SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2577. [2000 : 429] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

Let $a_1, a_2, \ldots, a_n$ $(n \geq 2)$ be positive integers. Determine the values of n and k $(2 \leq k \leq n)$ for which the following identity holds:

$$\gcd_{1 \leq i_1 < \ldots < i_k \leq n} \left(\operatorname{lcm}\{a_{i_1}, \ldots, a_{i_k}\}\right) = \operatorname{lcm}_{1 \leq i_1 < \ldots < i_k \leq n} \left(\gcd\{a_{i_1}, \ldots, a_{i_k}\}\right).$$

Solution by Michel Bataille, Rouen, France.

We show that the given identity holds if and only if n is odd and $k = \frac{1}{2}(n+1)$.

Suppose first that the equation

$$\gcd_{1 \leq i_1 < \ldots < i_k \leq n} \left(\operatorname{lcm}\{a_{i_1}, \ldots, a_{i_k}\}\right) = \operatorname{lcm}_{1 \leq i_1 < \ldots < i_k \leq n} \left(\gcd\{a_{i_1}, \ldots, a_{i_k}\}\right)$$

holds for all n–tuples $(a_1, a_2, \ldots, a_n)$ of positive integers.

Then it must hold, in particular, for $a_i = 2^i$ $(1 \leq i \leq n)$. In this case, $\operatorname{lcm}_{1 \leq i_1 < \ldots < i_k \leq n} \{a_{i_1}, \ldots, a_{i_k}\} = 2^{i_k} \geq 2^k$ (since $i_k \geq k$) with equality when $(i_1, i_2, \ldots, i_k) = (1, 2, \ldots, k)$.

Hence, $\gcd_{1 \leq i_1 < \ldots < i_k \leq n} \left(\operatorname{lcm}\{a_{i_1}, \ldots, a_{i_k}\}\right) = 2^k$.

Similarly, $\gcd_{1 \leq i_1 < \ldots < i_k \leq n} \{a_{i_1}, \ldots, a_{i_k}\} = 2^{i_1} \leq 2^{n-(k-1)}$ with equality when $(i_1, i_2, \ldots, i_k) = (n-k+1, \ldots, n)$.

Hence, $\operatorname{lcm}_{1 \leq i_1 < \ldots < i_k \leq n} \left(\gcd\{a_{i_1}, \ldots, a_{i_k}\}\right) = 2^{n-(k-1)}$.

It follows that $2^k = 2^{n-(k-1)}$, or $n = 2k-1$; that is, n is odd and $k = \frac{1}{2}(n+1)$.

Conversely, suppose that $n = 2m+1$ and $k = m+1$. When q is a prime integer, we will denote by $v_q(a)$ the exponent of the highest power of q which divides a.

Let $(a_1, a_2, \ldots, a_{2m+1})$ be any family of $n = 2m+1$ positive integers. We have to prove that

$$\begin{aligned} L &= \gcd_{1 \leq i_1 < \ldots < i_{m+1} \leq 2m+1} \left(\operatorname{lcm}\{a_{i_1}, \ldots, a_{i_{m+1}}\}\right) \\ &= \operatorname{lcm}_{1 \leq i_1 < \ldots < i_{m+1} \leq 2m+1} \left(\gcd\{a_{i_1}, \ldots, a_{i_{m+1}}\}\right) = R \end{aligned}$$

Fix an arbitrary prime p. We may assume, by changing the numbering if necessary, that $v_p(a_1) \leq v_p(a_2) \leq \ldots \leq v_p(a_{2m+1})$.

In any sub-family $(a_{i_1}, \ldots, a_{i_{m+1}})$, there is at least one number from $\{a_{m+1}, \ldots, a_{2m+1}\}$ such that $v_p(\text{lcm}\{a_{i_1}, \ldots, a_{i_{m+1}}\}) \geq v_p(a_{m+1})$.

Moreover, this value $v_p(a_{m+1})$ is attained for the sub-family $(a_1, a_2, \ldots, a_{m+1})$. It follows that $v_p(L) = v_p(a_{m+1})$.

Now, in $(a_{i_1}, \ldots, a_{i_{m+1}})$, there is also a number coming from $\{a_1, a_2, \ldots, a_{m+1}\}$. Hence, $v_p(\gcd\{a_{i_1}, \ldots, a_{i_{m+1}}\}) \leq v_p(a_{m+1})$, with equality for the sub-family $(a_{m+1}, \ldots, a_{2m+1})$. It follows that $v_p(R) = v_p(a_{m+1})$.

Thus, $v_p(L) = v_p(R)$, and since this result may be obtained for each prime integer p, we have $L = P$.

Also solved by CON AMORE PROBLEM GROUP, The Danish University of Education, Copenhagen, Denmark; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer.

The solutions offered by Janous, Leversha, Wildhagen and the proposer were very terse. The editor preferred to highlight a fuller solution for the sake of our younger readers. Janous ended his solution with the comment "Neat!".

2578. [2000 : 429] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

For each integer n, determine the hundreds and the units digits of the number $\dfrac{1+5^{2n+1}}{6}$.

Solution by Mihai Cipu, IMAR, Bucharest, Romania

First we note that n must be non-negative in order that

$$x_n := \frac{1+5^{2n+1}}{6} \geq 1 .$$

Thus for $n < 0$ the integer part of x_n is 0 and the hundreds and units digits are both 0. For $n = 0$ and $n = 1$ one gets $x_0 = 1$ and $x_1 = 21$, respectively. [*Ed.:* which means that the units digits is 1 and the hundreds digit is 0.] Let us suppose now that $n \geq 2$. Then

$$\begin{aligned} x_n &= 5^{2n} - 5^{2n-1} + 5^{2n-2} - \cdots - 5 + 1 = 4 \cdot 5^{2n-1} + x_{n-1} \\ &= 100 \cdot 5^{2n-3} + x_{n-1} . \end{aligned}$$

Hence, $x_n \equiv 500 + x_{n-1} \pmod{1000}$, and the hundreds digit is repeated with period 2, while the last digit is constant: $x_{2n} \equiv 21 \pmod{1000}$ and $x_{2n+1} \equiv 521 \pmod{1000}$.

Also solved by ANGELO STATE UNIVERSITY PROBLEM SOLVING GROUP, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; BRIAN BEASLEY, *Presbyterian College, Clinton, SC, USA; ROBERT BILINSKI, Outremont, Québec; PIERRE BORNSZTEIN, Pontoise, France; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; LAKE SUPERIOR STATE UNIVERSITY PROBLEM SOLVING GROUP; KEE-WAI LAU, Hong Kong, China; GERRY LEVERSHA, St. Paul's School, London, England; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; CARL LIBIS, Richard Stockton College of New Jersey, Pomona, NJ, USA; HENRY LIU, student, Trinity College, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HENRY J. PAN, student, East York Collegiate Institute, Toronto, Ontario; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; D.J. SMEENK, Zaltbommel, the Netherlands; SOUTHWEST MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; and CHRIS WILDHAGEN, Rotterdam, the Netherlands.*

Many solvers pointed out that it appeared strange to not ask for the tens digit as well, since it was there for the asking. Indeed, the proposer's original submission asked for only the last two digits of the expression for $n > 0$. Several solvers, including the proposer, solved this problem instead of the one in print. Those solvers were ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; LI ZHOU, Polk Community College, Winter Haven, FL; and the proposer. There was also one incorrect solution.

2579. [2000 : 430] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

The excircle on the side BC of triangle ABC touches AC and AB, respectively at Y_A and Z_A. Likewise, the one on CA touches BC and BA at X_B and Z_B, and the one on AB touches CA and CB at Y_C and X_C. Let A' be the intersection of Z_BX_B and X_CY_C, B' be that of X_CY_C and Y_AZ_A, and C' be that of Y_AZ_A and Z_BX_B. Show that AA', BB' and CC' are concurrent. What is the point of intersection of these three lines?

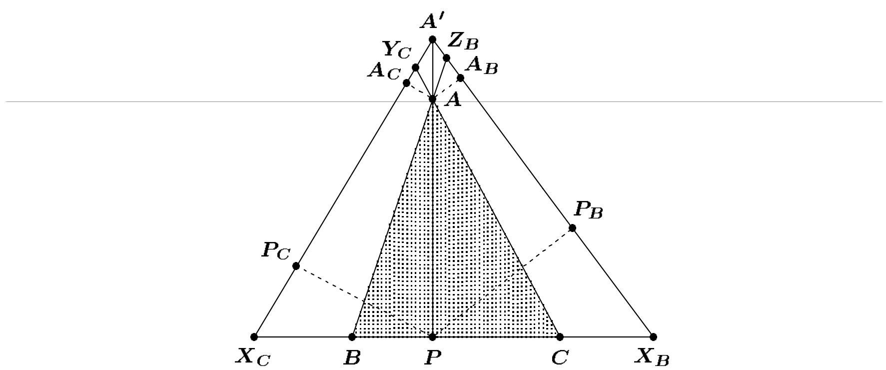
Solution by Niels Bejlegaard, Copenhagen, Denmark.

The sides of $\triangle A'B'C'$ intercept lengths equal to $s = \frac{a+b+c}{2}$ from each vertex of $\triangle ABC$; more precisely $s = CX_C = CY_C = BX_B = BZ_B$, so that

$$AY_C = s - b\,,\quad AZ_B = s - c\,,\ \text{and}\ BX_C = CX_B = s - a\,.$$

[See figure on page 540.]

Moreover, because the triangles CX_CY_C and BX_BZ_B are isosceles, X_CY_C is perpendicular to the bisector of $\angle C$ while X_BZ_B is perpendicular to the bisector of $\angle B$. Call P the point where AA' intersects BC; we shall see that P is the foot of the altitude from A in $\triangle ABC$. In the figure, A_C and P_C are the respective feet of the perpendiculars to X_CY_C from A and from P, while A_B and P_B are the feet of the perpendiculars to X_BZ_B from A and



from P. Hence, $AA_C = (s-b)\cos\frac{C}{2}$ and $AA_B = (s-c)\cos\frac{B}{2}$, so that for some $k > 0$, $PP_C = k(s-b)\cos\frac{C}{2}$ and $PP_B = k(s-c)\cos\frac{B}{2}$. As a consequence,

$$PB = \frac{PP_C}{\cos\frac{C}{2}} - (s-a) = k(s-b) - (s-a),$$

and similarly,

$$PC = k(s-c) - (s-a).$$

Since $a = PB + PC$, we deduce that $k = \dfrac{b+c}{a}$ and

$$\begin{aligned} PB &= \frac{b+c}{a}\cdot\frac{a+c-b}{2} - \frac{b+c-a}{2} \\ &= \frac{a^2+c^2-b^2}{2a} = \frac{2ac\cos B}{2a} = c\cos B\,. \end{aligned}$$

This indicates that P is the foot of the altitude from A in $\triangle ABC$, as claimed. Similarly, BB' and CC' are the other altitudes of $\triangle ABC$, so the three given lines concur at the orthocentre of $\triangle ABC$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2580. [2000 : 430] *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Suppose that a, b and c are positive real numbers. Prove that

$$\frac{b+c}{a^2+bc}+\frac{c+a}{b^2+ac}+\frac{a+b}{c^2+ab} \leq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}\,.$$

Solution by Richard Eden, Ateneo de Manila University, Philippines; John Fremlin, student, Colchester Royal Grammar School, Colchester, England; and Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let $D = abc(a^2+bc)(b^2+ac)(c^2+ab)$. Clearly, $D > 0$. We have

$$\begin{aligned}
&\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{b+c}{a^2+bc}-\frac{c+a}{b^2+ac}-\frac{a+b}{c^2+ab} \\
&\qquad = \frac{a^4b^4+b^4c^4+c^4a^4-a^4b^2c^2-b^4c^2a^2-c^4a^2b^2}{D} \\
&\qquad = \frac{(a^2b^2-b^2c^2)^2+(b^2c^2-c^2a^2)^2+(c^2a^2-a^2b^2)^2}{2D} \geq 0\,,
\end{aligned}$$

which shows that the given inequality is true. Equality holds if and only if $a = b = c$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); MICHEL BATAILLE, Rouen, France; NIELS BEJLEGAARD, Copenhagen, Denmark; MIHÁLY BENCZE, Brasov, Romania; PIERRE BORNSZTEIN, Pontoise, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; MIHAI CIPU, IMAR, Bucharest, Romania; CON AMORE PROBLEM GROUP, The Danish University of Education, Copenhagen, Denmark; JOSÉ LUÍS DIAZ-BARRERO, Universitat Politècnica de Catalunya, Terrassa, Spain; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; THOMAS JANG, Southwest Missouri State University, Springfield, MO, USA; D. KIPP JOHNSON, Valley Catholic High School, Gaston, OR, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; DAVID LOEFFLER, student, Cotham School, Bristol, UK; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; JOEL TAY, Anglo-Chinese School, Singapore; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2581. [2000 : 430] *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Suppose that a, b and c are positive real numbers. Prove that

$$\frac{ab+c^2}{a+b}+\frac{bc+a^2}{b+c}+\frac{ca+b^2}{c+a} \geq a+b+c\,.$$

Solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

Let $D = (a+b)(b+c)(c+a)$. Clearly, $D > 0$. We show that the difference between the left-hand side and the right-hand side of the inequality is non-negative.

$$\begin{aligned}
&\frac{ab+c^2}{a+b} - c + \frac{bc+a^2}{b+c} - a + \frac{ca+b^2}{c+a} - b \\
&= \frac{c^2+ab-ac-bc}{a+b} + \frac{a^2+bc-ab-ac}{b+c} + \frac{b^2+ac-ab-bc}{c+a} \\
&= \frac{(c-a)(c-b)}{a+b} + \frac{(a-b)(a-c)}{b+c} + \frac{(b-a)(b-c)}{c+a} \\
&= \frac{(c^2-a^2)(c^2-b^2)+(a^2-b^2)(a^2-c^2)+(b^2-a^2)(b^2-c^2)}{D} \\
&= \frac{a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2}{D} \\
&= \frac{[(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2]}{2D} \ \geq \ 0\,.
\end{aligned}$$

Equality holds if and only if $a = b = c$.

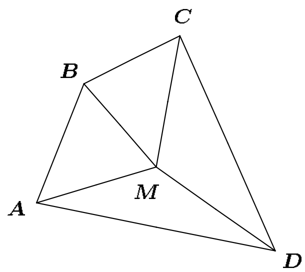
Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (four other solutions); MICHEL BATAILLE, Rouen, France; NIELS BEJLEGAARD, Denmark; MIHÁLY BENCZE, Brasov, Romania; PIERRE BORNSZTEIN, Pontoise, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; MIHAI CIPU, IMAR, Bucharest, Romania; CON AMORE PROBLEM GROUP, The Danish University of Education, Copenhagen, Denmark; JOSÉ LUÍS DIAZ-BARRERO, Universitat Politècnica de Catalunya, Terrassa, Spain; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD EDEN, Ateneo de Manila University, Philippines; JOHN FREMLIN, student, Colchester Royal Grammar School, Colchester, England; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA (two solutions); THOMAS JANG, Southwest Missouri State University, Springfield, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Valley Catholic High School, Gaston, OR, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; HENRY LIU, Trinity College, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; JOEL TAY, Anglo-Chinese School, Singapore; PANOS E. TSAOUSSOGLOU, Athens, Greece; CHRIS WILDHAGEN, Rotterdam, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Most solutions were similar to the above one. Bornsztein and Klamkin noted that the given inequality can be reduced to particular cases of Schur's Inequality

$$x^t(x-y)(x-z) + y^t(y-z)(y-x) + z^t(z-x)(z-y) \ \geq \ 0\,,$$

where t is a real number and x, y, z are non-negative real numbers.

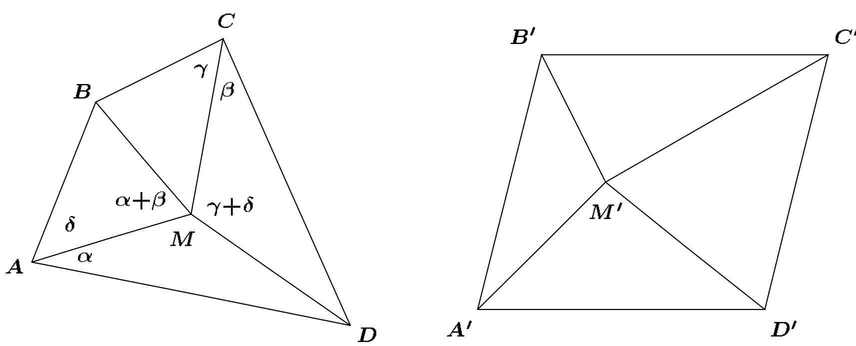
2583. [2000 : 430] *Proposed by Nairi M. Sedrakyan, Yerevan, Armenia.*



Given a point M inside the convex quadrangle (see diagram), such that $\angle AMB = \angle MAD + \angle MCD$, $\angle CMD = \angle MCB + \angle MAB$ and $MA = MC$.

Prove that $AB \cdot CM = BC \cdot MD$.

I. *Solution by the proposer.*



Consider the points A', B', C', D', and M' such that the following conditions are satisfied (see the diagram above): $A'B' = AB \cdot CM$, $B'M' = BM \cdot CM$, $A'M' = AM \cdot CM$, $B'C' = BM \cdot AD$, $C'M' = BM \cdot MD$, $C'D' = BC \cdot MD$, $M'D' = MA \cdot MD$, $A'D' = MA \cdot CD$.

Then we have $\triangle ABM \sim \triangle A'B'M'$, $\triangle BMC \sim \triangle C'M'D'$, $\triangle CMD \sim \triangle A'M'D'$, and $\triangle AMD \sim \triangle B'M'C'$. Hence,

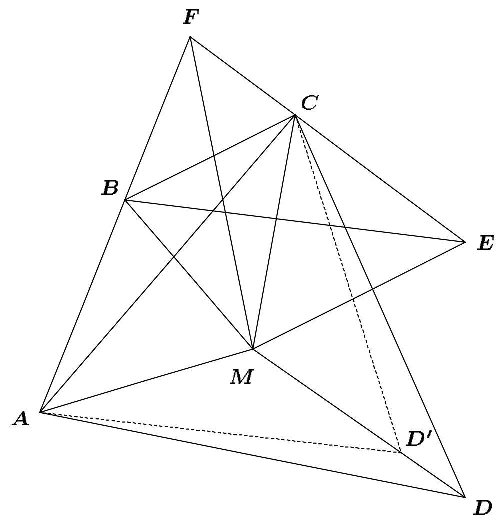
$$\angle A'M'B' = \angle AMB = \angle MAD + \angle MCD = \angle M'B'C' + \angle M'A'D',$$

from which it follows that $\angle D'A'B' + \angle A'B'C' = 180°$, implying that $B'C' \parallel A'D'$. We also have

$$\angle A'M'D' = \angle CMD = \angle MCB + \angle MAB = \angle M'D'C' + \angle M'A'B',$$

from which we see that $\angle B'A'D' + \angle A'D'C' = 180°$, implying that $A'B' \parallel C'D'$. Therefore, the quadrilateral $A'B'C'D'$ is a parallelogram. Thus, $A'B' = C'D'$, from which follows $AB \cdot CM = BC \cdot MD$.

II. *Solution by Nikolaos Dergiades, Thessaloniki, Greece.*



On MD we take the point D' such that

$$\frac{MD'}{CM} = \frac{AB}{BC}. \qquad (1)$$

We will prove that $D' \equiv D$. See the diagram above. Since

$$\begin{aligned} \angle AMB + \angle BMC &= \angle MAD + \angle MCD + \angle ADC \\ \text{and} \quad \angle CMD + \angle AMD &= \angle MCB + \angle MAB + \angle ABC \end{aligned}$$

we get from the hypothesis that

$$\begin{aligned} \angle BMC &= \angle ADC & (2) \\ \text{and} \quad \angle AMD &= \angle ABC & (3) \end{aligned}$$

By rotation around M we move the point A onto C and $\triangle MAB$ onto $\triangle MCE$, and we have

$$\angle BCE = \angle CMD. \qquad (4)$$

Let AB and EC meet at F. Since $\angle MAB = \angle MCE$, the points M, A, F, C are concyclic. A similar argument applies to the points M, B, F, E. From (1), (3) and $MA = MC$ we have $\triangle D'MA \sim \triangle ABC$, which implies

$$\angle MD'A = \angle BAC = \angle FMC. \qquad (5)$$

From (1) and (4) we have $\triangle BCE \sim \triangle CMD'$, implying

$$\angle MD'C = \angle BEC = \angle FMB. \qquad (6)$$

From (2), (5), and (6) we get $\angle AD'C = \angle BMC = \angle ADC$ which implies $D' \equiv D$, and by (1) we have $AB \cdot CM = BC \cdot MD$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina. His solution was essentially the same as solution I *above.*

2584. [2000 : 430] *Proposed by Nairi M. Sedrakyan, Yerevan, Armenia.*

You are given that X, Y, Z and T are points on the chord AB of the circle Γ. Circles Γ_1 and Γ_2 pass through the points X and Y, and touch the circle Γ at points P and S, respectively, while the circles Γ_3 and Γ_4 pass through the points Z and T, and touch the circle Γ at points Q and R, respectively. Also, Q belongs to the arc APB and the segments XY and ZT do not have common points. Prove that the segments PR, QS and AB intersect at the same points.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

Let the tangents to Γ at P and S intersect at K. [If these tangents are parallel then we shall say that K is at infinity.] Then K is the radical centre of Γ, Γ_1, Γ_2, and it therefore lies on the extension of the chord AB. Similarly, if M is the intersection point of the tangents to Γ at R and Q, then M also lies on AB. In other words, if N is the point where KP meets MQ and L is the point where KS meets MR, then AB is a portion of the diagonal KM of the quadrilateral $KLMN$. Since Γ is inscribed in the hexagon $KLRMNP$, Brianchon's theorem implies that the diagonals KM, LN, PR pass through the same point. Similarly QS passes through the point of intersection of KM and LN; therefore PR, QS, and AB intersect in the same point.

Editor's comment. From the first lines of Dergiades' argument we see that our problem reduces to the familiar projective result,

> *If the vertices of a complete quadrangle lie on a conic, the tangents at a pair of vertices meet in a point of the line joining the diagonal points of the quadrangle that are not on the side joining the two vertices.*

Dergiades shows that this is simply a degenerate case of Brianchon's theorem. Woo provides another quick argument: there is a projective transformation that takes $PQRS$ to a rectangle inscribed in a circle [where AB is part of some diameter], in which case the desired conclusion immediately follows.

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; PETER Y. WOO, Biola University, La Mirada, CA, USA (3 proofs); and the proposer.

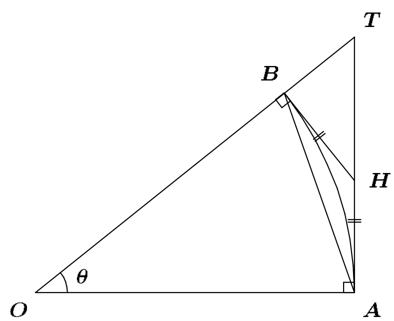
2585. [2000 : 430] *Proposed by Vedula N. Murty, Visakhapatnam, India.*

Prove that, for $0 < \theta < \pi/2$,

$$\tan\theta + \sin\theta \;>\; 2\theta\,.$$

I. *Solution by Toshio Seimiya, Kawasaki, Japan.*

Let OAB be a sector with centre O, radius 1 and $\angle AOB = \theta$, where $0 < \theta < \frac{\pi}{2}$.



Let T be the point of intersection of OB and the tangent to the arc AB at A. Let H be the point of intersection of AT and the tangent to the arc AB at B. Then $\angle OAH = \angle OBH = 90^\circ$ and $HA = HB$. Since $\angle TBH = 90^\circ$, $TH > HB = HA$. Let $[F]$ denote the area of a plane figure F. Then $HA < TH$ implies

$$[HAB] \;<\; [HBT]\,,$$

and therefore,

$$[\text{sector } OAB] - [OAB] < [OAT] - [\text{sector } OAB]\,.$$

The last inequality can be written as

$$\frac{1}{2}\theta - \frac{1}{2}\sin\theta \;<\; \frac{1}{2}\tan\theta - \frac{1}{2}\theta\,,$$

or

$$2\theta \;<\; \tan\theta - \sin\theta\,,$$

which is the desired inequality.

II. *Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

More generally, we show that for $0 < \theta < \frac{\pi}{2}$ and any pair of real numbers a and b with $a^2 + b^2 > 0$,

$$a^2\tan\theta + b^2\sin\theta \;>\; 2ab\theta\,.$$

Consider the function $f(\theta) = a^2 \tan\theta + b^2 \sin\theta - 2ab\theta$ on the interval $(0, \frac{\pi}{2})$. We have

$$\begin{aligned} f'(\theta) &= a^2 \sec^2\theta + b^2 \cos\theta - 2ab \\ &> a^2 \sec^2\theta + b^2 \cos^2\theta - 2ab \\ &= (a\sec\theta - b\cos\theta)^2 \;\geq\; 0\,. \end{aligned}$$

Thus $f(\theta)$ is strictly increasing on the interval $(0, \frac{\pi}{2})$. In addition, $f(0) = 0$ and $f(\theta)$ is increasing and continuous on $[0, \frac{\pi}{2})$. Therefore, $f(\theta) > f(0) = 0$ for every θ in $(0, \frac{\pi}{2})$, which gives the desired inequality.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (three solutions); MICHEL BATAILLE, Rouen, France; BRIAN BEASLEY, Presbyterian College, Clinton, SC, USA; MIHÁLY BENCZE, Brasov, Romania; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIHAI CIPU, IMAR, Bucharest, Romania; CON AMORE PROBLEM GROUP, The Danish University of Education, Copenhagen, Denmark; NIKOLAOS DERGIADES, Thessaloniki, Greece; JOSÉ LUÍS DIAZ-BARRERO, Universitat Politècnica de Catalunya, Terrassa, Spain; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; CHARLES HENDEE, Southwest Missouri State University, Springfield, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Valley Catholic High School, Gaston, OR, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong; HENRY LIU, Trinity College, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK (two solutions); PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; HENRY J. PAN, student, East York Collegiate Institute, Toronto, Ontario; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; NICHOLAS THAM, Raffles Junior College, Singapore; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Rotterdam, the Netherlands; CALVIN ZHIWEI, Singapore; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There were two incorrect solutions submitted.

Seimiya was the only solver who submitted a geometric solution. Janous noted that the following known fact can be used to generalize the given inequality: If a and b are positive real numbers, then

$$M_\lambda(a,b) \;=\; \begin{cases} (a^\lambda + b^\lambda)^{\frac{1}{\lambda}}\,, & \lambda \neq 0 \\ \sqrt{ab}\,, & \lambda = 0 \end{cases}$$

is an increasing function of λ. The given inequality is $M_1(\tan x, \sin x) > x$ for $x \in (0, \frac{\pi}{2})$. Janous then proceeded to show the stronger inequality $M_0(\tan x, \sin x) > x$ for $x \in (0, \frac{\pi}{2})$ and conjectured that $M_\lambda(\tan x, \sin x) > x$ for all $x \in (0, \frac{\pi}{2})$ if and only if $\lambda \geq -(\log 2)/\log(\frac{\pi}{2})$. The inequality $M_0(\tan x, \sin x) > x$ was also proven by Arslanagić in one of his solutions.

2586. [2000 : 431] *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Find all (real or complex) solutions of the system

$$\begin{aligned} 3x + x^3 &= y(1 + 3x^2)\,, \\ 3y + y^3 &= z(1 + 3y^2)\,, \\ 3z + z^3 &= w(1 + 3z^2)\,, \\ 3w + w^3 &= x(1 + 3w^2)\,. \end{aligned}$$

Solution by the Southwest Missouri State University Problem Solving Group, Springfield, MO, USA.

Since $3u+u^3$ and $1+3u^2$ cannot vanish simultaneously, we may rewrite the system as

$$y = \frac{3x + x^3}{1 + 3x^2}\,,\quad z = \frac{3y + y^3}{1 + 3y^2}\,,\quad w = \frac{3z + z^3}{1 + 3z^2}\,,\quad x = \frac{3w + w^3}{1 + 3w^2}\,.$$

It is an elementary fact from hyperbolic trigonometry that

$$\tanh 3t \;=\; \frac{3\tanh t + \tanh^3 t}{1 + 3\tanh^2 t}\,.$$

Now the hyperbolic tangent function takes on all complex values except 1 and -1. From the system above, it is immediate that if $x = 1$, then $y = z = w = 1$ and if $x = -1$, then $y = z = w = -1$. For any other value of x, let $x = \tanh t$. Applying our hyperbolic trigonometric identity to the system above yields: $y = \tanh 3t$, $z = \tanh 9t$, $w = \tanh 27t$, and $x = \tanh 81t$. This implies that $\tanh t = \tanh 81t$. Since the period of the hyperbolic tangent is πi, this forces $t + i\pi k = 81t$, where k is an integer, and hence $t = i\pi k/80$. Since $\tanh iu = i\tanh u$ and it suffices to take t over a full period of the hyperbolic tangent function, the set of all solutions to the original system is

$$\begin{aligned} \{x, y, z, w\} \;&=\; \pm\{1, 1, 1, 1\} \quad \text{or} \\ &\{i\tan(\pi k/80), i\tan(3\pi k/80), i\tan(9\pi k/80, i\tan(27\pi k/80)\} \\ &\qquad \text{for } k = -39, \ldots, 39. \end{aligned}$$

Note that a similar argument will work for analogous systems in any number of variables.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE and ROGER ZARNOWSKI, Angelo State University, San Angelo, TX, USA; RICHARD B. EDEN, Ateno de Manila University, Philippines; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer.

Most submissions were variations of the above one. Of the 81 *solutions, three are real. Several solvers missed the solutions* $(1, 1, 1, 1)$ *and* $(-1, -1, -1, -1)$*, which may be found by inspection. There was one incomplete solution.*

2587. [2000 : 431] *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

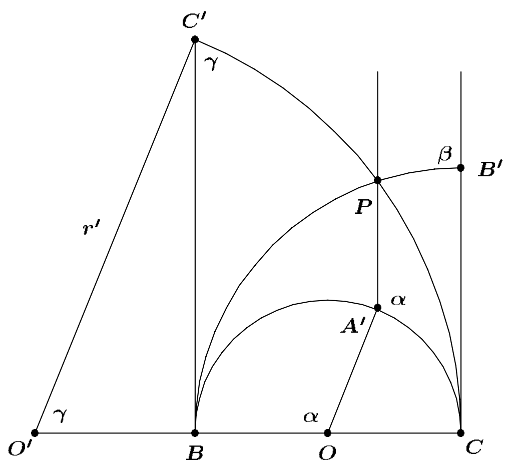
In the half plane $Z = \{(x,y) : y \geq 0\}$, let f be the union of the set of all semicircles lying in Z with diameters on the x–axis, with the set of all lines in Z perpendicular to the x–axis.

Denote by f_{XY} the unique member of f that goes through any two points X and Y in Z. For any three points A, B and C in Z, denote by $\triangle ABC$ the curvilinear triangle formed by the arcs f_{AB}, f_{BC} and f_{CA}.

Let A, B and C be any three points on the x–axis. Let P be any point in the interior of $\triangle ABC$. Let $A' = f_{AP} \cap f_{BC}$, $B' = f_{BP} \cap f_{CA}$ and $C' = f_{CP} \cap f_{AB}$. Let α be the angle at A', interior to $\triangle CAA'$, let β be the angle at B' interior to $\triangle ABB'$, and let γ be the angle at C' interior to $\triangle BCC'$.

Prove that $\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)\tan\left(\frac{\gamma}{2}\right) = 1$.

I. *Solution by the proposer.*



Invert the figure in a circle whose centre is A to get the accompanying diagram — the point A goes to infinity and the semicircles f_{AP}, f_{AB}, and f_{AC} all become halflines perpendicular to BC. Introduce coordinates so that the origin O is the mid-point of BC, and B is $(-1,0)$ while C is $(1,0)$. Let O' and r' be the centre and radius of the arc $f_{CC'}$. Then $\angle CO'C' = \gamma$, $O'B = r'\cos\gamma$, and $BC = 2 = r'(1-\cos\gamma)$. Therefore,

$$r' = \frac{2}{1-\cos\gamma} = \csc^2\frac{\gamma}{2}.$$

If x' is the x–coordinate of O', then

$$-x' = OO' = CO' - CO = r' - 1 = \csc^2\frac{\gamma}{2} - 1 = \cot^2\frac{\gamma}{2},$$

and the points of $f_{CC'}$ satisfy

$$\left(x+\cot^2\frac{\gamma}{2}\right)^2+y^2 \;=\; \left(\csc^2\frac{\gamma}{2}\right)^2 ,$$

or

$$x^2+y^2+2x\cot^2\frac{\gamma}{2} \;=\; \csc^4\frac{\gamma}{2}-\cot^4\frac{\gamma}{2} \;=\; \csc^2\frac{\gamma}{2}+\cot^2\frac{\gamma}{2} ;$$

that is

$$x^2+y^2+2x\cot^2\frac{\gamma}{2} \;=\; 1+2\cot^2\frac{\gamma}{2} .$$

Similarly, $f_{BB'}$ satisfies

$$x^2+y^2-2x\tan^2\frac{\beta}{2} \;=\; 1+2\tan^2\frac{\beta}{2} .$$

Subtracting, we find that the x–coordinate x'' of P satisfies

$$x'' \;=\; \frac{\cot^2\frac{\gamma}{2}-\tan^2\frac{\beta}{2}}{\cot^2\frac{\gamma}{2}+\tan^2\frac{\beta}{2}} .$$

From the diagram we see that

$$x'' \;=\; -\cos\angle A'OB \;=\; -\cos\alpha .$$

Therefore,

$$\cot^2\frac{\alpha}{2} \;=\; \frac{1+\cos\alpha}{1-\cos\alpha} \;=\; \frac{1-x''}{1+x''} \;=\; \frac{\tan^2\frac{\beta}{2}}{\cot^2\frac{\gamma}{2}} .$$

Consequently, $\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)\tan\left(\frac{\gamma}{2}\right) \;=\; 1$ as desired.

II. *Outline of the solution by J. Chris Fisher, University of Regina, Sakatchewan.*

In the half-plane model of the hyperbolic plane (with BC as the line at infinity), the curvilinear triangle of the problem becomes the trebly asymptotic triangle ABC of hyperbolic geometry. This means that each side of the triangle is parallel to the other two, so that the three sides meet in pairs at the improper points A, B, and C. We can define a *cevian* of an asymptotic triangle to be a halfline from a point on a side to the opposite improper vertex. Our problem (extended to include the converse) requires proving a hyperbolic version of Ceva's theorem for asymptotic triangles; specifically,

The three cevians of a trebly asymptotic triangle intersect in a point if and only if the equally oriented angles α, β, γ that the sides make with their cevians satisfy

$$\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)\tan\left(\frac{\gamma}{2}\right) \;=\; 1 .$$

Our result makes use of Lobachevsky's formula for the relationship between the acute angles U and V of a right triangle, namely

$$\cos U \;=\; \frac{\sin V}{\sin \Pi(u)}\,,$$

or equivalently,

$$\tan\frac{U}{2} \;=\; \sqrt{\frac{\sin\Pi(u) - \sin V}{\sin\Pi(u) + \sin V}}\,.$$

Here, $\Pi(u)$ denotes the angle of parallelism that corresponds to the length u of the side opposite the angle U. Assume that the cevians all intersect at the point P. Denote the feet of the perpendiculars from P to the sides of the given asymptotic triangle by A^*, B^*, C^*, and let the distances be $a = PA^*$, $b = PB^*$, and $c = PC^*$. We now have three right triangles. In the right triangle $PA'A^*$, for example, the angle at A' is either α or its supplement (whichever is acute), and the side opposite has length a. To compute $\tan\frac{\alpha}{2}$, it suffices to show that the angle at P is $\left|\Pi(b) - \Pi(c)\right|$. This follows quickly from the fact that the angles formed by the half lines that meet at P are $2\Pi(a)$, $2\Pi(b)$, and $2\Pi(c)$ (and consequently, $\Pi(a) + \Pi(b) + \Pi(c) \;=\; \pi$). Thus, for the first triangle we have

$$\tan\frac{\alpha}{2} \;=\; \sqrt{\frac{\sin\Pi(a) \pm \sin(\Pi(b) - \Pi(c))}{\sin\Pi(a) \mp \sin(\Pi(b) - \Pi(c))}}\,.$$

Multiplying this by the similar equations for the other two tangents produces an expression that reduces to 1.

No other solutions were submitted.

2588. [2000 : 431] *Proposed by Niels Bejlegaard, Copenhagen, Denmark.*

Each positive whole integer a_k $(1 \leq k \leq n)$ is less than a given positive integer N. The least common multiple of any two of the numbers a_k is greater than N.

(a) Show that $\displaystyle\sum_{k=1}^{n}\frac{1}{a_k} \;<\; 2$.

(b)* Show that $\displaystyle\sum_{k=1}^{n}\frac{1}{a_k} \;<\; \frac{6}{5}$.

(c)* Find the smallest real number γ such that $\displaystyle\sum_{k=1}^{n}\frac{1}{a_k} \;<\; \gamma$.

Editor's comment.

There was one solution to part (a) submitted, by Michel Bataille, Rouen, France. Pierre Bornsztein, Pontoise, France, and Walther Janous, Innsbruck, Austria, both pointed out that part (a) was proposed by Paul Erdős as Problem 4365 in the American Mathematical Monthly, Vol. 56 (1949). Part (b) was settled by R. S. Lehman in the same publication, Vol. 58 (1951), pp. 345–346. Janous also noted that part (c) is entirely solved. Indeed, $\sum_{k=1}^{n} \frac{1}{a_k} \leq \frac{31}{32}$, with $\frac{31}{32}$ best possible. This was proved in A. Schinzel et G. Szeueres, Sur un problème de M. Paul Erdős, Acta Sci Math. (Szeged) Vol. 20 (1959), pp. 221–229. Finally, Janous noted that several further results "in the vicinity" of the Erdős one can be found in D. S. Mitrinović and M. S. Popadić, Inequalities in Number Theory, Naučni Podmladak, Nič 1978, Chapter 2, pp. 26–37.

2589. [2000 : 497] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

For $n = 2, 3, \ldots$, evaluate $\sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{k}\binom{n}{k-1}$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let S_n denote the given sum. Note first that $S_n = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k}\binom{n}{n+1-k}$.

Since $(1+x)^{2n} = (1+x)^n(1+x)^n = \left(\sum_{i=0}^{n} \binom{n}{i} x^i\right) \left(\sum_{j=0}^{n} \binom{n}{j} x^j\right)$,

we have, by comparing the coefficients of the term x^{n+1}, that

$$\binom{2n}{n+1} = \sum_{k=1}^{n} \binom{n}{k}\binom{n}{n+1-k}. \qquad (1)$$

If $n = 2m$ is even, where $m \geq 1$, then by letting $j = n - k + 1$ we have

$$\sum_{k=m+1}^{n} \binom{n}{k}\binom{n}{n+1-k} = \sum_{j=1}^{m} \binom{n}{n+1-j}\binom{n}{j}. \qquad (2)$$

From (1) and (2) we then get

$$\begin{aligned}\binom{2n}{n+1} &= \sum_{k=1}^{m}\binom{n}{k}\binom{n}{n+1-k}+\sum_{k=m+1}^{n}\binom{n}{k}\binom{n}{n+1-k}\\ &= 2\sum_{k=1}^{m}\binom{n}{k}\binom{n}{n+1-k},\end{aligned}$$

whence $S_n = \dfrac{1}{2}\dbinom{2n}{n+1}$.

If $n = 2m+1$ is odd, where $m \geq 0$, then the same change of index $j = n-k+1$ would yield

$$\sum_{k=m+2}^{n}\binom{n}{k}\binom{n}{n+1-k} = \sum_{j=1}^{m}\binom{n}{n+1-j}\binom{n}{j}. \qquad (3)$$

From (1) and (3) we then get

$$\begin{aligned}\binom{2n}{n+1} &= 2\sum_{k=1}^{m}\binom{n}{k}\binom{n}{n+1-k} + \binom{n}{m+1}\binom{n}{n-m}\\ &= 2\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{k}\binom{n}{n+1-k} + \binom{n}{m+1}^2,\\ \text{whence } S_n &= \frac{1}{2}\left(\binom{2n}{n+1}-\binom{n}{\frac{n+1}{2}}^2\right).\end{aligned}$$

Taking both cases into account, S_n can be expressed as follows:

$$S_n = \frac{1}{2}\left(\binom{2n}{n+1}-\frac{1-(-1)^n}{2}\binom{n}{\frac{n+1}{2}}^2\right).$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OLEG IVRII, (grade 8) student, Cummer Valley Middle School, North York, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; HENRY LIU, student, Trinity College, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; WILLIAM MOSER, McGill University, Quebec; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; CHRIS WILDHAGEN, Rotterdam, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and proposer.

All the submitted solutions are either very similar to, or are minor variations of, the one given above, though many solvers made claims, without any justifications, like "$S_n = \frac{1}{2}\sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1}$, if n is even." The identity in (1) above is of course just a special case of the Vandermonde's Convolution Formula. This was pointed out by Bradley and

Seiffert (and used directly in their proofs). Lewis, Moser, and Zhou all pointed out that this identity can also be established by a simple and well-known combinatorial argument. For the binomial coefficient $\binom{n}{\frac{n(n+1)}{2}}$ given in the solution above (for the case when n is odd), three different, but equivalent forms were given by various solvers: namely, $\binom{n}{\frac{n-1}{2}}$, $\binom{n}{\lfloor\frac{n}{2}\rfloor}$ and $\binom{n}{\lfloor\frac{n}{2}\rfloor+1}$.

2590. [2000 : 497] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

For $n = 1, 2, \ldots$, prove that $\displaystyle\prod_{k=1}^{n}\binom{n}{k}^2 \leq \left(\frac{1}{n+1}\binom{2n}{n}\right)^n$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

By the solution to problem #2589 [Ed: see equation (1) of the published solution to #2589 above] and the AM–GM inequality, we get that

$$\begin{aligned}\frac{1}{n+1}\binom{2n}{n} &= \frac{1}{n}\binom{2n}{n+1} = \frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1}\\ &\geq \left[\prod_{k=1}^{n}\binom{n}{k}\binom{n}{k-1}\right]^{\frac{1}{n}} = \left[\prod_{k=1}^{n}\binom{n}{k}^2\right]^{\frac{1}{n}}\end{aligned}$$

which completes the proof.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Trinity College, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; STEFFEN WEBER, Merseburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Besides Zhou, essentially the same solution based on the proof of #2589 was also given by Arslanagić, Bataille, Loeffler, and the proposer. It is easy to see that equality holds if and only if $n = 1$ or 2. This was pointed out by Arslanagić, Diminnie, and Loeffler.

2591. [2000 : 497] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

Two players, A and B, each toss n fair coins, and two other players, C and D, toss $n-1$ and $n+1$ fair coins, respectively.

For each $n = 2, 3, \ldots$, prove that the two events:

A gets exactly one head more than B

and

C and D get exactly the same number of heads

are equally likely.

Find the probability of these events.

Solution by Gerry Leversha, St. Paul's School, London, England.
The probability that A gets exactly one head more than B is

$$\frac{1}{2^{2n}}\sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1} = \frac{1}{2^{2n}}\sum_{k=1}^{n}\binom{n}{k}\binom{n}{n-k+1} = \frac{1}{2^{2n}}\binom{2n}{n+1},$$

where the second equality is obtained by considering the coefficient of the term x^{n+1} on both sides of the identity $(1+x)^n(1+x)^n \equiv (1+x)^{2n}$.

The probability that C and D get exactly the same number of heads is

$$\begin{aligned}\frac{1}{2^{2n}}\sum_{k=1}^{n-1}\binom{n-1}{k}\binom{n+1}{k} &= \frac{1}{2^{2n}}\sum_{k=1}^{n-1}\binom{n-1}{k}\binom{n+1}{n+1-k} \\ &= \frac{1}{2^{2n}}\binom{2n}{n+1},\end{aligned}$$

where the second equality is obtained by considering the coefficient of the term x^{n+1} on both sides of the identity $(1+x)^{n-1}(1+x)^{n+1} \equiv (1+x)^{2n}$.

The events are thus equally likely.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; OLEG IVRII, (grade 8) student, Cummer Valley Middle School, North York, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; HENRY LIU, Trinity College, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2592. [2000 : 498] *Proposed by Nairi M. Sedrakyan, Yerevan, Armenia.*

Describe all numbers, which can be represented in the form of $\dfrac{a^3+b^3}{c^3+d^3}$, where a, b, c, d are natural numbers.

Amalgamated solutions of Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina and the proposer.

We first claim that every rational number from the interval $(1, 2)$ can be represented in the form $\dfrac{a^3+b^3}{a^3+d^3}$. Indeed, let $\dfrac{m}{n} \in (1, 2)$, where m and n are natural numbers. We will choose a, b, d such that $b \neq d$ and $a^2 - ab + b^2 = a^2 - ad + d^2$; that is $b + d = a$. In that case

$$\frac{a^3 + b^3}{a^3 + d^3} = \frac{a + b}{a + d} = \frac{a + b}{2a - b}.$$

Taking now $a+b = 3m$ and $2a-b = 3n$ (that is, $a = m+n$ and $b = 2m-n$) the claim is proved.

Now we will prove that any positive rational number $r > 0$ can be represented in the given form. Let $r > 0$ be any positive rational number. Select positive integers p and q such that

$$1 < \frac{p^3}{q^3}r < 2.$$

From the above, there exist positive integers a, b, d such that

$$\frac{p^3}{q^3}r = \frac{a^3 + b^3}{a^3 + d^3}.$$

Hence,

$$r = \frac{(aq)^3 + (bq)^3}{(ap)^3 + (dp)^3}.$$

Also solved by MOHAMMED AASSILA, Strasbourg, France; PIERRE BORNSZTEIN, Pontoise, France; and the SOUTHWEST MISSOURI STATE PROBLEM SOLVING GROUP. There was one incorrect solution.

2593. [2000 : 498] *Proposed by Nairi M. Sedrakyan, Yerevan, Armenia.*

Let $S(a)$ denote the sum of the digits of the natural number a. Let k and n be natural numbers with $(n, 3) = 1$. Prove that there exists a natural number m which is divisible by n and $S(m) = k$ if either

(a) $k > n - 2$; or

(b) $k > S^2(n) + 7S(n) - 9$.

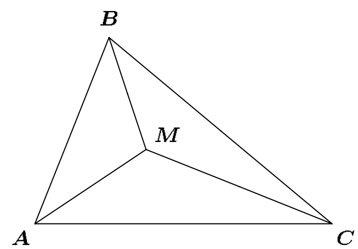
Editor's comment.

There have been no solutions submitted except that of the proposer. The editor has had some difficulty easily reading it, the problem is non-trivial, and the proposer's solution is difficult to follow. Since there were no other solutions submitted, we are keeping the problem open in the hope that you, our readers, will find a simpler solution.

2594. [2000 : 498] *Proposed by Nairi M. Sedrakyan, Yerevan, Armenia.*

Given a point M inside the triangle ABC (see diagram), prove that

$$\min(MA,\, MB,\, MC) + MA + MB + MC \;<\; AB + BC + AC\,.$$



Editor's Note: A number of correspondents noted that this is a known problem that has appeared in a number of places. Christopher J. Bradley, Clifton College, Bristol, UK, recalls that the problem was submitted to the IMO in Bucharest, but was not selected by the jury. Walther Janous, Ursulinengymnasium, Innsbruck, Austria, notes further that this was used as a problem at the Romanian Olympiad, and provides a reference to a recent article which uses complex numbers to give a proof (New Geometrical Inequality for Interior Point of a Triangle; M. Diuca and M. Bencze; Octogon, vol. 9, No. 1, 2001, pp. 437–440). Finally, as was pointed out by David Loeffler, student, Cotham School, Bristol, UK, this was Problem 2 in the final selection round of the 2000 British Mathematics Olympiad.

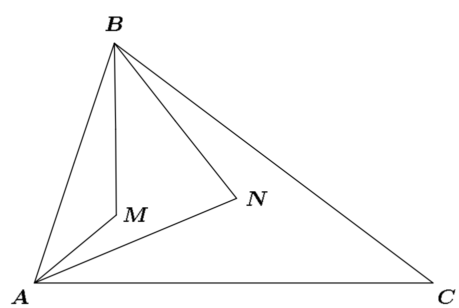
We refer any interested readers to these sources for a solution.

Solutions were submitted by MOHAMMED AASSILA, Strasbourg, France; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; C. FESTRAETS-HAMOIR, Brussels, Belgium; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2595. [2000 : 498] *Proposed by Nairi M. Sedrakyan, Yerevan, Armenia.*

Given that M and N are points inside the triangle ABC such that $\angle MAB = \angle NAC$ and $\angle MBA = \angle NBC$, prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} \;=\; 1\,.$$



Solution by Pierre Bornsztein, Pontoise, France.

Let K be the point on the half-line from B towards N such that $\angle BCK = \angle BMA$. Since M is inside $\triangle ABC$, we have $\angle BMA > \angle BCA$, which implies that K is outside the triangle. Moreover, we have that $\angle MBA = \angle NBC = \angle KBC$. It follows that $\triangle ABM$ and $\triangle KBC$ are similar. Thus,

$$\frac{AB}{KB} = \frac{BM}{BC} = \frac{AM}{KC}. \tag{1}$$

Since N is inside $\triangle ABC$ we have

$$\begin{aligned}\angle ABK &= \angle ABN = \angle ABC - \angle NBC \\ &= \angle ABC - \angle MBA \\ &= \angle MBC;\end{aligned}$$

together with $\dfrac{AB}{KB} = \dfrac{BM}{BC}$ this implies that $\triangle ABK$ is similar to $\triangle MBC$ so that

$$\frac{AB}{MB} = \frac{BK}{BC} = \frac{AK}{MC}. \tag{2}$$

Since in the first pair of similar triangles $\angle BKC = \angle BAM$, we have $\angle NKC = \angle BKC = \angle MAB = \angle NAC$. It follows that $AKCN$ is cyclic, so that Ptolemy's Theorem leads to $AC \cdot NK = AK \cdot NC + AN \cdot CK$; that is,

$$AC \cdot (BK - BN) = AK \cdot NC + AN \cdot CK. \tag{3}$$

From (1) and (2) we have

$$AK = \frac{AB \cdot MC}{BM}, \quad BK = \frac{AB \cdot BC}{BM}, \quad CK = \frac{AM \cdot BC}{BM}.$$

Substituting these expressions in (3), and multiplying through by $\dfrac{BM}{AB \cdot BC \cdot CA}$, we obtain

$$1 - \frac{BM \cdot BN}{BA \cdot BC} = \frac{AM \cdot AN}{AB \cdot AC} + \frac{CM \cdot CN}{CA \cdot CB}.$$

The desired conclusion follows immediately.

Remark. This problem was proposed but not used by the jury at the IMO in 1998.

Also solved by MOHAMMED AASSILA, Strasbourg, France; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HENRY LIU, student, Cambridge, England; TOSHIO SEIMIYA, Kawasaki, Japan; and by the proposer.

Arslanagić and the proposer both provided solutions that were essentially identical to the featured solution. Two others exploited the fact that M and N are isogonal conjugates with respect to $\triangle ABC$, but this observation did not provide any substantial advantage.

2596. [2000 : 498] *Proposed by Clark Kimberling, University of Evansville, Evansville, IN, USA.*

Write $r \prec s$ if there is an integer k satisfying $r < k < s$. Find, as a function of n ($n \geq 2$) the least positive integer k satisfying

$$\frac{k}{n} \prec \frac{k}{n-1} \prec \frac{k}{n-2} \prec \ldots \prec \frac{k}{2} \prec k\,.$$

This problem has actually appeared before! We printed it as problem 2272★ [1997 : 365]. At that time, the editor had a piece of paper without the name of the proposer (and no solution). We printed a solution in [1998 : 438]. Thanks to Michel Bataille for pointing this out. Now we know that the "anonymous" proposer is Clark Kimberling!

2597. [2000 : 499] *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Let P be an arbitrary interior point of an equilateral triangle ABC. Prove that $|\angle PBC - \angle PCB| \;\leq\; \arcsin\left(2\sin\left(\frac{|\angle PAB - \angle PAC|}{2}\right)\right) - \left(\frac{|\angle PAB - \angle PAC|}{2}\right) \;\leq\; |\angle PAB - \angle PAC|$.

Show that the left inequality cannot be improved in the sense that there is a position Q of P on the ray AP giving an equality.

(Thus, the inequality in **2255** is improved.)

Editor's Note: No solutions (other than the proposer's) have been received. We await the answer to this challenge from you, our readers!

2598. [2000 : 499, 2001 : 267] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Suppose that AD, BE and CF are the internal angle bisectors of $\triangle ABC$, with D on BC, E on CA and F on AB. Write $a = BC$, $b = CA$, $c = AB$, $x = AE$ and $y = AF$. We are given that $x + y = a$. Prove that:

(a) $a^2 = bc$;

(b) $\frac{1}{x} - \frac{1}{y} \;=\; \frac{1}{b} - \frac{1}{c}$;

(c) $\frac{1}{x} + \frac{1}{y} \;=\; \left(\frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}\right)^2$;

(d) $AD < a$.

Solution de C. Festraets-Hamoir, Brussels, Belgium.

$$\frac{x}{b-x} = \frac{c}{a} \iff ax = bc - cx \iff x = \frac{bc}{a+c}.$$

$$\frac{y}{c-y} = \frac{b}{a} \iff ay = bc - by \iff y = \frac{bc}{a+b}.$$

(a)
$$x + y = a = \frac{bc}{a+c} + \frac{bc}{a+b}.$$

$$\begin{aligned} a^3 + a^2b + a^2c + abc &= bca + b^2c + bca + bc^2\,, \\ a^2(a+b+c) &= bc(a+b+c)\,, \\ a^2 &= bc\,. \end{aligned}$$

(b)
$$\frac{1}{x} = \frac{a+c}{bc} \quad \text{et} \quad \frac{1}{y} = \frac{a+b}{bc}.$$

$$\frac{1}{x} - \frac{1}{y} = \frac{c-b}{bc} = \frac{1}{b} - \frac{1}{c}\,.$$

(c)
$$\frac{1}{x} + \frac{1}{y} = \frac{2a+b+c}{bc} = \frac{2\sqrt{bc}+b+c}{bc} = \left(\frac{\sqrt{b}+\sqrt{c}}{\sqrt{bc}}\right)^2 = \left(\frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}\right)^2\,.$$

(d)

$$\begin{aligned} AD &= \frac{2bc}{b+c}\cos\left(\tfrac{A}{2}\right) \\ &\le \sqrt{bc}\cos\left(\tfrac{A}{2}\right) && \text{(MH<MG)} \\ &< \sqrt{bc} = a && \text{car } \cos\left(\tfrac{A}{2}\right) < 1. \end{aligned}$$

Parts (a), (b) *and* (c) *were also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Trinity College, Cambridge, UK; DAVID LOEFFLER; student, Cotham School, Bristol, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; PANOS E. TSAOUSSOGLOU,* *Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.*

All solvers realized that (d) *was incorrect as originally printed. Some gave actual counter-examples. Bataille, Bornsztein, Bradley, Engelhaupt, Leversha, Liu, Loeffler, Seimiya and Zhou all proved the correct version.*

The editor feels that this is, by far, the most minimally French solution ever to have been submitted in the history of ***CRUX with MAYHEM****!*

2599. [2000 : 499] *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Let P be a point inside the triangle ABC and let AP, BP, CP meet the sides BC, CA, AB at L, M, N, respectively. Show that the following two conditions are equivalent:

$$\frac{1}{AP}+\frac{1}{PL} = \frac{1}{BP}+\frac{1}{PM} = \frac{1}{CP}+\frac{1}{PN};$$

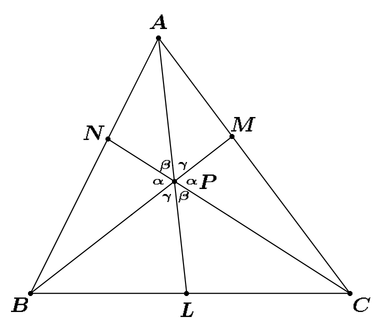
$$\angle APN = \angle NPB = \angle BPL = \angle LPC = \angle CPM = \angle MPA = 60^\circ.$$

Solution by Toshio Seimiya, Kawasaki, Japan.

Let $\angle BPN = \angle CPM = \alpha$, $\angle CPL = \angle APN = \beta$ and $\angle APM = \angle BPL = \gamma$. Then $\alpha+\beta+\gamma = \pi$.

Let $[F]$ denote the area of a plane figure F.

Since $[PBC] = [PBL] + [PCL]$, we have



$$\frac{1}{2}PB\cdot PC\sin(\beta+\gamma) = \frac{1}{2}PB\cdot PL\sin\gamma + \frac{1}{2}PC\cdot PL\sin\beta.$$

Dividing both sides by $\frac{PB\cdot PC\cdot PL}{2}$, we obtain

$$\frac{\sin(\beta+\gamma)}{PL} = \frac{\sin\gamma}{PC}+\frac{\sin\beta}{PB}.$$

Since $\alpha+\beta+\gamma=\pi$, we have $\sin(\beta+\gamma)=\sin\alpha$, and we can rewrite the above equality as

$$\frac{\sin\alpha}{PL} = \frac{\sin\gamma}{PC}+\frac{\sin\beta}{PB}.$$

Adding $(\sin\alpha/AP)$ to both sides, we get

$$\left(\frac{1}{AP}+\frac{1}{PL}\right)\sin\alpha \;=\; \frac{\sin\alpha}{AP}+\frac{\sin\beta}{BP}+\frac{\sin\gamma}{CP}\,.$$

Similarly,

$$\left(\frac{1}{BP}+\frac{1}{PM}\right)\sin\beta \;=\; \frac{\sin\alpha}{AP}+\frac{\sin\beta}{BP}+\frac{\sin\gamma}{CP}$$

and

$$\left(\frac{1}{CP}+\frac{1}{PN}\right)\sin\gamma \;=\; \frac{\sin\alpha}{AP}+\frac{\sin\beta}{BP}+\frac{\sin\gamma}{CP}\,.$$

Thus,

$$\left(\frac{1}{AP}+\frac{1}{PL}\right)\sin\alpha \;=\; \left(\frac{1}{BP}+\frac{1}{PM}\right)\sin\beta \;=\; \left(\frac{1}{CP}+\frac{1}{PN}\right)\sin\gamma\,.$$

Therefore,

$$\begin{aligned}
\frac{1}{AP}+\frac{1}{PL} \;=\; \frac{1}{BP}+\frac{1}{PM} \;&=\; \frac{1}{CP}+\frac{1}{PN}\\
\Longleftrightarrow\quad & \sin\alpha \;=\; \sin\beta \;=\; \sin\gamma\\
\Longleftrightarrow\quad & \alpha \;=\; \beta \;=\; \gamma\\
\Longleftrightarrow\quad & \angle APM \;=\; \angle BPL \;=\; \angle BPN \;=\; \angle CPM\\
& \qquad\qquad =\; \angle CPL \;=\; \angle APN\,.
\end{aligned}$$

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Cotham School, Bristol, UK; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

All solvers noticed the typo in the original statement of the problem ($1/PN$ was printed $1/CN$) and then solved the problem with the correct condition.

2600. [2000 : 499] *Proposed by Svetlozar Doichev, Stara Zagora, Bulgaria.*

Find all real numbers x such that, if a and b are the lengths of the sides of a triangle with medians from the mid-points of these sides of lengths m_a and m_b, respectively, then the equalities $a+xm_a=b+xm_b$ and $a=b$ are equivalent.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA (adapted by the editor).

Clearly, if $a=b$, then $m_a=m_b$, so that $a=b$ implies $a+xm_a=b+xm_b$ for all real x. Thus, $a+xm_a=b+xm_b$ and $a=b$ are equivalent if and only if $a+xm_a=b+xm_b$ implies $a=b$.

By the Cosine Law,

$$m_a^2 \;=\; b^2 + \frac{a^2}{4} - ba\cos C$$

and

$$m_b^2 \;=\; a^2 + \frac{b^2}{4} - ab\cos C\,,$$

so that

$$m_a^2 - m_b^2 \;=\; \frac{3}{4}(b^2 - a^2)\,.$$

Now,

$$\begin{aligned} a + xm_a \;&=\; b + xm_b \\ \iff\quad a - b \;&=\; x(m_b - m_a) \\ \iff\quad (a-b)(m_b + m_a) \;&=\; x(m_b^2 - m_a^2) \\ \iff\quad (a-b)(m_b + m_a) \;&=\; \frac{3}{4}x(a^2 - b^2) \\ \iff\quad (a-b)\left(m_a + m_b - \frac{3}{4}x(a+b)\right) \;&=\; 0 \\ \iff\quad (a-b)\left(x - \frac{4(m_a+m_b)}{3(a+b)}\right) \;&=\; 0\,. \end{aligned} \tag{1}$$

By the Triangle Inequality,

$$b - \frac{a}{2} \;<\; m_a \;<\; b + \frac{a}{2}$$

and

$$a - \frac{b}{2} \;<\; m_b \;<\; a + \frac{b}{2}\,.$$

Hence,

$$\frac{2}{3} \;<\; \frac{4(m_a+m_b)}{3(a+b)} \;<\; 2\,.$$

We next show that the ratio $4(m_a+m_b)/(3(a+b))$ can take any value in the interval $(2/3, 2)$. Consider the right triangle $(a,b,c) = (t, \sqrt{t^2+1}, 1)$. We have $m_a = (1/2)\sqrt{t^2+4}$ and $m_b = (1/2)\sqrt{t^2+1}$, so that

$$\frac{4(m_a+m_b)}{3(a+b)} \;=\; \frac{2(\sqrt{t^2+4} + \sqrt{t^2+1})}{3(t + \sqrt{t^2+1})}\,.$$

Let $f(t)$ denote the expression on the right side of the last equality. The function $f(t)$ is continuous for $t > 0$, $\lim_{t\to 0^+} f(t) = 2$ and $\lim_{t\to +\infty} f(t) = 2/3$. Hence the range of the function $f(t)$ is $(2/3, 2)$.

Therefore, the equality (1) implies $a = b$ if and only if

$$x \in (-\infty, 2/3] \cup [2, +\infty) .$$

Also solved by MICHEL BATAILLE, Rouen, France; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. There was one incorrect solution submitted.