayhem Problems

Proposals and solutions may be sent to **Mathematical Mayhem, c/o Faculty of Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, N2L 3G1**, or emailed to

`mayhem-editors@cms.math.ca`

Please include in all correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 May 2002*. Look for prizes for solutions in the new year.

M22. *Proposed by the Mayhem staff.*

George is walking across a bridge on the train track. When he is $\frac{5}{12}$ of the way across the bridge he notices a train bearing down on him at 90 km/h. If he can just escape death by running in either direction, how fast can George run?

M23. *Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Barcelona, Spain.*

Find all complex solutions of the following system of equations

$$\begin{aligned} x^3 + y^3 + z^3 + t^3 &= 12 \\ x^2 + y^2 + z^2 + t^2 &= 0 \\ xy + zt + (x+y)(z+t) &= 0 \\ xyzt &= 3\,. \end{aligned}$$

M24. *Proposed by the Mayhem staff.*

A school math club is deciding on a name for its mascot, a stuffed rabbit. They have narrowed the choices down to three: Euler, Galois and Ramanujan. To pick the name they have each of the 100 club members rank the names in order of preference. When the polls were totalled it was found that 60 people preferred Galois over Ramanujan and 62 preferred Ramanujan over Euler. It was suggested, by a Galois supporter, that Euler should be dropped. A staunch Eulerist objected and demanded the counting continue. When the final totals came in it was found that 68 preferred Euler over Galois! If each possible ranking was picked by at least one member, how many picked each name as their first choice?

M25. *Proposed by the Mayhem staff.*

What is the smallest number with the property that when the first digit (leftmost) is moved to the rightmost position, the new number is three times the original?

M26. *Proposed by the Mayhem staff.*

Find two isosceles triangles, with two sides 106 units long and the other side an integer, that have the same area.

M27. *Proposed by the Mayhem staff.*

Find $\sqrt{ab+1}$ where $a = \overbrace{111\cdots 11}^{\text{2002 1's}}$ and $b = 1\overbrace{00\cdots 00}^{\text{2001 0's}}5$.

M28. *Proposed by the Mayhem staff.*

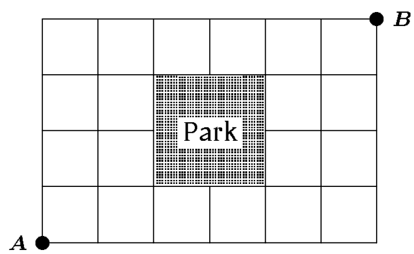
Shawn tosses 2001 fair coins and Bruce tosses 2002 fair coins. What is the probability that Bruce gets more heads than Shawn?

Problem of the Month

Jimmy Chui, student, University of Toronto

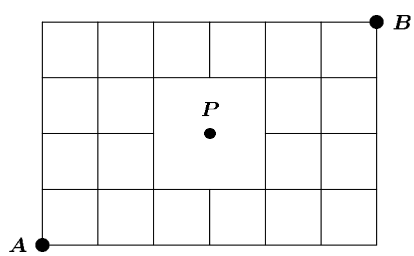
Problem.

Problem. A road map of Grid City is shown in the diagram. The perimeter of the park is a road but there is no road through the park. How many different shortest road routes are there from point A to point B?



(1996 COMC, Problem A5)

Solution 1. Point B is situated 6 blocks east and 4 blocks north of point A. Anyone can walk from point A to point B any way he or she chooses, except there is no pathway through the park. Now suppose a man by the name of Max Power wants to walk from point A to point B in the least time possible. And he absolutely refuses to walk through the park.



Let us call the centre of the park, point P. Now, point P is 3 blocks east and 2 blocks north of point A, and point B is also 3 blocks east and 2 blocks north of point P.

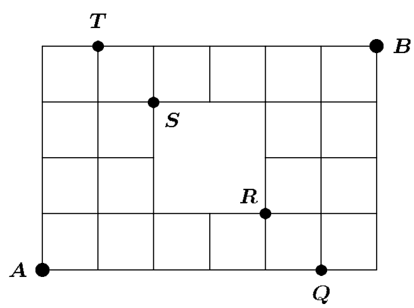
How can we count the total number of ways that Max can walk from point A to point B?

Method I. We can do a count as follows. If the park wasn't there, then Max is free to walk the shortest route (10 blocks) any way he chooses. As long as he chooses to go north 4 times and east 6 times, then he will reach his destination. This is exactly to $\binom{10}{4}$, or equivalently, $\binom{10}{6}$, which has the value of 210.

But in this calculation, the park disappeared and Max was free to walk through it. The only paths that are affected are the ones that pass through the park. The number of ways that Max can get from point A to point P would be $\binom{5}{2}$, and the number of ways that he could travel from point P to point B would also be $\binom{5}{2}$. Whatever choice of path Max takes to get to the park does not have anything to do with the choice from the park to point B. So the total number of paths that connect point A to point B, through point P, is $\binom{5}{2}\binom{5}{2} = 100$.

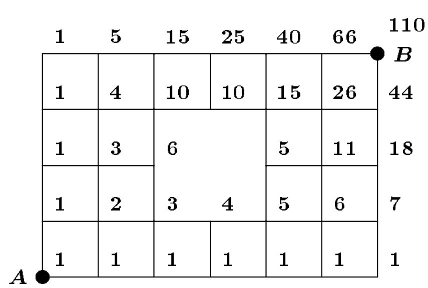
These 100 paths are impossible for Max to take. The number of paths feasible for Max to take would be $210 - 100 = 110$.

Method II. Another way is to note that there are a certain number of points that Max has to pass through. In the above diagram, Max **must** pass through one and only one of the points Q, R, S, and T.



The number of paths through point Q is $\binom{5}{0}\binom{5}{4} = 5$. Similarly, the number of paths through points R, S, and T, are respectively $\binom{5}{1}\binom{5}{3} = 50$, $\binom{5}{3}\binom{5}{1} = 50$, and $\binom{5}{4}\binom{5}{0} = 5$. And this means that Max has a total choice of $5 + 50 + 50 + 5 = 110$ paths.

Solution 2. We can do a manual count in a Pascal Triangle-esque manner. The diagram below shows the count.



Polya's Paragon

Shawn Godin

As a small child, I used to love playing with counting problems. I would spend hours on end enumerating this and counting that. I distinctly remember the moment when I realized that removing my shoes and socks allowed me to deal with even larger numbers. It was not until the end of my high school career that I ran into techniques that allowed me to count things without having to count them.

Two very basic ideas open doors to more complex situations.

The Fundamental Counting Theorem: If we can perform a certain action in a ways, a second action in b ways, a third action in c ways $\ldots$, then the number of ways of doing the first action **and** the second action **and** the third action **and** $\ldots$ is $a \times b \times c \times \ldots$

The Rule of Sum: If we can perform a certain action in a ways, a second action in b ways, a third action in c ways $\ldots$, and none of the actions can be performed simultaneously, then the number of ways of doing the first action **or** the second action **or** the third action **or** $\ldots$ is $a + b + c + \ldots$

Let us take a look at these two very simple ideas at work:

How many ways can 10 *people be lined up to take a club photo?*

By our first idea we have $10 \times 9 \times \cdots \times 2 \times 1 = 3628800$.

How many ways can a president, vice-president and treasurer be picked from a group of 10 *people?*

Similarly we have $10 \times 9 \times 8 = 720$.

These two types of problems arise often, so we have special notation for each of them.

Definition: The total number of ways to arrange n distinct objects is $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$, where $n!$ is read n **factorial**.

Thus for our first problem the result is $10!$. In the second case we have the first three terms of $10!$, which we will call $P(10,3)$. In general we have $P(n,r) = \dfrac{n!}{(n-r)!}$ where $P(n,r)$ is the number of arrangements of r of the n objects in a specific order. What happens if order is not important?

How many ways can a group of 3 *people be chosen from a group of* 10 *people?*

In this case the $P(10,3)$ from above counts 123, 132, 213, 231, 312, and 321 as different arrangements, but they represent the same group of three. Thus each group of three is repeated $3!$ times in our case above, so that the number of groups of three must be $\frac{P(10,3)}{3!} = \frac{10!}{7!3!} = 120$.

In general the number of ways to pick r of n objects, without regard to order is $\binom{n}{r} = \frac{n!}{(n-r)!r!}$. The symbol $\binom{n}{r}$ is read n **choose** r, and the collection of these objects is called the binomial coefficients since they show up in expanding powers of binomials.

We can put everything together with some problems.

1. How many unique arrangements are there of the letters in the word $EULER$?
2. How many ways can you arrange the letters $ABCDEFG$ such that the A and B are beside each other?
3. How many ways can a class of 20 people be divided into 4 teams of 5 people?

Who said the following?

Even fairly good students, when they have obtained the solution of the problem and written down neatly the argument, shut their books and look for something else. Doing so, they miss an important and instructive phase of the work. . . . A good teacher should understand and impress on his students the view that no problem whatever is completely exhausted. One of the first and foremost duties of the teacher is not to give his students the impression that mathematical problems have little connection with each other, and no connection at all with anything else. We have a natural opportunity to investigate the connections of a problem when looking back at its solution.

In order to translate a sentence from English into French, two things are necessary. First, we must understand thoroughly the English sentence. Second, we must be familiar with the forms of expression peculiar to the French language. The situation is very similar when we attempt to express in mathematical symbols a condition proposed in words. First, we must understand thoroughly the condition. Second, we must be familiar with the forms of mathematical expression.