PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (⋆) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}$"×11" or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 April 2002**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in LaTeX format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

We regret to have to admit that problem 2647 is a repeat of 2611. It must be a good problem if we should want to use it twice!

2689. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Given $\triangle ABC$ and a point P not on it, draw $BD \| AC$ such that D lies on AP, draw $AE \| CB$ such that E lies on CP, and draw $CF \| AB$ such that F lies on BP. Let X be the point of intersection of the lines AB and CD, let Y be the point of intersection of the lines AC and BE, and let Z be the point of intersection of the lines BC and AF. Prove that X, Y and Z are collinear.

2690. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let $\triangle ABC$ be such that $\angle A$ is the largest angle. Let r be the inradius and R the circumradius. Prove that

$$A \gtreqless 90^\circ \iff R + r \gtreqless \frac{b+c}{2}\,.$$

2691. *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

The length of one base of an isosceles trapezoid, the equal sides, and the equal diagonals, are all odd integers. Show that if the remaining base also has integer length, then it is divisible by 8.

2692. *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Let PQ be the distance between the mid-points of the diagonals of quadrilateral $ABCD$ with sides a, b, c, d and diagonals p and q. Give an example of such a quadrilateral where a, b, c, d, p, q and PQ are all positive integers.

2693. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given triangle ABC and a point P, the line through P parallel to BC, intersects AC, AB at Y_1, Z_1 respectively. Similarly, the parallel to CA intersects BC, AB at X_2, Z_2, and the parallel to AB intersects BC, AC at X_3, Y_3. Locate the point P for which the sum $Y_1P \cdot PZ_1 + Z_2P \cdot PX_2 + X_3P \cdot PY_3$ of products of *signed* lengths is maximal.

2694. *Proposed by Aaron Lee and Jason Wilson, students, Biola University, La Mirada, CA, USA.*

Given a line segment AB, construct a square $ABCD$ using four or fewer circular arcs and a straightedge. The construction should use fewer arcs than those usually given in texts.

2695. *Proposed by Aaron Lee and Jason Wilson, students, Biola University, La Mirada, CA, USA.*

Given a line ℓ and a point P not on it, construct the line through P parallel to ℓ, using two or fewer circular arcs and a straightedge. The construction should use fewer arcs than those usually given in texts.

2696. *Proposed by Aaron Lee and Jason Wilson, students, Biola University, La Mirada, CA, USA.*

Using only a straightedge, construct the tangents from a point outside a given circle (and its centre).

2697. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Find a closed form for $\sum_{k=1}^{n} k \sin^2(kx)$.

2698. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands (adapted by the editor).*

The perimeter of a right triangle with integer sides is a perfect square. The area of the triangle is the cube of an integer. Find the smallest triangle satisfying these conditions.

[Ed. Smeenk asked the case when the hypotenuse has length 240.]

2699. *Proposed by Maureen P. Cox and Albert White, St. Bonaventure University, St. Bonaventure, NY, USA.*

Evaluate $$\prod_{k=1}^{n}\left(\frac{4k+4n-3}{4n}\right)^{\frac{4n+4k-3}{4n^2}}.$$

2700. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Let n be a positive integer. Show that

$$\sum_{k=1}^{n}\frac{k}{n+k}\binom{n}{k} \;<\; \sum_{k=1}^{n}\binom{n}{k}\log\left(\frac{n+k}{n}\right) \;<\; 2^{n-1}.$$

[Ed. "log" is, of course, the natural logarithm.]

2700A. *Proposed by Paul Bracken, CRM, Université de Montréal, Montréal, Québec.*

Show that the function e^{-xn^2} can be written in the following form,

$$\begin{aligned} e^{-xn^2} &= \sum_{k=0}^{n-1}(-1)^k\frac{x^k n^{2k}}{k!}+(-1)^n\frac{x^n n^{2n}}{n!}\phi_x(n)\,, \quad \text{where} \\ \phi_x(n) &= 1-\int_0^{xn^2} e^{-t}\left(1-\frac{t}{xn^2}\right)^n dt\,. \end{aligned}$$

Determine the leading large n behaviour of $\phi_x(n)$, and show that

$$\lim_{n\to\infty} n\phi_x(n) \;=\; 1/x\,.$$

The solution of problems is one of the lowest forms of mathematical research, ... yet its educational value cannot be overestimated. It is the ladder by which the mind ascends into higher fields of original research and investigation. Many dormant minds have been aroused into activity through the mastery of a single problem.

Benjamin Franklin Finkel, in "The American Mathematical Monthly", no. 1.