MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 977 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z7 (NEW!)**. The electronic address is

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Mayhem Problems

Proposals and solutions may be sent to MATHEMATICAL MAYHEM, c/o Faculty of Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, ON, N2L 3G1, or emailed to

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Please include on any correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 March 2002*. Look for prizes for solutions in the new year.

M15. *Proposed by Richard Hoshino, Dalhousie University, Halifax, Nova Scotia.*

Let $ABCDEF$ be a regular hexagon with area S. Let T be the area of the hexagonal region common to both $\triangle ACE$ and $\triangle BDF$. Determine $\frac{S}{T}$.

M16. *Proposed by the Mayhem staff.*

Can an 8×9 checkerboard be completely covered by twelve 1×6 rectangles?

M17. *Proposed by Andy Liu, University of Alberta, Edmonton, Alberta.*

Seven theoretical and eleven experimental physicists were working in a low-temperature laboratory. Each day, a different physicist is in charge, and the cycle repeats when everyone has had a turn. One of the privileges of being in charge is the control of the thermostat, initially set at 0°C. When a theoretical physicist is in charge, she raises the temperature by 1.1°C. When an experimental physicist is in charge, he lowers it by 0.7°C. What is the probability that in an eighteen-day cycle, the temperature is below 0° for exactly nine days?

M18. *Proposed by Richard Hoshino, Dalhousie University, Halifax, Nova Scotia.*

For each integer n, let n^* be the integer n written backwards. For example if $n = 1234$ then $n^* = 4321$. We say that a four-digit integer n is ***magical*** if both $n + n^*$ and $n - n^*$ are (positive) palindromes. For example, 2001 is magical. If n is magical, determine all possible values of $n - n^*$.

M19. *Proposed by the Mayhem staff.*

On the magical island of Xurc, there lives a giant Ecurb. Ecurb has an unlimited supply of special coins that are worth one million dollars each. Ecurb allows people to go into his castle and take as many of these coins as they like, but, they must give some up in order to cross the bridges to leave his island. At each of the five bridges Ecurb demands that you give $\frac{99}{100}$ of a coin more than $\frac{99}{100}$ of the coins in your possession. Coins cannot be cut or broken in any way. If the demand cannot be met Ecurb takes all of your coins and eats one of your feet. How many coins do you have to start with in order to make it off the island with exactly one coin (and both feet)?

M20. *Proposed by Richard Hoshino, Dalhousie University, Halifax, Nova Scotia.*

Suppose that A, B and C are positive integers in arithmetic progression with $A^\circ < B^\circ < C^\circ < 180^\circ$.

If $\sin A^\circ + \sin B^\circ = \sin C^\circ$ and $\cos A^\circ - \cos B^\circ = \cos^\circ C$, determine the triplet (A, B, C).

M21. *Proposed by the Mayhem staff.*

Find all positive integers a, b, c, d, and e which satisfy

$$a! \;=\; b! + c! + d! + e!\,.$$

Problem of the Month

Jimmy Chui, student, University of Toronto

Problem. The sum of the series $\frac{25}{72} + \frac{25}{90} + \frac{25}{110} + \frac{25}{132} + \cdots + \frac{25}{9900}$ is

(a) 2.25 (b) 2.5 (c) 2.8125 (d) 2.875 (e) $3.044\dot{1}\dot{9}$

(1996 Cayley, Problem 24)

Solution. In this problem, we want to find a pattern, and hope that we can find a method to simplify the problem, and in effect, answer it.

The numerators are all equal to 25, and the denominators are $72 = 8 \cdot 9$, $90 = 9 \cdot 10$, $110 = 10 \cdot 11$, $132 = 11 \cdot 12$, $\cdots$, $9900 = 99 \cdot 100$.

If we try adding up the first two fractions, we get

$$\frac{25}{72} + \frac{25}{90} \;=\; 25 \cdot \left(\frac{10}{720} + \frac{8}{720}\right) \;=\; 25 \cdot \frac{18}{720} \;=\; 25 \cdot \frac{1}{40}\,.$$

If we try adding up the first three fractions, we have

$$\frac{25}{40} + \frac{25}{110} \;=\; 25 \cdot \left(\frac{11}{440} + \frac{4}{440}\right) \;=\; 25 \cdot \frac{15}{440} \;=\; 25 \cdot \frac{3}{88}\,.$$

There is a pattern emerging! The first fraction itself is $25 \cdot \frac{1}{72} = 25 \cdot \frac{1}{8 \cdot 9}$.

The first two fractions have a sum of $25 \cdot \frac{1}{40} = 25 \cdot \frac{2}{8 \cdot 10}$.

The first three fractions have a sum of $25 \cdot \frac{3}{88} = 25 \cdot \frac{3}{8 \cdot 11}$.

By examining this pattern, we can guess that the answer for the entire sum is

$$25 \cdot \frac{92}{8 \cdot 100} \;=\; \frac{92}{8 \cdot 4} \;=\; \frac{23}{8} \;=\; 2.875\,.$$

Hence, with pretty good confidence, we can say that the answer is (d)!

A more mathematical argument would show why the pattern works. For problems of this type (adding terms that exhibit a pattern), one possible approach is to look for some sort of telescoping series.

If we recognize that $\frac{1}{8 \cdot 9} = \frac{1}{8} - \frac{1}{9}$, $\frac{1}{9 \cdot 10} = \frac{1}{9} - \frac{1}{10}$, $\dots$, $\frac{1}{99 \cdot 100} = \frac{1}{99} - \frac{1}{100}$, then the expression is evaluated very quickly. Indeed, we have

$$\begin{aligned} \frac{25}{72} + \frac{25}{90} + \cdots + \frac{25}{9900} &= 25 \cdot \left\{\left(\frac{1}{8} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{10}\right) + \cdots + \left(\frac{1}{99} - \frac{1}{100}\right)\right\} \\ &= 25 \cdot \left(\frac{1}{8} - \frac{1}{100}\right) \\ &= 25 \cdot \frac{92}{8 \cdot 100}\,, \end{aligned}$$

which is identical to what we guessed above, and evaluates to 2.875.

Polya's Paragon

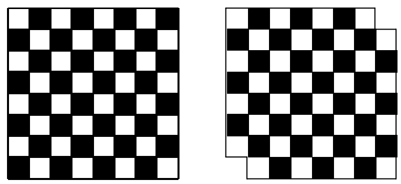
Shawn Godin

"It is amazing how a splash of colour makes anything look fresh and new," my wife says to me, handing me a can of paint. Walls of a bedroom, a bench on the porch, or the deck stand out with their new clean look. In many cases a splash of colour can shed some light on a mathematical problem as well.

As an example, let us look at a well-known problem that has appeared in many books:

A checkerboard is an 8×8 grid of squares. If a domino covers exactly 2 squares, we can easily cover the 64 squares with 32 dominoes. Suppose that two diagonally opposite squares are removed, can the new board be covered by 31 dominoes?

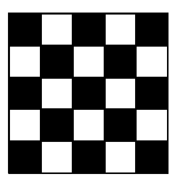
If we colour the checkerboard in the standard fashion, as in the diagram below, we notice that the opposite corners are the same colour. Thus, the board started with 32 black squares and 32 white squares, but ended with 30 black squares and 32 white squares. Since a domino covers a black and a white square we can fit only 30 dominoes on the board, and 2 white squares will be left uncovered.



Consider the following related problem.

A class of 25 students is arranged in 5 rows and 5 columns. If a student can move to a desk directly in front, behind, to the left or to the right of his own desk, can all students move and occupy a new desk?

It would seem that you should be able to do it, but if we colour the desks in the same pattern as the checkerboard we notice that there are 13 black desks and 12 white desks. The allowed moves take a person to a desk of a different colour. Thus, we will not be able to place all of students that started in black desks, and our task is impossible.



This method of colouring is closely linked to the idea of **parity**, the evenness or oddness of a number. In the last example, if we numbered the desks 1 through 25, as below, we see that a person starting in an even numbered desk must move to an odd numbered desk and vice versa. Since we started with 12 even and 13 odd numbered desks one person would be left out seatless or unable to change.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

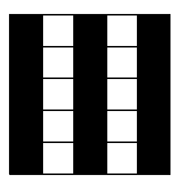
One of the problems with colouring proofs is that they are generally good for proving that something is impossible, but they do not prove that something is possible although, in some cases, it may give us a hint.

Consider again the class of 25 students in 5 rows and 5 columns. Imagine this time that they must move to a neighbour diagonally. Using the previous colouring we see that we remain on the same colour (or alternately, if we had an even numbered seat, we move to another even numbered seat, and similarly for odd numbered seats). You may be convinced at this time that the task is thus possible. Go ahead and try to do it, I will just wait here . . . .

Back so soon? What is that, you have had no luck? That is too bad. Sit down, and let us take another look. Suppose we decided to be really artistic and paint our classroom desks in four different colours. I will number the colours 1, 2, 3 and 4 because they will not let me print this section of Mayhem in multiple colours! The things a true *artist* has to deal with!

1	2	1	2	1
4	3	4	3	4
1	2	1	2	1
4	3	4	3	4
1	2	1	2	1

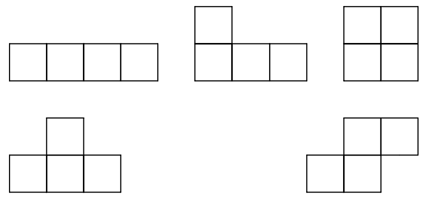
We see here that anything that started on a 1 (which I would have coloured teal) ends up on a 3 (chartreuse) and vice versa. Similarly the number 2's (magenta) swap places with the number 4's (periwinkle). The even numbers pose no problems; there are 6 of each which seems OK. But there are nine teal coloured squares (1's) and only 4 chartreuse (3's). Thus we see that this too is an impossibility. (For a less artistic colouring you can consider the one below).



Here are some problems to ponder:

1. Can an 8×8 board be covered with twenty one 3×1 rectangles and one 1×1 rectangle? If so, find a tiling.

2. A tetromino is four squares glued together along the edges. There are 5 tetrominoes pictured below. Can a 4×5 board be covered by one complete set of tetrominoes? If so, find a tiling.

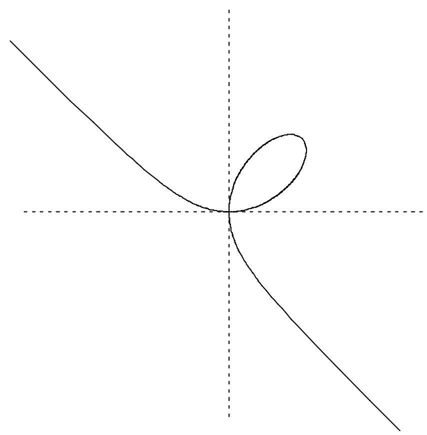


3. A rectangular floor is covered by 3×3 and 1×9 tiles. One of the tiles got smashed, but there is a tile of the other type available. Show that the floor cannot be covered by rearranging the tiles.

Challenge Board Solutions

David Savitt

C95. Prove that the curve $x^3 + y^3 = 3xy$, has a horizontal tangent at the origin. (This curve is known as the Folium of Descartes.)



Solution.

Consider the map P from the (X, Y)–plane to the (x, y)–plane given by sending (X, Y) to (X, XY).

Lemma. If $Y = f(X)$ is a C^1 function and $f(0) = 0$, then the image of the graph of f under the map P has a horizontal tangent at the origin.

Proof. For each point $(X, f(X))$ of the graph of f in the (X, Y)–plane, the image $P(X, f(X))$ in the (x, y)–plane is the point $(X, Xf(X))$. Hence the image under P of the graph of f is the graph $y = xf(x)$. This function passes through the origin, and since f is C^1, we may check that this graph has a horizontal tangent at the origin by differentiating: indeed, $y' = xf'(x) + f(x)$, and since $f(0) = 0$ we evidently have $y'(0) = 0$.

Now we can solve the given problem. Consider the function $X = g(Y) = 3Y/(1+Y^3)$. It is continuously differentiable and one-to-one in a neighbourhood of the origin, and $g'(0) \neq 0$, so by the Inverse Function Theorem, we may write $Y = f(X)$ with f a C^1 function in a neighbourhood of the origin, and $f(0) = 0$. By the lemma, the image of the graph $(X, f(X))$ under P in a neighbourhood of the origin — and so the image of the graph of $(g(Y), Y)$ under P as well — has a horizontal tangent at the origin.

But the points of the graph of $X = g(Y)$ satisfy $X(1 + Y^3) = 3Y$. Multiplying through by X^2, we get $X^3 + (XY)^3 = 3X(XY)$, so that we see that the image of this graph under P satisfies $x^3 + y^3 = 3xy$. Hence the latter has a horizontal tangent at the origin.

C96. Recall that a *bipartite graph* is a graph whose vertices may be divided into two non-empty disjoint sets (call them L and R, for left and right) so that all of the edges of the graph connect a vertex in L to a vertex in R. In other words, no two vertices in L are joined by an edge, and similarly for R. Let G be a bipartite graph with 27 edges and in which L and R each contain exactly 9 vertices. Show that we can find three vertices l_0, l_1, $l_2 \in L$ and three vertices r_0, r_1, $r_2 \in R$ such that at least six of the nine potential edges l_0r_0, l_0r_1, l_0r_2, l_1r_0, l_1r_1, l_1r_2, l_2r_0, l_2r_1, l_2r_2 are indeed edges of G.

Solution.

Assume a counterexample. We will steadily eliminate possibilities for the list of degrees of the vertices. (Recall that the degree of a vertex is the number of edges touching that vertex.) As this list becomes more and more restricted, the problem becomes more and more rigid, until it is easy to see that a counterexample cannot exist. We will refer to the vertices l_m as left-hand vertices, and to the r_i as right-hand vertices.

We begin by eliminating the possibility that a vertex has large degree. For example, suppose l_1 has degree exactly 5, with edges from l_1 to each of $r_1, \ldots, r_5$. If some other l_m has edges to two vertices r_i and r_j with $1 \leq i, j \leq 5$, then no other l_n could have an edge to any r_k with $1 \leq k \leq 5$ or else there would be six edges between l_1, l_m, l_n, r_i, r_j, and r_k. Hence there are at most 8 edges from $l_2, \ldots, l_9$ to $r_1, \ldots, r_5$, and the total of the degrees of r_1 through r_5 is at most 13. It follows that $r_6, \ldots, r_9$ have degrees summing to at least 14, and therefore the three of those with largest degree have degrees summing to at least 11. But these 11 edges connect to the eight vertices l_2 through l_9 and not to l_1, so by the Pigeonhole Principle, at least 6 of these 11 edges connect to a subset of only three left-hand vertices. Thus, we have found the expected three left-hand and three right-hand vertices with 6 edges between them. Therefore, there is no vertex of degree exactly 5. One can use (easier versions of) the same argument to see as well that no vertex may have degree larger than 5.

Next, we show that there cannot be two left-hand vertices of degree 4. Certainly there can be at most two: if there were three left-hand vertices of degree 4, then these three vertices would have total degree 12, and so by the same pigeonhole argument as above, there would be three right-hand vertices with 6 edges to these three left-hand vertices. Therefore, if there are two left-hand vertices of degree 4, there must also be at least five left-hand vertices of degree exactly 3. Suppose first that l_1 connects to $r_1, \ldots,$ r_4 and l_2 connects to $r_5, \ldots, r_8$. If some other l_m connected to three of $r_1, \ldots, r_8$, then those three right-hand vertices, together with l_1, l_2, and l_m, would provide the sought-for subgraph. Therefore each of the five l_m of degree exactly 3 must have an edge to r_9. This is a contradiction, since r_9 would then have degree at least 5. Hence the neighbourhoods of l_1 and l_2 must not be disjoint. (Recall that the neighbourhood of a vertex is the set of vertices to which it is joined by an edge.)

However, if l_1 connects to $r_1, \ldots, r_4$ and l_2 connects to $r_4, \ldots, r_7$, this leaves 19 edges emanating from $l_3, \ldots, l_9$, at most 9 of which may connect to r_8 and r_9. (Otherwise, since neither joins to l_1 or l_2, the Pigeonhole Principle would guarantee that the neighbourhoods of r_8 and r_9 have intersection of size at least three, giving the expected subgraph with 6 edges.) This leaves at least 10 edges joining $l_3, \ldots, l_9$ to $r_1, \ldots, r_7$. Evidently (more than) one l_m joins to two vertices r_i and r_j, with $3 \leq m \leq 9$ and $1 \leq i, j \leq 7$. Then there are six edges between l_1, l_2, l_m and r_i, r_j, r_4 (or, if either $i = 4$ or $j = 4$, between l_1, l_2, l_m and r_i, r_j, r_k with k distinct from i and j and $1 \leq k \leq 7$). This shows that the neighbourhoods of l_1 and l_2 must have at least two vertices in common. Yet if l_1 connects to $r_1, \ldots, r_4$ and l_2 connects to $r_3, \ldots, r_6$, it is easy to see that more than one r_i, $i = 1$, 2, 5, 6, must connect to a vertex l_m besides l_1 and l_2 (for example, because otherwise there would be three vertices of degree 2, requiring either a vertex of degree 5 or three of degree 4). Then there are six edges between l_1, l_2, l_m and r_3, r_4, r_i. Since plainly the neighbourhoods of l_1 and l_2 cannot have three vertices in common, this concludes the demonstration that there are not two left-hand vertices of degree exactly 4.

This leaves only two possibilities for the list of degrees of the left-hand (and, similarly, the right-hand) vertices: $4, 3, 3, 3, 3, 3, 3, 3, 2$ and $3, 3, 3, 3, 3, 3, 3, 3, 3$. Either way, there are at least seven left-hand vertices of degree exactly 3, at most four of which can connect to a vertex of degree 4, and at most two of which can connect to a vertex of degree 2. Therefore, there exists a vertex of degree 3 which connects to three vertices of degree 3; without loss of generality, suppose that l_1 connects to r_1, r_2, r_3, all of degree exactly 3. If, say, r_1 and r_2 both connected to l_m, with $m \neq 1$, then choosing any other l_n connected to r_1, r_2, or r_3 gives 6 edges between l_1, l_m, l_n and r_1, r_2, r_3. Without loss of generality, we can therefore suppose that r_1 connects to l_1, l_2, l_3, that r_2 connects to l_1, l_4, l_5, and that r_3 connects to l_1, l_6, l_7. If some other r_i were connected to two l_m and l_n out of $l_2, \ldots, l_7$, then if l_m and l_n connect to r_j and r_k respectively, $1 \leq j, k \leq 3$, we would have six edges between l_1, l_m, l_n and r_i, r_j, r_k. Therefore each of $r_4, \ldots, r_9$ connects to at most one of $l_1, \ldots, l_7$, and the total of the degrees of l_1 through l_7 is at most $9 + 6 = 15$. Hence the degrees of l_8 and l_9 sum to at least 12. This is certainly a contradiction, completing our solution.

C97. Given a positive integer n, let $\overline{0}, \overline{1}, \ldots, \overline{n-1}$ denote the integers modulo n (so that $\overline{a}$ is the reduction of a modulo n). Find all positive integers n with the property that the set

$$\{\overline{a} \mid 0 < a < n/2 \text{ with } a \text{ and } n \text{ relatively prime}\}$$

is a group under multiplication.

Solution by Kiran Kedlaya, University of California-Berkeley, Berkeley, CA, USA.

If $n = 2k+1$ is odd and $n \geq 5$, then 2 and k are relatively prime to n and lie below $n/2$, while the reduction mod n of $2 \cdot k$ is $\overline{n-1}$. Therefore, the only odd n with the desired property is $n = 3$.

Henceforth, assume n is even and greater than 2. If n satisfies the desired condition, then every integer in the interval $(n/2, n)$ which is coprime to n must in fact be a prime, and therefore must divide $\binom{n}{n/2}$. It follows that

$$2^n \;\geq\; \binom{n}{n/2} \;\geq\; (n/2)^{\phi(n)/2},$$

where ϕ denotes Euler's ϕ-function: $\phi(n)$ is the number of integers between 1 and n which are relatively prime to n. Now

$$\phi(n) \;=\; n \prod_{i=1}^{k} \left(1 - \frac{1}{p_k}\right),$$

where $p_1, \ldots, p_k$ are the distinct primes dividing n. This product is bounded below by

$$\left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right) \cdots \left(1 - \frac{1}{k+1}\right) \;=\; \frac{1}{k+1},$$

and combining this with the previous inequality yields

$$2^n \;\geq\; (n/2)^{n/2(k+1)}$$

or

$$2k + 3 \;\geq\; \log_2(n).$$

To bound k from above in terms of n, we have the inequality $n \geq 2^k$. This will not be of any help, but similarly we always have $n \geq 2 \cdot 3^{k-1}$, and $n \geq 2 \cdot 3 \cdot 5^{k-2}$, and so on. For example, $n \geq 2 \cdot 3 \cdots 13 \cdot 17^{k-6}$, and therefore,

$$k \;\leq\; 6 + \log_{17}(n/30030).$$

Hence,

$$15 + 2\log_{17}(n/30030) \;\geq\; \log_2 n$$

which amounts to $7.7220\ldots \geq 0.7367\ldots \cdot \log_e(n)$, yielding

$$n \;\leq\; 35625\,.$$

With this tractable upper-bound on n, we can apply a computer search. For example, we have already observed that if n has the desired property, then every integer between $n/2$ and n which is coprime to n must be prime. Thus, if $\pi(x)$ denotes the number of primes less than or equal to x, we must have $\pi(n) - \pi(n/2) = \phi(n)/2$ for $n > 2$. For integers in the range under consideration, these functions are quickly computed, and a one-line program in any mathematical computation package (such as Maple, Mathematica, etc.) yields the possibilities $n = 4, 6, 8, 12, 18, 20, 24, 30$. Of these, both $n = 18$ ($7 \cdot 7 = 49 \equiv 13$) and $n = 30$ ($11 \cdot 13 = 143 \equiv 23$) fail, but the remainder succeed. Hence, the complete list of integers which satisfy the conditions of the problem is $n = 2, 3, 4, 6, 8, 12, 20, 24$.

C98. Find all pairs of integers (x, y) which satisfy the equation

$$x^2 - 34y^2 = -1\,.$$

Solution.

The equation has no integer solutions. We will examine the equation $x^2 - 34y^2 = \pm 1$, and we will make a great deal of use of the factorization

$$(x + \sqrt{34}y)(x - \sqrt{34}y) \;=\; \pm 1\,.$$

Notice that if (x, y) is a solution to the equation, then so are $(\pm x, \pm y)$. Suppose that x, $y > 0$. Then $x + y\sqrt{34} > 1$, so

$$|x - y\sqrt{34}| \;=\; 1/|x + y\sqrt{34}| \;<\; 1\,.$$

It follows that if $x^2 - 34y^2 = \pm 1$, then $x + \sqrt{34}y$ is greater than 1 if and only if x and y are both positive (and less than -1 if and only if they are both negative, and between -1 and 1 if and only if they have mixed signs).

It follows that if $x^2 - 34y^2 = \pm 1$ has any solution with $x + \sqrt{34}y > 1$, there must be such a solution (X, Y) with $X + \sqrt{34}Y > 1$ minimal. For example, $35^2 - 34 \cdot 6^2 = 1$, and $35 + 6 \cdot \sqrt{34} < 70$. Therefore, any solution (x, y) with smaller $x + \sqrt{34}y > 1$ would have to have $1 \leq x < 70$ and $1 \leq y \leq 70/\sqrt{34} < 13$; since there are only a finite number of possibilities, a minimal solution must exist. It is easy to verify that, in fact, $\epsilon = 35 + 6 \cdot \sqrt{34}$ is minimal. Note crucially that $35^2 - 34 \cdot 6^2 = 1$, not -1.

Now if (x, y) is any solution with x, $y > 0$, if $(x, y) \neq (35, 6)$, we have $x + y\sqrt{34} > \epsilon$. Considering the product

$$\begin{aligned}(x + y\sqrt{34})\epsilon^{-1} &= (x + y\sqrt{34})(35 - 6\sqrt{34}) \\ &= (35x - 204y) + (35y - 6x)\sqrt{34} > 1\end{aligned}$$

we find that $(35x - 204y, 35y - 6x)$ must be a smaller solution with $35x - 204y, 35y - 6x > 0$. Repeating this process, it follows eventually that $(x + y\sqrt{34})\epsilon^{-n} = \epsilon$; that is, $x + y\sqrt{34} = (35 + 6\sqrt{34})^{n+1} = \epsilon^{n+1}$. But then $x - y\sqrt{34} = (35 - 6\sqrt{34})^{n+1} = \epsilon^{-(n+1)}$, so $x^2 - 34y^2 = \epsilon^{n+1}\epsilon^{-(n+1)} = 1$. It follows, easily, that the equation $x^2 - 34y^2 = \pm 1$ has solutions only such that $x^2 - 34y^2 = 1$.

Remark 1.

This equation is a special case of a well-studied class of equations known as Pell's equation: $x^2 - Dy^2 = \pm 1$. The reason this special case is of particular interest is that while in fact $x^2 - 34y^2 = -1$ has no integer solutions, it *does* have rational solutions, for example $(3/5)^2 - 34(1/5)^2 = -1$. The existence of this rational solution, combined with a little bit of work for powers of 5, implies that $x^2 - 34y^2 = -1$ has a solution modulo n for any integer n. Thus we have found an example of an equation with an integer solution mod n for any n, but no integer solution. (Incidentally, while Pell may have studied this equation, it appears that Euler mis-attributed to Pell a solution due to William Brouncker. Thus, the equation is likely ill-named!)

Remark 2.

The argument we have given shows in general that for Pell's equation $x^2 - Dy^2 = \pm 1$, if there exists at least one non-trivial solution, then there is a positive solution (X, Y) so that $\epsilon = X + Y\sqrt{D} > 1$ is minimal, and every solution (x, y) of the equation is given via $x + \sqrt{D}y = \pm\epsilon^n$ for integer n. Moreover, the argument shows that a solution with $x^2 - Dy^2 = -1$ exists only when $X^2 - DY^2 = -1$; in that case, the solutions $x^2 - Dy^2 = -1$ correspond precisely to odd powers of ϵ.

Remark 3.

In fact for any D, Pell's equation always has a non-trivial solution. For example, one way to find the solution $35^2 - 34 \cdot 6^2 = 1$ is as follows: the continued fraction expansion of $\sqrt{34}$ is $[5, 1, 4, 1, 10, 1, 4, 1, 10, \dots] = [5, \overline{1, 4, 1, 10}]$, and the minimal solution ϵ corresponds to the convergent

$$[5, 1, 4, 1] = 5 + \cfrac{1}{1 + \frac{1}{4 + \frac{1}{1}}} = 35/6\,.$$

The fact that the solution yields $+1$ instead of -1 corresponds to the fact that the number of terms in this convergent is even. It is, so far as I know, an unsolved problem to characterize those D for which the continued fraction of $\sqrt{D}$ has odd length; that is, for which $x^2 - Dy^2 = -1$ has a solution.

C99. Find all collections of polynomials $p_{11}, p_{12}, p_{21}, p_{22}$ with complex coefficients satisfying the relation

$$\begin{pmatrix} p_{11}(XY) & p_{12}(XY) \\ p_{21}(XY) & p_{22}(XY) \end{pmatrix} = \begin{pmatrix} p_{11}(X) & p_{12}(X) \\ p_{21}(X) & p_{22}(X) \end{pmatrix} \cdot \begin{pmatrix} p_{11}(Y) & p_{12}(Y) \\ p_{21}(Y) & p_{22}(Y) \end{pmatrix}.$$

Solution.
We begin with the observation that if

$$f(XY) \;=\; \sum_{i=1}^{n} g_i(X)h_i(Y)\,,$$

then the polynomial f has at most n non-zero terms. To see this, first note that for any polynomial in two variables $q(X,Y) = \sum_{j,k} a_{j,k}X^jY^k$ we may consider the matrix of coefficients $(a_{j,k})$. For the products $g_i(X)h_i(Y)$, this matrix has rank 1, and so the matrix of coefficients of the polynomial $\sum_{i=1}^{n} g_i(X)h_i(Y)$ has rank at most n. However, the rank of the matrix associated to $f(XY)$ is exactly equal to the number of non-zero terms of f, and so this number is at most n.

For example, suppose $f(XY) = f(X)f(Y)$. Applying our observation, either $f = 0$ or f is a monomial $f(X) = cX^n$, and in the latter case we see $c(XY)^n = cX^ncY^n$ and so $c = 1$. Hence $f(X) = 0$ or $f(X) = X^n$ for some non-negative integer n.

Let $P(X)$ denote our two-by-two matrix of polynomials

$$\begin{pmatrix} p_{11}(X) & p_{12}(X) \\ p_{21}(X) & p_{22}(X) \end{pmatrix} .$$

Then $P(X)P(Y) = P(XY)$ implies that $P(1)^2 = P(1)$. This leaves three possibilities for the minimal polynomial for $P(1)$: the minimal polynomial is either $P(1) = 0$, $P(1)(P(1) - I) = 0$, or $P(1) - I = 0$. In the middle case, $P(1)$ has eigenvalues 0 and 1, and therefore, there exists a matrix A with complex entries such that

$$P(1) \;=\; A^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A\,.$$

We now consider the three cases in turn.

Case 1. If $P(1) = 0$, then $P(X) = P(X)P(1) = 0$. Hence P is identically 0.

Case 2. If $P(1)$ satisfies

$$P(1) \;=\; A^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A\,,$$

we consider the conjugate matrix $Q(X) = AP(X)A^{-1}$. Then it is still the case that $Q(XY) = Q(X)Q(Y)$, and now

$$Q(1) \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} .$$

But then

$$Q(X) \;=\; Q(1)Q(X) \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \;=\; \begin{pmatrix} q_{11} & q_{12} \\ 0 & 0 \end{pmatrix}$$

and similarly

$$Q(X) \;=\; Q(X)Q(1) \;=\; \begin{pmatrix} q_{11} & 0 \\ q_{21} & 0 \end{pmatrix}$$

from which it follows that

$$Q(X) \;=\; \begin{pmatrix} q_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

and $q_{11}(XY) = q_{11}(X)q_{11}(Y)$. By our earlier observation, it follows that $q_{11}(X) = X^m$ for some non-negative m, and in summary we have obtained:

$$P(X) \;=\; A^{-1}\begin{pmatrix} X^m & 0 \\ 0 & 0 \end{pmatrix} A$$

for a complex matrix A.

Case 3. If we have $P(1) = I$, our solution follows the method of Case 2, but is more complicated. In this case $p_{ij}(XY) = p_{i1}(X)p_{1j}(Y) + p_{i2}(X)p_{2j}(Y)$, and by our initial observation, it follows that each p_{ij} has at most two terms. However, we can say even more: for an invertible complex matrix A, the conjugate $Q(X) = A^{-1}P(X)A$ still satisfies $Q(XY) = Q(X)Q(Y)$, so the entries of $A^{-1}P(X)A$ must also have at most two terms each. Writing down the entries of $A^{-1}P(X)A$ explicitly in terms of the entries of A, it is easy to see that among the various terms of $p_{11}, p_{12}, p_{21}, p_{22}$ there can be terms of at most two different degrees—otherwise, it would be possible to arrange a choice of A so that $A^{-1}P(X)A$ had an entry which had three terms.

Each p_{ij} may therefore be written $p_{ij} = a_{ij}X^m + b_{ij}X^n$ for some common pair of distinct integers m and n. Using $P(1) = I$, we see moreover that $P(X)$ can be written

$$P(X) \;=\; \begin{pmatrix} aX^m + (1-a)X^n & b(X^m - X^n) \\ c(X^m - X^n) & dX^m + (1-d)X^n \end{pmatrix},$$

which we rewrite as

$$P(X) \;=\; X^n I + (X^m - X^n)M$$

for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Expanding the condition $P(XY) = P(X)P(Y)$ in terms of the above expression, we quickly obtain

$$(X^m - X^n)(Y^m - Y^n)M^2 \;=\; (X^m - X^n)(Y^m - Y^n)M\,,$$

and thus, $M^2 = M$. As earlier, it follows that either $M = 0$, $M = I$, or $M = A^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A$. In these three cases we see, respectively, that $P(X) = X^n I$, $P(X) = X^m I$, or

$$P(X) \;=\; A^{-1}\left(X^n I + (X^m - X^n)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) A \;=\; A^{-1}\begin{pmatrix} X^m & 0 \\ 0 & X^n \end{pmatrix} A\,.$$

In summary, we have shown that $P(X)$ must be of the form:

$$P(X) = 0,$$

$$P(X) = A^{-1}\begin{pmatrix} X^m & 0 \\ 0 & 0 \end{pmatrix} A,$$

or

$$P(X) = A^{-1}\begin{pmatrix} X^m & 0 \\ 0 & X^n \end{pmatrix} A,$$

for some invertible complex matrix A.

C100. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Let $x_1, x_2, \ldots, x_n$ be positive real numbers, let $S = \sum_{k=1}^{n} x_k$, and suppose that $(n-1)x_k < S$ for all k. Prove that

$$\prod_{k=1}^{n} (S-(n-1)x_k) \leq \prod_{k=1}^{n} x_k.$$

When does equality occur?

Solution.

(Solved by Michel Bataille, Rouen, France, and David Loeffler, student, Trinity College, Cambridge, UK)

Observe that

$$\sum_{k=1}^{n}(S-(n-1)x_k) = nS-(n-1)\sum_{k=1}^{n} x_k = S,$$

and therefore,

$$\sum_{k\neq j}(S-(n-1)x_k) = S-(S-(n-1)x_j) = (n-1)x_j.$$

Noting that each term in the sum is positive, it follows from the AM–GM inequality that

$$x_j \geq \prod_{k\neq j}(S-(n-1)x_k)^{1/(n-1)}.$$

Multiplying together these inequalities for $j = 1, \ldots, n$, yields the desired inequality.

To determine when equality holds, note from our use of the AM–GM inequality that we must have $x_j = S-(n-1)x_k$ for all $k \neq j$. This implies that any selection of $n-1$ of the x_k's must all be equal, and so for $n > 2$, equality holds if and only if the x_k are all equal. Moreover, one checks easily that equality always holds if $n = 1$ or $n = 2$.