THE ACADEMY CORNER

No. 45

Bruce Shawyer

All communications about this column should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7

In this issue, we present the hints and answers to the The Bernoulli Trials 2001, which we gave in [2001 : 353].

The Bernoulli Trials 2001
by
Ian VanderBurgh and Christopher G. Small

Hints:

1. To miss on the 501^{st} question, the first 500 questions must have been selected from the 1500 that were not too hard, and the 501^{st} from the 501 that were too hard.

2. Consider Tony and Maria as each running 2100 m at constant speeds (the first 1400 m being the "uphill" portion).

3.
$$\frac{EF}{BC} = \frac{AE}{AB} = \frac{AE/AD}{AB/AD} = \frac{\cos 30^\circ}{\sec 30^\circ}\,.$$

4. Note that
$$H_n = \frac{1}{n}\sum_{i=0}^{n-1} f\left(\frac{i}{n}\right)\,,$$
where $f(x) = 1/(1+3x)$. Show that $\lim H_n = (\ln 4)/3$. No calculators are allowed for the last step!

5. Since $2001 = 3 \times 23 \times 29$, it follows that $d(2001^n) = (n+1)^3$. Then, $(n+1)^3 = 2kn+1$ reduces to
$$n^2 + 3n + (3-2k) = 0\,.$$
Thus,
$$n = \frac{-3+\sqrt{8k-3}}{2}$$
which will be a positive integer provided $8k-3$ is a perfect square.

6. Divide the numbers into pairs as

$$\{2n-1, 2n\}$$

for $n = 1, \ldots, 1000$. This leaves 2001 by itself. Suppose Alice chooses 2001 as her starting number. For each number that Barbara chooses, Alice can choose the other number of the pair.

7. Let the side length of each of the solids be s. Notice that the octahedron is the combination two square based pyramids with all side lengths s. Therefore, the volume of the octahedron is

$$2 \times \frac{1}{3} \times (\text{base of pyramid}) \times (\text{height of pyramid}) .$$

Write this as a function of s. Show that the volume of the tetrahedron is

$$\frac{\sqrt{2}}{12}s^3 ,$$

which is 1, by assumption.

8. Consider

$$f(x) = \frac{2001}{x^2} .$$

9. Let $x =$ FOR and let $y =$ WAT. Then

$$6(1000x + y) = 7(1000y + x) .$$

This reduces to

$$5993x = 6994y .$$

But 5993 and 6994 are both divisible by 13. Thus, $461x = 538y$. However, 461 and 538 are relatively prime!

10. For the angles of a triangle,

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C .$$

Thus, if x, $1 + x$ and $1 - x$ were the tangents, then we would have

$$\begin{aligned} x + 1 + x + 1 - x &= x(1-x)(1+x) \\ x + 2 &= x - x^3 \\ x^3 &= -2 \\ x &= -\sqrt[3]{2} . \end{aligned}$$

Thus, $x < 0$ and $1 + x < 0$. Is this possible?

11. This would be trivial with an illegal calculator. However, ten minutes is enough time to number crunch this, even without a calculator. To do so is to miss the fact that there is a more mathematically interesting argument.

Obviously there are no solutions for $0 < x \leq 1$. Since

$$f(x) \;=\; x^{x^{2001}}$$

is increasing for $x > 1$, there is at most one solution for $x > 1$. It is easy to check that

$$x \;=\; 2001^{1/2001}$$

is a solution. But

$$2001^{1/2001} \;=\; \left(\sqrt[3]{2001}\right)^{1/667} \;>\; 12.5^{1/667} \;>\; 12.5^{0.0014}\,.$$

12. We can write

$$\begin{aligned}
&\lfloor(1+\sqrt{3})^{2001}\rfloor \\
&\quad= \;(1+\sqrt{3})^{2001} + (1-\sqrt{3})^{2001} \\
&\quad= \;(1+\sqrt{3})(4+2\sqrt{3})^{1000} + (1-\sqrt{3})(4-2\sqrt{3})^{1000} \\
&\quad= \;2^{1000}[(1+\sqrt{3})(2+\sqrt{3})^{1000} + (1-\sqrt{3})(2-\sqrt{3})^{1000}]\,.
\end{aligned}$$

Can an additional factor of 2 be pulled out?

Answers:

1.	True.	2.	False.	3.	True.	4.	False.
5.	False.	6.	False.	7.	True.	8.	False.
9.	True.	10.	False.	11.	True.	12.	True.