THE OLYMPIAD CORNER

No. 218

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Oops! We have noticed a duplication — the problems of the 38$^{\text{th}}$ National Mathematical Olympiad of Slovenia 1994, which have just appeared in the previous issue [2001 : 359] have already appeared on [1998 : 132] with all the solutions published in [1999 : .208–211] and [1999 : 266–269]. Sorry.

We begin this number with a contest from France, from the "Concours Général des lyceés" and the Composition de Mathématiques (Classe terminale S) 1999. Thanks go to Michel Bataille, Rouen, France for forwarding the set to me.

COMPOSITION DE MATHÉMATIQUES 1999

Classe terminale S

Durée : 5 heures

1. Quel est le volume maximum d'un cylindre, ayant même axe de révolution qu'un cône donné et intérieur à ce cône ?

Quel est le volume maximum d'une boule, centrée sur cet axe et intérieure au cône ?

Comparer les deux maximums trouvés.

2. Résoudre dans $\mathbb{N}$ l'équation en n :

$$(n+3)^n = \sum_{k=3}^{n+2} k^n .$$

3. Pour quels triangles aux angles tous aigus le quotient du plus petit côté par le rayon du cercle inscrit est-il maximum ?

4. Sur une table trônent 1 999 bonbons rouges et 6 661 bonbons jaunes rendus indiscernables par des emballages uniformes. Un gourmand applique jusqu'à épuisement du stock l'algorithme ci-dessous :

(a) s'il reste des bonbons, il en tire un au hasard, note sa couleur, le mange et va en (b) ;

(b) s'il reste des bonbons, il en tire un au hasard et note sa couleur :

— si elle est la même que celle du dernier bonbon avalé, il le mange et retourne en (b),

— sinon, il le remmaillote, le pose et retourne en (a).

Montrer que tous les bonbons seront mangés et donner la probabilité pour que le dernier bonbon mangé soit rouge.

5. Montrer que les symétriques de chaque sommet d'un triangle par rapport au côté opposé sont alignés si, et seulement si, la distance de l'orthocentre au centre du cercle circonscrit est égale à son diamètre.

Next we give the problems of the three rounds of the Iranian Mathematical Olympiad 1998-1999. Thanks go to Ed Barbeau, Canadian Team Leader to the IMO at Bucharest for collecting them for our use.

16th IRANIAN MATHEMATICAL OLYMPIAD 1998-1999

First Round

1. Suppose that $a_1 < a_2 < \cdots < a_n$ are real numbers. Prove that:

$$a_1 a_2^4 + a_2 a_3^4 + \cdots + a_{n-1} a_n^4 + a_n a_1^4 \ \geq \ a_2 a_1^4 + a_3 a_2^4 + \cdots + a_n a_{n-1}^4 + a_1 a_n^4 \,.$$

2. Suppose that n is a natural number. The n–tuple $(a_1, a_2, \ldots, a_n)$ is said to be *good*, if $a_1 + a_2 + \cdots + a_n = 2n$ and furthermore, no subset of $\{a_1, \ldots, a_n\}$ has a sum equal to n. Find all good n–tuples.

3. Let I be the incentre of the triangle ABC and AI meet the circumcircle of ABC at point D. Denote the foot of perpendiculars dropped from I on BD and CD by E and F respectively. If $IE + IF = \frac{1}{2}AD$, find the value of $\angle BAC$.

4. Let ABC be a triangle with $BC > CA > AB$. Select points D on BC and E on the extension of AB such that $BD = BE = AC$. The circumcircle of BED intersects AC at point P and BP meets the circumcircle of ABC at point Q. Show that $AQ + CQ = BP$.

5. Suppose that n is a positive integer and $d_1 < d_2 < d_3 < d_4$ are the four smallest positive integers, dividing n. Find all integers n satisfying $n = d_1^2 + d_2^2 + d_3^2 + d_4^2$.

6. Suppose that $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ are two 0/1 sequences. The distance of A from B is defined to be the number of i for which $a_i \neq b_i$ $(1 \leq i \leq n)$ and is denoted by $d(A, B)$.

Suppose that A, B, C are three $0/1$ sequences and $d(A,B) = d(A,C) = d(B,C) = \delta$.

(a) Prove that δ is an even number.

(b) Prove that there exists a $0/1$ sequence D such that

$$d(D,A) \;=\; d(D,B) \;=\; d(D,C) \;=\; \frac{1}{2}\delta\,.$$

Second Round

1. Define the sequence $\{x_i\}_{i=0}^{\infty}$ by $x_0 = 0$ and,

$$\begin{aligned} x_n &= x_{n-1} + \frac{3^r - 1}{2}\,, \quad \text{if } n = 3^{r-1}(3k+1)\,, \\ x_n &= x_{n-1} - \frac{3^r + 1}{2}\,, \quad \text{if } n = 3^{r-1}(3k+2)\,, \end{aligned}$$

where k and r are integers. Prove that every integer occurs exactly once in this sequence.

2. Suppose that $n(r)$ denotes the number of points with integer coordinates on a circle of radius $r > 1$. Prove that,

$$n(r) < 6\sqrt[3]{\pi r^2}\,.$$

3. Suppose that $ABCDEF$ is a convex hexagon with $AB = BC$, $CD = DE$, and $EF = FA$. Prove that

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{3}{2}\,.$$

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying,

$$f(f(x) + y) \;=\; f(x^2 - y) + 4f(x)y\,,$$

for all real numbers x, $y \in \mathbb{R}$.

5. In triangle ABC, the angle bisector of $\angle BAC$ meets BC at point D. Suppose that Γ is the circle which is tangent to BC at D and passes through the point A. Let M be the second point of intersection of Γ and AC and BM meets the circle at P. Prove that AP is a median of triangle ABD.

6. Suppose that ABC is a triangle. If we paint the points of the plane in red and green, prove that there exist either two red points which are one unit apart or three green points forming a triangle equal to ABC.

Third Round

1. Suppose that $X = \{1, 2, \ldots, n\}$ and $A_1, A_2, \ldots, A_k$ are subsets of X such that for every $1 \leq i_1, i_2, i_3, i_4 \leq k$, we have,

$$|A_{i_1} \cup A_{i_2} \cup A_{i_3} \cup A_{i_4}| \;\leq\; n - 2\,.$$

Prove that $k \leq 2^{n-2}$.

2. Suppose that a circle passing through the points A and C of triangle ABC meets AB and BC at points D and E respectively. In the arcwise triangle EBD, inscribe a circle Γ with centre S. Suppose that Γ is tangent to arc DE at point M. Prove that the angle bisector of $\angle AMC$ passes through the incentre of triangle ABC.

3. Suppose that $C_1, \ldots, C_n$ are circles of radius one in the plane such that no two of them are tangent, and the subset of the plane, formed by the union of these circles, is connected. If $S = \{C_i \cap C_j \mid 1 \leq i < j \leq n\}$, prove that $|S| \geq n$.

4. Suppose that $x_1, \ldots, x_n$ are real numbers and $-1 \leq x_i \leq 1$ and $x_1 + \cdots + x_n = 0$. Prove that there exists a permutation σ such that for all $1 \leq p \leq q \leq n$, we have,

$$|x_{\sigma(p)} + \cdots + x_{\sigma(q)}| \;\leq\; 2 - \frac{1}{n}\,.$$

Prove that the right side cannot be replaced by $2 - \frac{4}{n}$.

5. Suppose that $ABCDEF$ is a convex hexagon with $\angle B + \angle D + \angle F = 360^\circ$ and

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} \;=\; 1\,.$$

Prove that

$$\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} \;=\; 1\,.$$

6. Suppose that $r_1, \ldots, r_n$ are real numbers. Prove that there exists $I \subseteq \{1, 2, \ldots, n\}$ such that I meets $\{i, i+1, i+2\}$ in at least one and at most two elements, for $1 \leq i \leq n-2$ and

$$\left|\sum_{i \in I} r_i\right| \;\geq\; \frac{1}{6}\sum_{i=1}^{n} |r_i|\,.$$

As a final set of problems for the year we give the 1999 Chinese Mathematical Olympiad. Thanks again go to Ed Barbeau, Canadian Team Leader at the IMO at Bucharest, for collecting them for the *Corner*.

1999 CHINESE MATHEMATICAL OLYMPIAD

First Day

January 11, 1999 — Time: 4.5 hours

1. In acute triangle $\triangle ABC$, $\angle ACB > \angle ABC$. Point D is on BC such that $\angle ADB$ is obtuse. Let H be the orthocentre of $\triangle ABD$. Suppose point F is inside $\triangle ABC$ and on the circumcircle of $\triangle ABD$. Prove that point F is the orthocentre of $\triangle ABC$, if and only if HD is parallel to CF and H is on the circumcircle of $\triangle ABC$.

2. For a given real number a, suppose the sequence of real coefficient polynomials $\{f_n(x)\}$ satisfies

$$\begin{cases} f_0(x) = 1 \\ f_{n+1}(x) = xf_n(x) + f_n(ax), \quad n = 0, 1, 2, \ldots. \end{cases}$$

(a) Prove that $f_n(x) = x^n f_n(\frac{1}{x})$, $n = 0, 1, 2, \ldots$;

(b) Give explicit formulas for $f_n(x)$.

3. Space city MO consists of 99 space stations. Each two stations are connected by a tube passage. Among all these tubes, 99 are two-way and the others are strictly one-way. For any 4 stations, we call them a strongly connected quadruple if, from each of the 4 stations, one can get to the other 3 through the tubes connecting these 4 stations.

Design a scheme for the space city MO such that it has the maximal number of strongly connected quadruples. (Please give the maximal number and prove your conclusion.)

Second Day

January 12, 1999 — Time: 4.5 hours

4. Let m be a given integer. Prove that there exist integers a, b and k such that both a, b are not divisible by 2, $k \geq 0$, and

$$2m = a^{19} + b^{99} + k \cdot 2^{1999}.$$

5. Find the maximal real number λ such that, whenever

$$f(x) = x^3 + ax^2 + bx - c$$

is a real polynomial and all of its roots are non-negative real numbers, we always have

$$f(x) \geq \lambda(x - a)^3, \quad \forall x \geq 0.$$

When does equality hold?

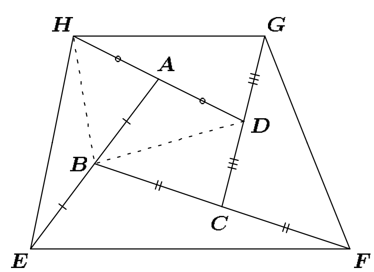
6. A big cube of dimensions $4 \times 4 \times 4$ consists of 64 unit cubes. Paint 16 of the unit cubes with the colour red in such a way that, in each cuboid of size $1 \times 1 \times 4$, $1 \times 4 \times 1$ or $4 \times 1 \times 1$ of the big cube, there exists exactly one red unit cube. What is the total number of ways to do this painting? Give your explanation.

Now we turn to solutions and comments by our readers to problems of the South African Mathematics Olympiad, Section B, September 1995 [1999: 392].

SECTION B.

1. The convex quadrilateral $ABCD$ has area 1, and AB is produced to E, BC to F, CD to G and DA to H, such that $AB = BE$, $BC = CF$, $CD = DG$ and $DA = AH$. Find the area of the quadrilateral $EFGH$.

Solutions by Christopher J. Bradley, Clifton College, Bristol, UK; and by Toshio Seimiya, Kawasaki, Japan. We give the solution by Seimiya.



We denote the area of n-gon $P_1P_2\ldots P_n$ by $[P_1P_2\ldots P_n]$. Since B and A are the mid-points of AE and DH respectively, we get $[HEB] = [HAB] = [ABD]$, so that

$$[AEH] = 2[ABD].$$

Similarly we have $[CFG] = 2[CBD]$. Thus,

$$\begin{aligned}[AEH] + [CFG] &= 2[ABD] + 2[CBD] \\ &= 2[ABCD] \\ &= 2.\end{aligned}$$

Similarly, we get

$$[BEF] + [DGH] = 2.$$

Thus, we have

$$\begin{aligned}[EFGH] &= [ABCD] + \{[AEH] + [CFG]\} + \{[BEF] + [DGH]\} \\ &= 1 + 2 + 2 \\ &= 5.\end{aligned}$$

2. Find all pairs (m,n) of natural numbers with $m < n$ such that m^2+1 is a multiple of n and n^2+1 is a multiple of m.

Solutions by Pierre Bornsztein, Pontoise, France; and by Christopher J. Bradley, Clifton College, Bristol, UK. We give Bornsztein's solution and remark.

We will prove that the solutions are the pairs of the form (f_{2n-1}, f_{2n+1}) for $n \in \mathbb{N}$, where $\{f_k\}$ is the Fibonacci sequence ($f_0 = 0$, $f_1 = 1$ and $f_{n+2} = f_n + f_{n+1}$).

Let (m,n) be such that m, $n \in \mathbb{N}^*$, $m < n$, and m^2+1 is a multiple of n and n^2+1 is a multiple of m.

If $m = 1$ then $n \geq 2$ and n divides 2. Thus, $n = 2$.

Conversely, $(1,2)$ is a solution.

Claim: (m,n) is a solution with $m > 1$ if and only if $\left(\frac{m^2+1}{n}, m\right)$ is a solution.

Proof of the Claim: First note that $(m,(m^2+1)) = 1$ and $(n,(n^2+1)) = 1$. It follows that

$$\begin{aligned} & (m,n) \text{ is a solution} \\ \iff & \frac{(m^2+1)(n^2+1)}{mn} \text{ is an integer} \\ \iff & \frac{m^2+n^2+1}{mn} \text{ is an integer} \\ \iff & \text{there exists } k \in \mathbb{N}^* \text{ such that } m^2+n^2+1 = kmn \\ \iff & \text{there exists } k \in \mathbb{N}^* \text{ such that } n \text{ is a solution of the} \\ & \text{equation } (E_k) : X^2 - kmX + m^2 + 1 = 0\,. \end{aligned}$$

In the same way: (m,n) is a solution $\iff$ there exists $k \in \mathbb{N}^*$ such that m is a solution of the equation $X^2 - knX + n^2 + 1 = 0$.

Since (E_k) is a quadratic equation with integer coefficients and leading coefficient equal to 1, one of its solutions is an integer if and only if the other is as well. Moreover n is a solution of (E_k) if and only if $\frac{m^2+1}{n}$ is a solution of (E_k) (from the product of the roots of (E_k)).

It follows that (m,n) is a solution $\iff$ there exists $k \in \mathbb{N}^*$ such that $\frac{m^2+1}{n}$ is a solution of (E_k).

- If $m = 1$ then $n = 2$ and $\frac{m^2+1}{n} = m$.
- If $m > 1$ then, since $m+1 \leq n$, we have

$$mn \;\geq\; m(m+1) \;>\; m^2+1\,.$$

Thus, $\frac{m^2+1}{n} < m$.

We deduce that (m, n) is a solution with $m > 1 \iff \left(\frac{m^2+1}{n}, m\right)$ is a solution, and the claim is proved.

Let (m, n) be a solution, with $m > 1$. From the claim, we may find another solution (m', n') with $m' < n' = m < n$. Repeating this process, we construct a sequence (m_i, n_i) of solutions, and the sequence $\{n_i\}$ is a strictly decreasing sequence of positive integers. This sequence has to be finite. But the only way to stop the process is to obtain $m_i = 1$ (and then $n_i = 2$) for some k.

It follows that each solution of the problem is generated from $(1, 2)$ by using a finite number of applications of the function $(m, n) \to \left(n, \frac{n^2+1}{m}\right)$.

Conversely, if $m_1 = 1$ and $n_1 = 2$, and for $n \in \mathbb{N}^*$,

$$\left\{\begin{array}{rcl} m_{i+1} & = & n_i\,, \\ n_{i+1} & = & \frac{n_i^2+1}{m_i}\,. \end{array}\right. \tag{1}$$

From the claim, we know that (m_i, n_i) is a solution for all $i \geq 1$.

Then, the solutions of the problem are pairs (m_i, n_i) defined by (1), for all $i \geq 2$: $n_{i+1} = \frac{n_i^2+1}{m_i} = \frac{n_i^2+1}{n_{i-1}}$; that is

$$n_{i+1}n_{i-1} - 1 \;=\; n_i^2\,. \tag{2}$$

It looks like a well-known property of the Fibonacci sequence

$$f_n^2 \;=\; f_{n+1}f_{n-1} + (-1)^{n-1}$$

where $f_0 = 0$, $f_1 = 1$, $f_{n+2} = f_n + f_{n+1}$. It is also well known that for all $n \geq 0$

$$f_n \;=\; \frac{1}{\sqrt{5}}\left(\varphi^n - \left(-\frac{1}{\varphi}\right)^n\right) \quad \text{where} \quad \varphi \;=\; \frac{1+\sqrt{5}}{2} \tag{3}$$

Using (3) it is not difficult to see that f_{2k+1} satisfies (2). Since $f_1 = m_1$ (if we define $n_0 = m_1$ we then have $n_0 = f_1$), and $f_3 = n_1$, $f_5 = n_2$, it follows that for all $i \geq 1$

$$n_i \;=\; f_{2i+1}$$

and then

$$m_i \;=\; n_{i-1} \;=\; f_{2i-1}\,,$$

And we are done.

Comments: A generalization of this problem may be found in [1]:

"Let $q \in \mathbb{N}^*$. Let $\mathcal{P}$ be the set of pairs (m, n) of coprime positive integers m, n such that m divides $n^2 + q^2$ and n divides $m^2 + q^2$, with $m \geq n$.

Let f_a be the a^{th} generalized Fibonacci number ($f_0 = 0$, $f_1 = 1$, $f_{n+2} = qf_{n+1} + pf_n$, where $p = 1$ in this case).

Then:

• if a is odd, $(f_a, f_{a-2}) \in \mathcal{P}$; and

• $\mathcal{P} = \{(f_a, f_{a-2}) : a \text{ odd}\}$ if and only if the only pairs $(m, n) \in \mathcal{P}$ with $n \leq q^2$ are the pairs

$$(1, 1)\,, \quad (q^2 + 1, 1)\,."$$

Reference:

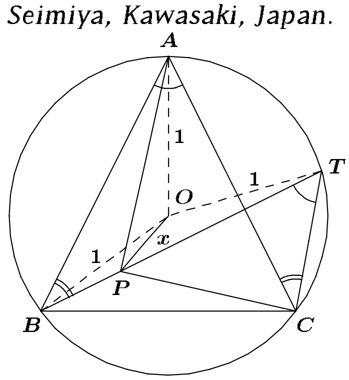
1. P. Hilton, J. Pedersen, *A fresh look at old favourites: the Fibonacci and Lucas sequences revisited*, *Australian Mathematical Society Gazette*, Vol 25, no 3 (August 1998), p. 146–160.

3. The circumcircle of $\triangle ABC$ has radius 1 and centre O, and P is a point inside the triangle such that $OP = x$. Prove that

$$AP \cdot BP \cdot CP \;\leq\; (1 + x)^2(1 - x)\,,$$

with equality only if $P = O$.

Solution by Toshio Seimiya, Kawasaki, Japan.



If we assume that

$$\angle PBA + \angle PCA \;<\; \angle BAC\,, \tag{1}$$

that

$$\angle PAB + \angle PCB \;<\; \angle ABC\,, \tag{2}$$

and that

$$\angle PAC + \angle PBC \;<\; \angle ACB\,, \tag{3}$$

then, adding (1) and (2) to (3), we have

$$\angle BAC + \angle ABC + \angle ACB \;<\; \angle BAC + \angle ABC + \angle ACB\,.$$

This is a contradiction. Therefore, (1), (2) and (3) do not hold simultaneously, so that at least one of them is not true.

Without loss of generality we may assume that (1) is not true, so that we have

$$\angle PBA + \angle PCA \;\geq\; \angle BAC \,.$$

Let T be the second intersection of BP with the circumcircle of $\triangle ABC$. Then, we have $\angle BTC = \angle BAC$ and $\angle ACT = \angle ABT$.

Thus, $\angle PCT = \angle ACP + \angle ACT = \angle ACP + \angle ABT = \angle ACP + \angle ABP \geq \angle BAC = \angle BTC = \angle PTC$; that is, $\angle PCT \geq \angle PTC$. Thus, we have $PT \geq CP$. Since

$$BP \cdot PT \;=\; OT^2 - OP^2 \;=\; 1 - x^2 \,,$$

we have

$$BP \cdot CP \;\leq\; BP \cdot PT \;=\; 1 - x^2 \,. \tag{4}$$

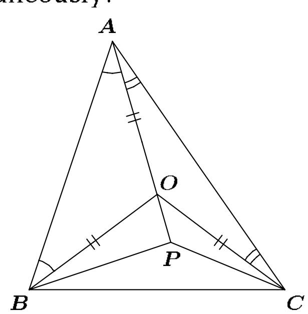
Since

$$AP \;\leq\; AO + OP \;=\; 1 + x \,, \tag{5}$$

we obtain, from (4) and (5), that

$$AP \cdot BP \cdot CP \;\leq\; (1+x)(1-x^2) \;=\; (1+x)^2(1-x) \,. \tag{6}$$

Next, we consider when equality in (6) holds. This is when equalities in (4) and (5) hold simultaneously.



In (5), equality holds if and only if P is a point on AO produced beyond O or $P = O$.

If $\triangle ABC$ is not an acute triangle, O is not an interior point of $\triangle ABC$, so that equality in (5) does not hold.

When $\triangle ABC$ is an acute triangle, let P be a point on AO produced beyond O. Then $\angle ABP + \angle ACP > \angle ABO + \angle ACO = \angle BAO + \angle CAO = \angle BAC$. Thus, $BP \cdot CP < 1 - x^2$. Thus, equality in (4) does not hold.

If $P = O$, in (4), (5) and (6), then all equalities hold.

Therefore, in (6) equality holds if and only if $P = O$.

Remark by Pierre Bornsztein, Pontoise, France.

This problem is solved in American Mathematical Monthly 1995, problem 10282, p. 468. In fact, the problem of the Monthly is, with notations as above:

"Show that $PA \cdot PB \cdot PC < \frac{32}{27}$."

by Paul Erdős.

Note that $\frac{32}{27}$ is the maximum value of $(1+x)^2(1-x)$ in the interval $[0,1]$. The editor comments

"A generalization to an n–gon inscribed in a circle was obtained. The bound in this case is the maximum value of $(1-x)(1+x)^{n-1}$, which is $(\frac{2}{n})^n(n-1)^{n-1}$."

Now we turn to solutions to problems of the Taiwan Mathematical Olympiad 1996 [1999 : 392–393].

1. Let the angles α, β, γ be such that $0 < \alpha, \beta, \gamma < \frac{\pi}{2}$ and $\alpha+\beta+\gamma = \frac{\pi}{4}$. Suppose that

$$\tan\alpha = \frac{1}{a}, \quad \tan\beta = \frac{1}{b}, \quad \tan\gamma = \frac{1}{c},$$

where a, b, c are positive integers. Determine the values of a, b, c.

Solutions by Pierre Bornsztein, Pontoise, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Klamkin's solution.

Since

$$\begin{aligned} 1 = \tan\frac{\pi}{4} &= \tan(\alpha+\beta+\gamma) \\ &= \frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\beta\tan\gamma-\tan\gamma\tan\alpha-\tan\alpha\tan\beta} \\ &= \frac{bc+ca+ab-1}{abc-a-b-c}, \end{aligned}$$

we solve for b and get

$$b = 1 + \frac{2a+2c}{(c-1)a-(c+1)}.$$

We now successively let $c = 2, 3, \ldots$, and determine a such that b is an integer. These are given only by permutations of

$$(c,a,b) = (2,4,13), \ (2,5,8), \ \text{ and } \ (3,3,7).$$

2. Let a be a real number such that $0 < a \leq 1$ and $a \leq a_j \leq \frac{1}{a}$, for $j = 1, 2, \ldots, 1996$. Show that for any non-negative real numbers λ_j ($j = 1, 2, \ldots, 1996$), with

$$\sum_{j=1}^{1996} \lambda_j \;=\; 1\,,$$

one has

$$\left(\sum_{i=1}^{1996} \lambda_i a_i\right)\left(\sum_{j=1}^{1996} \lambda_j a_j^{-1}\right) \;\leq\; \frac{1}{4}\left(a + \frac{1}{a}\right)^2\,.$$

Solutions by Michel Bataille, Rouen, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea. We give the solution by Bataille.

Let $n = 1996$ (or n be any positive integer, actually). From the AM–GM Inequality,

$$\begin{aligned}\left(\sum_{i=1}^{n} \lambda_i a_i\right)^{1/2}\left(\sum_{j=1}^{n} \lambda_j a_j^{-1}\right)^{1/2} \;&\leq\; \frac{1}{2}\left(\sum_{i=1}^{n} \lambda_i a_i + \sum_{j=1}^{n} \lambda_j a_j^{-1}\right) \\ &=\; \frac{1}{2}\left(\sum_{j=1}^{n} \lambda_j\left(a_j + \frac{1}{a_j}\right)\right)\,.\end{aligned}$$

Now, if t satisfies $a \leq t \leq \frac{1}{a}$, then $\left(t + \frac{1}{t}\right) - \left(a + \frac{1}{a}\right) = (t - a)\left(1 - \frac{1}{at}\right) \leq 0$, so that $t + \frac{1}{t} \leq a + \frac{1}{a}$. It follows that $a_j + \frac{1}{a_j} \leq a + \frac{1}{a}$ ($j = 1, 2, \ldots, n$) and, since $\lambda_j \geq 0$, $\sum_{j=1}^{n} \lambda_j\left(a_j + \frac{1}{a_j}\right) \leq \left(a + \frac{1}{a}\right)\sum_{j=1}^{n} \lambda_j = a + \frac{1}{a}$.

From this, we get

$$\left(\sum_{i=1}^{n} \lambda_i a_i\right)^{1/2}\left(\sum_{j=1}^{n} \lambda_j a_j^{-1}\right)^{1/2} \;\leq\; \frac{1}{2}\left(a + \frac{1}{a}\right)$$

and the required result by squaring both sides.

Generalization: Replacing the hypothesis on the a_i by $0 < m \leq a_i \leq M$, we get the following inequality:

$$\left(\sum_{i=1}^{n} \lambda_i a_i\right)\left(\sum_{j=1}^{n} \lambda_j a_j^{-1}\right) \;\leq\; \frac{1}{4}\frac{(m + M)^2}{mM}\,.$$

It suffices to remark that $\sqrt{\frac{m}{M}} \leq \frac{a_i}{\sqrt{mM}} \leq \sqrt{\frac{M}{m}}$ and apply the result above with $a = \sqrt{\frac{m}{M}}$ and $\frac{a_i}{\sqrt{mM}}$ instead of a_i.

We also give the "quick" solution of Klamkin.

The result follows immediately by applying the following inequality of Polya and Szego [1]:

$$4\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) \;\le\; \left(\sqrt{\frac{M_1M_2}{m_1m_2}} \;+\; \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2 \left(\sum_{k=1}^{n} a_kb_k\right)^2$$

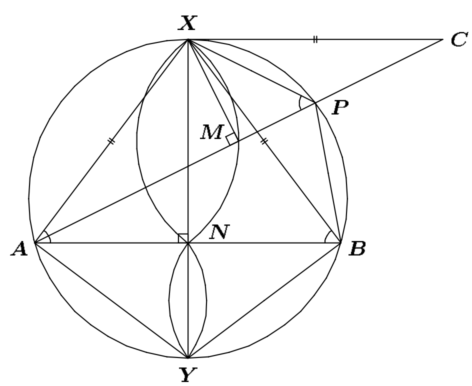
where $0 < m_1 \le a_k \le M_1$, $0 < m_2 \le b_k \le M_2$ for $k = 1, 2, \ldots, n$.

Reference:

1. D.S. Mitrinovic, *Analytic Inequalities*, Springer-Verlag, Heidelberg, 1970, p. 60.

3. Let A and B be two fixed points on a fixed circle. Let a point P move on this circle and let M be a corresponding point such that either M is on the segment PA with $AM = MP + PB$ or M is on the segment PB with $AP + MP = PB$. Determine the locus of such points P.

Solution by Toshio Seimiya, Kawasaki, Japan.



The perpendicular bisector of AB meets the circle at X and Y. Then $XA = XB$ and $YA = YB$.

Let P be a point on the minor arc XB. Then $PA \ge PB$.

Let C be a point on AP produced beyond P such that $PC = PB$. Since $AM = MP + PB = MP + PC = MC$, M is the mid-point of AC.

Since $\angle XPA = \angle XBA = \angle XAB$, we have

$$\angle XPC \;=\; 180^\circ - \angle XPA \;=\; 180^\circ - \angle XAB \;=\; \angle XPB\,.$$

Since $PC = PB$ and $XP = XP$ we get $\triangle XPC \equiv \triangle XPB$. Hence, $XC = XB = XA$.

Since M is the mid-point of AC we have $XM \perp AC$; that is, $\angle XMA = 90^\circ$.

Let N be the mid-point of AB. Then $\angle XNA = 90°$.

If P moves on the minor arc BX from B to X, then M moves on the minor arc NX of the circle with diameter XA. Similarly, if P moves on the minor arc XA from X to A, then M moves on the minor arc XN of the circle with diameter XB.

And if P moves on the minor arc AY, then m moves on the minor arc NY of the circle with diameter BY.

And if P moves on the minor arc YB, then M moves on the minor arc NY of the circle with diameter AY.

Thus, the locus of M is the union of the four minor arcs as shown in the figure as a shape of a "figure-eight" loop.

4. Show that for any real numbers a_3, a_4, ..., a_{85}, the roots of the equation

$$a_{85}x^{85} + a_{84}x^{84} + \cdots + a_3x^3 + 3x^2 + 2x + 1 \;=\; 0$$

are not real.

Solutions by Pierre Bornsztein, Pontoise, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Bornsztein's write-up.

Let $P(x) = a_{85}x^{85} + \cdots + a_3x^3 + 3x^2 + 2x + 1$. Since $P(0) = 1$, then 0 is not a root of P.

Let r_1, ..., r_{85} be the complex roots of P.

For $i = 1, \ldots, 85$, denote $s_i = \frac{1}{r_i}$. Then the s_i's are the complex roots of the polynomial $Q(y) = y^{85} + 2y^{84} + 3y^{83} + a_3y^{82} + \cdots + a_{84}y + a_{85}$. It follows that

$$\sum_{i=1}^{85} s_i \;=\; -2 \quad \text{and} \quad \sum_{i<j} s_is_j \;=\; 3\,.$$

Then

$$\sum_{i=1}^{85} s_i^2 \;=\; \left(\sum_{i=1}^{85} s_i\right)^2 - 2\sum_{i<j} s_is_j \;=\; -2 \;<\; 0\,.$$

Thus, the s_i's are not all real, and then the r_i's are not all real.

Remark. The conclusion holds for all real numbers a_0, a_1, a_2 such that $a_0 \neq 0$ and $a_1^2 < 2a_0a_2$.

6. Let q_0, q_1, q_2, ... be a sequence of integers such that

(a) for any $m > n$, $m - n$ is a factor of $q_m - q_n$, and

(b) $|q_n| \leq n^{10}$ for all integers $n \geq 0$.

Show that there exists a polynomial $Q(x)$ satisfying $Q(n) = q_n$ for all n.

Comment by Pierre Bornsztein, Pontoise, France.

This is a particular case of problem 4 of the 1995 USAMO, where it was asked:

"Suppose q_0, q_1, ... is an infinite sequence of integers satisfying the following two conditions:

(i) $m - n$ divides $q_m - q_n$ for $m > n \geq 0$.

(ii) there is a polynomial P such that $|q_n| < P(n)$ for all n.

Show there is a polynomial Q such that $q_n = Q(n)$ for all n."

Reference:

1. *Math. Magazine*, Vol 69, no 3, June 1996, p. 235.

To complete this number of the *Corner* we turn to solutions of problems of the Croatian National Mathematics Competition, IV Class [1999 : 393–394].

1. Is there any solution of the equation

$$\lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 4x \rfloor + \lfloor 8x \rfloor + \lfloor 16x \rfloor + \lfloor 32x \rfloor \;=\; 12345\,?$$

($\lfloor x \rfloor$ denotes the greatest integer which does not exceed x.)

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Bornsztein's solution.

The equation

$$\lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 4x \rfloor + \lfloor 8x \rfloor + \lfloor 16x \rfloor + \lfloor 32x \rfloor \;=\; 12345 \qquad (1)$$

has no real solution. Suppose, for a contradiction, that $x \in \mathbb{R}$ satisfies (1).

Then $x > 0$ and we may write

$$x \;=\; N + \frac{a}{2} + \frac{b}{4} + \frac{c}{8} + \frac{d}{16} + \frac{e}{32} + f$$

where N is a non-negative integer,

$$a,\, b,\, c,\, d,\, e \in \{0, 1\} \quad \text{and} \quad f \in \left[0, \tfrac{1}{32}\right)\,.$$

From (1) we obtain

$$63N + 31a + 15b + 7c + 3d + e \;=\; 12345\,.$$

Thus,

$$63N \;\leq\; 12345 \;\leq\; 63N + 31 + 15 + 7 + 3 + 1\,.$$

That is,

$$\frac{12288}{63} \;\leq\; N \;\leq\; \frac{12345}{63}\,.$$

Then, $195 < N < 196$, which is impossible if N is supposed to be an integer. Thus, (1) has no solution, as claimed.

2. Determine all pairs of numbers λ_1, $\lambda_2 \in \mathbb{R}$ for which every solution of the equation

$$(x + i\lambda_1)^n + (x + i\lambda_2)^n \;=\; 0$$

is real. Find the solutions.

Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Pontoise, France. We give Bataille's solution.

Let x be any complex number solution. Then, $(x+i\lambda_1)^n = -(x+i\lambda_2)^n$ so that

$$|x + i\lambda_1|^n \;=\; |x + i\lambda_2|^n \quad \text{and} \quad |x + i\lambda_1| \;=\; |x + i\lambda_2|\,. \qquad (1)$$

Denote by A_1, A_2 the points in the complex plane corresponding to $-i\lambda_1$, $-i\lambda_2$ respectively. The relation (1) means that $MA_1 = MA_2$, where M corresponds to x. Thus, M is on the perpendicular bisector Δ of segment A_1A_2. Note that Δ is a line perpendicular to the imaginary axis. It follows that every solution is a real number if and only if Δ coincides with the real axis; that is, $\lambda_1 + \lambda_2 = 0$.

Conversely, suppose $\lambda_1 = -\lambda_2 = \lambda \in \mathbb{R}$. The given equation becomes $(x+i\lambda)^n = -(x-i\lambda)^n$. If $\lambda = 0$, then $x = 0$ is the only solution. Assuming now that $\lambda \neq 0$, our equation is equivalent to:

$$\left(\frac{x + i\lambda}{x - i\lambda}\right)^n \;=\; -1 \quad \text{or} \quad \frac{x + i\lambda}{x - i\lambda} \;=\; u_k$$

where $u_k = \exp(i(\frac{\pi}{n} + \frac{2k\pi}{n}))$ for $k = 0, 1, \ldots, n-1$. This gives $x = i\lambda\frac{u_k+1}{u_k-1}$ or, by an easy computation $x = \lambda\cot\left(\frac{(2k+1)\pi}{2n}\right)$.

Thus, the solutions are the n real numbers $x_k = \lambda\cot\left(\frac{(2k+1)\pi}{2n}\right)$, $(k = 0, 1, \ldots, n-1)$.

3. Determine all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous at 0, which satisfy the following relation

$$f(x) - 2f(tx) + f(t^2x) \;=\; x^2 \quad \text{for all} \quad x \in \mathbb{R}\,,$$

where $t \in (0,1)$ is a given number.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; and by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea. We give the write-up by Lee.

First, we introduce the function $g(x) = f(x) - f(tx)$ defined on $\mathbb{R}$. Then g is also continuous at 0.

Since $g(tx) = f(tx) - f(t^2x)$, we easily get $g(x) - g(tx) = x^2$. Therefore, we have

$$g(x) - g(t^{n+1}x) \;=\; \sum_{i=1}^{n+1}\{g(t^{i-1}x) - g(t^i x)\} \;=\; \sum_{i=1}^{n+1}(t^{i-1}x)^2\,,$$

or

$$g(x) \;=\; g(t^{n+1}x) + \left(\sum_{i=1}^{n+1}(t^2)^{i-1}\right) x^2\,.$$

We note that $\lim_{n\to\infty} t^{n+1} = 0$ since $t \in (0,1)$. By the continuity of g at 0, we have

$$\begin{aligned} g(x) &= \lim_{n\to\infty} g(t^{n+1}x) + \lim_{n\to\infty}\left(\sum_{i=1}^{n+1}(t^2)^{i-1}\right) x^2 \\ &= g(0) + \frac{1}{1-t^2}x^2\,. \end{aligned}$$

Since $g(0) = f(0) - f(t\cdot 0) = 0$, we obtain $g(x) = \frac{1}{1-t^2}x^2$. Thus, we have $f(x) - f(tx) = \frac{1}{1-t^2}x^2$.

Also, we get

$$f(x) - f(t^{n+1}x) \;=\; \sum_{i=1}^{n+1}\{f(t^{i-1}x) - f(t^i x)\} \;=\; \frac{1}{1-t^2}\sum_{i=1}^{n+1}(t^{i-1}x)^2\,.$$

By the continuity of f at 0, we have

$$f(x) \;=\; \lim_{n\to\infty} f(t^{n+1}x) + \frac{1}{1-t^2}\lim_{n\to\infty}\sum_{i=1}^{n+1}(t^{i-1}x)^2$$

or

$$f(x) \;=\; f(0) + \frac{1}{(1-t^2)^2}x^2\,.$$

Therefore, $f(x) = \frac{1}{(1-t^2)^2}x^2 + C$, $(C \in \mathbb{R})$.

4. Let α and β be positive irrational numbers such that $\frac{1}{\alpha}+\frac{1}{\beta}=1$ and $A=\{\lfloor n\alpha\rfloor \mid n\in\mathbb{N}\}$, $B=\{\lfloor n\beta\rfloor \mid n\in\mathbb{N}\}$. Prove that $A\cup B=\mathbb{N}$ and $A\cap B=\emptyset$.

Remark: You can prove the following equivalent assertion: For a function $\pi:\mathbb{N}\to\mathbb{N}$ defined by

$$\pi(m) = \mathrm{Card}\{k \mid k\in\mathbb{N},\, k\le m, k\in A\}+\mathrm{Card}\{k \mid k\in\mathbb{N},\, k\le m,\, k\in B\}$$

one has $\pi(m)=m$, $\forall m\in\mathbb{N}$. ($\lfloor x\rfloor$ denotes the greatest integer which does not exceed x.)

Comment by Pierre Bornsztein, Pontoise, France.

This problem is well known as Beatty's problem. See [1], [2] or [3] for an elementary proof and related results.

It is also well known that the converse is true (see [3]).

For a positive real number x define $S(x)=\{\lfloor nx\rfloor : n\in\mathbb{N}^*\}$.

Then a necessary and sufficient condition for $\mathbb{N}^*$ to be the disjoint union of $S(\alpha)$ and $S(\beta)$ is that α and β are irrationals such that $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

Moreover, Uspensky proved in 1927 that:

> "There do not exist three or more positive numbers α_1, α_2, ..., α_n such that $\mathbb{N}^*$ is the union of the pairwise disjoint sets $S(\alpha_1)$, $S(\alpha_2)$, ..., $S(\alpha_n)$."

For the special case $n=3$, see [4]. For the general case, see [3].

References:

1. Ross Honsberger, *Ingenuity in Mathematics*, M.A.A. p. 93–110.

2. A.M. Gleason, R.E. Greenwood, L.M. Kelly, *The William Lowell Putnam Mathematical Competition — Problems and Solutions 1938–1964*, M.A.A., Afternoon session 1959 problem 6, p. 513.

3. Joe Roberts, *Elementary Number Theory: A Problem Oriented Approach*, M.I.T. Press 1977, p. 38–45 and 475–585.

4. *The William Lowell Putnam Mathematical Competition 1995*, *American Mathematical Monthly* 1996, pp. 676–677 (problem B.6).

Comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

This problem has quite a history and appeared as problem 6 in the 20$^{\text{th}}$ Putnam Competition, 1959 [1]. Although $A\cap B=\emptyset$ was not asked for, it was part of the solution. At the end of the given solution is the following remark:

This is sometimes called Beatty's problem, after Samuel Beatty (1881–1970). In a slightly different form it appeared as Problem 3117, *American Mathematical Monthly*, Vol. 34 (1927), pp. 158–159. Howard Grossman, *A*

Set Containing All Integers, *American Mathematical Monthly*, Vol. 69 (1962), pp. 532–533, gives a proof by analyzing lattice points. A.S. Fraenkel, *The Bracket Function and Complementary Sets of Integers*, *Canadian Journal of Mathematics*, Vol. 21 (Jan. 1969), pp. 6–27, gives a history, a bibliography, and a generalization of the problem.

Reference:

1. A.M. Gleason, R.E. Greenwood, L.M. Kelly, *The William Lowell Putnam Mathematical Competition: Problems and Solutions 1938–1964*, M.A.A., Washington, D.C., 1980, pp. 513–514.

As a final solutions set this issue, we present the solutions to problems posed at the Additional Competition (Croatian National Mathematica Competition) for Selection of the IMO Team [1999 : 394].

1. (a) $n = 2k + 1$ points are given in the plane. Construct an n–gon such that these points are mid-points of its sides.

(b) Arbitrary $n = 2k$, $k > 1$, points are given in the plane. Prove that it is impossible to construct an n–gon, in each case, such that these points are mid-points of its sides.

Solutions by Michel Bataille, Rouen, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Bataille's solution.

We will denote by H_A the half-turn around point A and by $T_{\mathbf{U}}$ the translation with vector $\mathbf{U}$. Note the following formula: $H_B \circ H_A = T_{\mathbf{U}}$ where $\mathbf{U} = 2\overrightarrow{AB}$.

Suppose now that $A_1, A_2, \ldots, A_n$ are the mid-points of the sides of the n–gon $M_1M_2\ldots M_n$ (with A_1 the mid-point of M_1M_2, etc. ...). Then

$$\begin{aligned} M_1 &= H_{A_n}(M_n) = H_{A_n} \circ H_{A_{n-1}}(M_{n-1}) \\ &= \cdots = H_{A_n} \circ H_{A_{n-1}} \circ \cdots \circ H_{A_1}(M_1) \end{aligned}$$

so that M_1 is invariant under the transformation $H = H_{A_n} \circ H_{A_{n-1}} \circ \cdots \circ H_{A_1}$.

If $n = 2k$, then $H = T_{\mathbf{U}_k} \circ T_{\mathbf{U}_{k-1}} \circ \cdots \circ T_{\mathbf{U}_1}$ where $\mathbf{U}_i = 2\overrightarrow{A_{2t-1}A_{2t}}$ $(i = 1, \ldots, k)$. Hence, H is the translation with vector $\mathbf{U} = \mathbf{U}_1 + \cdots + \mathbf{U}_k$. Unless $\mathbf{U} = \overrightarrow{0}$, H has no invariant point so that no n–gon $M_1M_2\ldots M_n$ can be obtained. (If $\mathbf{U} = \overrightarrow{0}$, M_1 can be any point in the plane and we obtain an infinity of solutions; see figure 2.)

If $n = 2k + 1$, then

$$\begin{aligned} H &= H_{A_n} \circ T_{\mathbf{U}} \quad \text{(with the notation above)} \\ &= H_{A_n} \circ H_{A_n} \circ H_B \quad \text{where } B \text{ is such that } 2\overrightarrow{BA_n} = \mathbf{U} \\ &= H_B\,. \end{aligned}$$

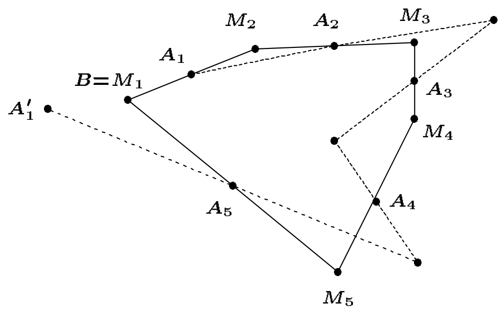


figure 1 (for $n = 5$)

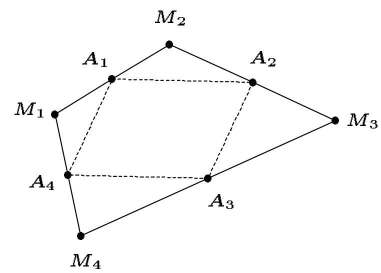


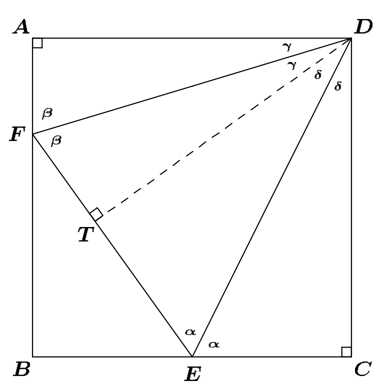
figure 2 (for $n = 4$ and $U = \overrightarrow{0}$)

Hence, H is the half-turn around B and, necessarily, $M_1 = B$. Note that B is easily constructed: just construct $A'_1 = H(A_1)$ and B is the mid-point of $A_1A'_1$.

Conversely, taking $M_1 = B$ and constructing successively the points $M_2 = H_{A_1}(M_1), \ldots, M_n = H_{A_{n-1}}(M_{n-1})$, we obtain a suitable n-gon $M_1M_2\ldots M_n$ (since $H_{A_n}(M_n) = H(M_1) = M_1$). Thus, when n is odd, there is a unique solution (see figure 1).

2. The side-length of the square $ABCD$ equals a. Two points E and F are given on sides $\overline{BC}$ and $\overline{AB}$ such that the perimeter of the triangle BEF equals $2a$. Determine the angle $\angle EDF$.

Solutions by Pierre Bornsztein, Pontoise, France; by Christopher J. Bradley, Clifton College, Bristol, UK; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by Toshio Seimiya, Kawasaki, Japan. We give the solution by Seimiya.



Since $BA = BC = a = \frac{1}{2}(BE + EF + FB)$, the excircle of $\triangle BEF$ opposite to B touches BA and BC at A and C respectively. Since $DA \perp AB$

and $DC \perp BC$, D is the excentre, so that DF and DE are the bisectors of $\angle AFE$ and $\angle FEC$, respectively.

Let T be the foot of the perpendicular from D to EF. Since $\angle AFD = \angle TFD$, and $\angle DAF = \angle DTF$ $(= 90°)$, we get

$$\angle ADF = \angle TDF .$$

Similarly we have $\angle CDE = \angle TDE$.

Therefore, $\angle ADF + \angle CDE = \angle TDF + \angle TDE = \angle EDF$. Thus, $\angle ADC = 2\angle EDF$.

Therefore, $\angle EDF = \frac{1}{2}\angle ADC = 45°$.

3. Find all pairs of consecutive integers the difference of whose cubes is a full square.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; by Christopher J. Bradley, Clifton College, Bristol, UK; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We use Bornsztein's write-up.

Let a, b be two integers.

$$\begin{aligned} (a+1)^3 - a^3 \;=\; b^2 &\iff 3a^2 + 3a + 1 \;=\; b^2 &(1)\\ &\iff 3(4a^2 + 4a + 1) + 1 \;=\; 4b^2 \\ &\iff (2b)^2 - 3(2a+1)^2 \;=\; 1 \\ &\iff (2b, 2a+1) \quad \text{is a solution} \end{aligned}$$

of Pell's equation

$$X^2 - 3Y^2 \;=\; 1 . \qquad (2)$$

The minimal non-trivial solution of (2) is $(2, 1)$. It is then well known that the solutions of (2) are the pairs $(\pm x_n, \pm y_n)$ where $x_0 = 1$, $y_0 = 0$, and for all $n \geq 0$

$$\left\{ \begin{array}{rcl} x_{n+1} &=& 2x_n + 3y_n \\ y_{n+1} &=& 2y_n + x_n \end{array} \right. .$$

But we want only those with x_n even and y_n odd.

It is easy to see that if x_n is even and y_n is odd then x_{n+1} is odd and y_{n+1} is even, and then x_{n+2} is even and y_{n+2} is odd.

Thus, since $x_1 = 2$ and $y_1 = 1$, we consider only the pairs (x_{2n+1}, y_{2n+1}). Since, for all $n \geq 0$

$$\left\{ \begin{array}{rcl} x_{n+2} &=& 7x_n + 12y_n \\ y_{n+2} &=& 4x_n + 7y_n \end{array} \right. .$$

Then, the solutions of (1) are the pairs (a, b) of the form

$$\left(\frac{-1 \pm V_n}{2}, \pm\frac{U_n}{2} \right) \quad \text{where} \quad U_1 \;=\; 2\,,\; V_1 \;=\; 1$$

and for all $n \geq 1$

$$\begin{cases} U_{n+1} &= 7U_n + 12V_n \\ V_{n+1} &= 4U_n + 7V_n \end{cases} .$$

For example, we first note that $(a+1)^3 - a^3 = b^2$ if and only if $(-a)^3 - (-a-1)^3 = b^2$. Then we give only the first positive values of a and b.

n	U_n	V_n	a	b
1	2	1	0	1
2	26	16	7	13
3	362	209	104	181
4	5042	2911	1455	2521
5	70226	40545	20272	35113
⋮				

4. Let $A_1, A_2, \ldots, A_n$ be a regular n–gon inscribed in the circle of radius 1 with the centre at O. A point M is given on the ray OA_1 outside the n–gon. Prove that

$$\sum_{k=1}^{n} \frac{1}{|MA_k|} \geq \frac{n}{|OM|} .$$

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We first give the solution by Bataille.

We may suppose that a system of coordinates has been chosen so that the complex numbers associated to $O, A_1, A_2, \ldots, A_n, M$ are respectively $0, 1, u, \ldots, u^{n-1}, r$, where $u = \exp(\frac{2\pi i}{n})$ and r is a real number > 1.

Note that $1, u, \ldots, u^{n-1}$ are the n^{th} roots of unity. Hence we have the identity

$$z^n - 1 = (z-1)(z-u)\ldots(z-u^{n-1}) . \tag{1}$$

Now,

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n} \frac{1}{|MA_k|} &= \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{|r-u^k|} \\ &\geq \sqrt[n]{\frac{1}{|r-1|} \cdot \frac{1}{|r-u|} \cdot \cdots \cdot \frac{1}{|r-u^{n-1}|}} \quad \text{(by AM-GM)} \\ &= \sqrt[n]{\frac{1}{|r^n-1|}} \quad \text{(using (1))} \\ &= \sqrt[n]{\frac{1}{r^n-1}} > \sqrt[n]{\frac{1}{r^n}} = \frac{1}{r} = \frac{1}{|OM|} . \end{aligned}$$

The result follows. [Ed.: Note that the proof gives a strict inequality.]

Next we give Klamkin's solution that provides some interesting connections.

By applying the AM-GM Inequality, it suffices to show that

$$|OM|^n \;\geq\; \prod_{k=1}^{n} |MA_k|\,. \qquad (1)$$

This will follow from *de Moivre's* property; that is, "if $A_0A_1A_2\ldots A_{n-1}$ is a regular polygon inscribed in a circle centre O, radius a, and P is a point such that $OP = x$, $\angle(OA_0, OP) = \theta$ then

$$\angle(OA_r, OP) \;=\; \theta + \frac{2r\pi}{n} \quad \text{and} \quad PA_r^2 \;=\; x^2 + a^2 - 2xa\cos\left(\theta + \frac{2r\pi}{n}\right).$$

Also,

$$PA_0^2 \cdot PA_1^2 \cdots PA_{n-1}^2 \;=\; \prod_{r=0}^{n-1}\left(x^2 - 2xa\cos\left(\theta + \frac{2r\pi}{n}\right) + a^2\right)$$

or

$$PA_0 \cdot PA_1 \cdots PA_{n-1} \;=\; \sqrt{x^{2n} - 2x^n a^n \cos n\theta + a^{2n}}\,.$$

If P lies on OA_0 so that $\theta = 0$, then $PA_0 \cdot PA_1 \cdots PA_{n-1} = |x^n - a^n|$. If OP bisects $\angle A_{n-1}OA_0$, so that $\theta = \frac{\pi}{n}$, then $PA_0 \cdot PA_1 \cdots PA_{n-1} = x^n + a^n$. These special results are called *Cotes'* properties".

Applying this to (1), we get $|OM|^n \geq |OM|^n - 1$, from which the result is now obvious.

For a simple derivation of the first *Cotes'* property using complex numbers, see the solution of problem 2 of the 15th Putnam Competition, 1955 [1].

Reference:

1. A.M. Gleason, R.E. Greenwood, L.M. Kelly, *The William Lowell Putnam Mathematical Competition: Problems and Solutions 1938–1964*, M.A.A., Washington, D.C., 1980, p. 403.

That completes the *Corner* for this number. Send me your contests, nice solutions, and comments.