A Couple of Pretty Mean Questions

Clifford Wagner

The general public uses the word average for a concept that mathematicians prefer to call the arithmetic mean. Fortunately, mathematicians bring more than jargon to a discussion of averages. This note discusses two questions about arithmetic means. See how well you can answer them before reading the hints and answers.

Question 1: Suppose the Internal Revenue Service reports that last year's returns with taxable income had an average adjusted gross income (AGI) of $46,000 and an average effective tax rate of 15 percent. The average effective tax rate was obtained by taking each return's tax as a percentage of AGI and averaging these percentages. What can one say about the average tax paid per taxable return? Clearly state any assumptions.

Hint 1: The average tax paid is not necessarily $6,900 (15 percent of $46,000). If necessary, let the number of tax returns be $n = 3$ and create some examples that match the question.

Hint 2: Let us suppose that the various AGIs can be sorted to create a sequence, $a_1 \leq a_2 \leq \ldots \leq a_n$, with corresponding tax rates, $b_1 \leq b_2 \leq \ldots \leq b_n$. Although the tax code is said to be progressive, this pairing of ordered sequences must be considered an assumption.

Hint 3: One can use the Chebyshev Inequality for Arithmetic Means [Mitrinović, 1970], which states that given two increasing sequences $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$, with, $a_1 \leq a_2 \leq \ldots \leq a_n$ and $b_1 \leq b_2 \leq \ldots \leq b_n$, the mean of the product sequence $\{a_1 b_1, a_2 b_2, \ldots, a_n b_n\}$ is at least as great as the product of the means of the two given sequences. That is,

$$\frac{1}{n}\sum_{i=1}^{n} a_i b_i \;\geq\; \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right) .$$

This inequality is not to be confused with the other well-known Chebyshev Inequality concerning the variance of a random variable.

Answer to Question 1: By the Chebyshev Inequality, the mean tax is **at least** $6,900.

Note: [Mitrinović, 1970] shows that Chebyshev's Inequality is equivalent to the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n} (a_i - a_j)(b_i - b_j) \;\geq\; 0 .$$

Thus, when both sequences are increasing (or both decreasing), the observation

$$(a_i - a_j)(b_i - b_j) \ \geq 0 \ \text{ for all } i,\, j = 1, \ldots, n$$

immediately leads to Chebyshev's Inequality, and also shows that equality occurs if and only if at least one sequence is constant.

Question 2: Suppose I surveyed prices for regular grade gasoline at three competing gas stations on a summer weekend. I observed that the average posted price was \$1.50 per gallon, and I determined that the average sales volume for regular gasoline was $5,000$ gallons per station. What can one say about the average revenue from regular gasoline at these three stations on that particular weekend? Clearly state any assumptions.

Hint 1: The average revenue is not necessarily \$7,500 ($5,000$ gallons at \$1.50 per gallon). If necessary, create some examples that match the question.

Hint 2: Assume that the various sales volumes can be sorted so as to constitute a sequence, $a_1 \leq a_2 \leq a_3$, with corresponding prices $b_1 \geq b_2 \geq b_3$. The assumption here is that sales volume and price charged are inversely related.

Hint 3: Use the previous note regarding the Chebyshev Inequality to recognize that when one sequence is increasing and the other decreasing we have

$$(a_i - a_j)(b_i - b_j) \ \leq \ 0 \ \text{ for all } i,\, j = 1, \ldots, n\,,$$

and this causes a reversal of the sign in the Chebyshev Inequality.

Answer to Question 2: By the reversed Chebyshev Inequality, the mean revenue is **at most** \$7,500.

Reference

1. Mitrinović, D.S., *Analytic Inequalities*, Springer-Verlag, 1970, pp. 36–37.

Clifford H. Wagner
Mathematics and Computer Science
Penn State University at Harrisburg
Middletown, PA 17057-4898, USA
w44@mail.psu.edu