Summation of Finite Series of Integers

C.-S. Lin

If someone asks me how to verify the equality

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$

immediately I would say it is easy by the Principle of Mathematical Induction, and that is true. If I am further asked how to get the equality in the first place, I would be probably hesitative for a while and I might fail to answer, unless I already knew a method. In fact, some strikingly original algebraic proofs are due to Archimedes and Fibonacci [1, p. 104 and 102]. In numerical analysis we use factorial polynomials and the telescoping method [5, Chap. 17], and a graphical expression (not a proof, though) can be found in [4, p. 77]. Mathematical induction is a great common tool used for checking an equality like the one above. But it is imperfect in the sense that we have to know the equality in question beforehand. In this article, motivated by the method of proof of the equality above due to Chorlton [2, p. 305], we shall use finite sums of sine and cosine functions to produce several types of formulas involving summations of finite series of integers by way of differentiation. The next result is our basic tool.

Lemma. For any integer $n \geq 1$, the following sine-series and cosine-series hold.

(1) $$\sin x + \sin 2x + \cdots + \sin(n-1)x + \sin nx = \frac{\cos\frac{1}{2}x - \cos\left(n+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}.$$

(2) $$\cos x + \cos 2x + \cdots + \cos(n-1)x + \cos nx = \frac{\sin\left(n+\frac{1}{2}\right)x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}.$$

(3) $$\sin x - \sin 2x + \sin 3x - \cdots \mp \sin(n-1)x \pm \sin nx = \frac{\sin\frac{1}{2}x \pm \sin\left(n+\frac{1}{2}\right)x}{2\cos\frac{1}{2}x}.$$

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The upper and lower signs depend on whether n is odd or even, respectively.

(4) $$\cos x - \cos 2x + \cos 3x - \cdots \mp \cos(n-1)x \pm \cos nx = \frac{\cos\frac{1}{2}x \pm \cos\left(n+\frac{1}{2}\right)x}{2\cos\frac{1}{2}x}.$$

The upper and lower signs depend on whether n is odd or even, respectively.

Proof. First, we note that the technique of the proof of each equality is the same and is known [3, p. 289], but let us prove it for the sake of completeness.

(1) $$[\cos\tfrac{1}{2}x - \cos\tfrac{3}{2}x] + [\cos\tfrac{3}{2}x - \cos\tfrac{5}{2}x] + \cdots + [\cos\left(n-\tfrac{1}{2}\right)x - \cos\left(n+\tfrac{1}{2}\right)x] = 2\sin\tfrac{1}{2}x\,[\sin x + \sin 2x + \cdots + \sin(n-1)x + \sin nx],$$

and the required equality follows easily.

(2) was proved in [2] by the same method as above.

(3) Let n be odd first. Then

$$[\sin\tfrac{1}{2}x + \sin\tfrac{3}{2}x] - [\sin\tfrac{3}{2}x + \sin\tfrac{5}{2}x] + \cdots + [\sin(n-\tfrac{1}{2})x + \sin(n+\tfrac{1}{2})x] = 2\cos\tfrac{1}{2}x\,[\sin x - \sin 2x + \cdots - \sin(n-1)x + \sin nx],$$

and we have the case of odd n. The case of even n and equalities (4) are treated in a similar fashion, and we shall omit the details.

Proposition. For any integer $n \geq 1$ the following summations of integers hold.

(1) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, and $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$, etc.

(2) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, and $\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$, etc.

(3) $$\sum_{k=1}^{n}(-1)^{k+1}k = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd, and} \\ \frac{-n}{2} & \text{if } n \text{ is even; and} \end{cases}$$

$$\sum_{k=1}^{n}(-1)^{k+1}k^3 = \begin{cases} \frac{(n+1)^2(2n-1)}{4} & \text{if } n \text{ is odd, and} \\ \frac{-n^2(2n+3)}{4} & \text{if } n \text{ is even, etc.} \end{cases}$$

(4) $$\sum_{k=1}^{n}(-1)^{k+1}k^2 = \begin{cases} \frac{n(n+1)}{2} & \text{if } n \text{ is odd, and} \\ \frac{-n(n+1)}{2} & \text{if } n \text{ is even; and} \end{cases}$$

$$\sum_{k=1}^{n}(-1)^{k+1}k^4 = \begin{cases} \frac{n(n+1)(n^2+n-1)}{2} & \text{if } n \text{ is odd, and} \\ \frac{-n(n+1)(n^2+n-1)}{2} & \text{if } n \text{ is even, etc.} \end{cases}$$

Proof. (1) Let $f(x) = \sin x + \sin 2x + \cdots + \sin(n-1)x + \sin nx$. Then

$f'(x) = \cos x + 2\cos 2x + \cdots + (n-1)\cos(n-1)x + n\cos nx$;

$f''(x) = -\left[\sin x + 2^2 \sin 2x + \cdots + (n-1)^2 \sin(n-1)x + n^2 \sin nx\right]$;

and

$f'''(x) = -\left[\cos x + 2^3 \cos 2x + \cdots + (n-1)^3 \cos(n-1)x + n^3 \cos nx\right]$,

which yield $f(0) = 0$, $f'(0) = \sum_1^n k$, $f''(0) = 0$, and $f'''(0) = -\sum_1^n k^3$. Now, we have from (1) in the Lemma,

$$(\alpha) \qquad 2\left(\sin \tfrac{1}{2}x\right) f(x) = \cos \tfrac{1}{2}x - \cos\left(n + \tfrac{1}{2}\right)x.$$

Straightforward computation shows that the first equality in (1) follows by differentiating the identity (α) twice and substituting 0 for x. If it is done four times at $x = 0$, then we get the second equality in (1). The similar further process can be used for $\sum_1^n k^5$ and $\sum_1^n k^7$, etc.

(2) Let $g(x) = \cos x + \cos 2x + \cdots + \cos(n-1)x + \cos nx$. Then $g(0) = n$, $g'(0) = 0$, $g''(0) = -\sum_1^n k^2$, $g'''(0) = 0$, and $g^{(4)}(0) = \sum_{k=1}^{n} k^4$. We need the next equality, which is from (2) in the Lemma:

$$(\beta) \qquad 2\left(\sin \tfrac{1}{2}x\right) g(x) = \sin\left(n + \tfrac{1}{2}\right)x - \sin \tfrac{1}{2}x.$$

Repeated differentiation of the identity (β) three times, and letting $x = 0$, we get the first equality in (2). If it is done five times at $x = 0$, then we have the second equality in (2). The similar further process implies formulas for $\sum_{k=1}^{n} k^6$ and $\sum_{k=1}^{n} k^8$, etc.

(3) We use $h(x) = \sin x - \sin 2x + \sin 3x - \cdots \mp \sin(n-1)x \pm \sin nx$. Differentiate the identity obtained from (3) in the Lemma; that is,

$$(\gamma) \qquad 2\left(\cos \tfrac{1}{2}x\right) h(x) = \sin \tfrac{1}{2}x \pm \sin\left(n + \tfrac{1}{2}\right)x,$$

depending on odd or even n, we then obtain the formulas in (3), and the formulas for $\sum_{k=1}^{n} (-1)^{k+1} k^5$ and $\sum_{k=1}^{n} (-1)^{k+1} k^7$, etc., as well.

(4) The function $j(x) = \cos x - \cos 2x + \cos 3x - \cdots \mp \cos(n-1)x \pm \cos nx$, together with the identity from (4) in the Lemma (that is,

$$(\delta) \qquad 2\left(\cos \tfrac{1}{2}x\right) j(x) = \cos \tfrac{1}{2}x \pm \cos\left(n + \tfrac{1}{2}\right)x,$$

depending on odd or even n), lead to the desired conclusion, and for formulas of types $\sum_{k=1}^{n} (-1)^{k+1} k^6$ and $\sum_{k=1}^{n} (-1)^{k+1} k^8$, etc., as well.

Remarks.

(A) We see that the method used in this paper (that is, applying finite sums of sine and cosine functions and by way of differentiation) is a unified approach to sums of integers. This same idea can also be extended to other types of sum. For example, the sine-series $\sin x + \sin 3x + \cdots + \sin(2n-1)$ produces formulas of types $\sum_{k=1}^{n}(2k-1)$ and $\sum_{k=1}^{n}(2k-1)^3$, etc.

(B) Some calculations in the proof of the Proposition are relatively long and laborious. Therefore, it is recommended to make use of a computer to simplify, expand and factor expressions.

References

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C.-S. Lin
Department of Mathematics
Bishop's University
Lennoxville, P.Q.
J1M 1Z7, Canada.
plin@ubishops.ca