SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2555. [2000 : 304] *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

In any triangle ABC, show that

$$\sum_{\text{cyclic}} \frac{1}{\tan^3\frac{A}{2} + \left(\tan\frac{B}{2} + \tan\frac{C}{2}\right)^3} \;<\; \frac{4\sqrt{3}}{3}\,.$$

Solution by Henry Liu, Trinity College, Cambridge, England.

The function x^3 is convex on $(0, +\infty)$. Thus, by Jensen's inequality,

$$\begin{aligned} \frac{x^3+y^3}{2} &\ge \left(\frac{x+y}{2}\right)^3 \\ \Longrightarrow\quad \frac{1}{x^3+y^3} &\le \frac{4}{(x+y)^3}\,. \end{aligned}$$

Similarly, the function $\tan x$ is convex on $\left(0, \frac{\pi}{2}\right)$, so that

$$\begin{aligned} \frac{\tan x + \tan y + \tan z}{3} &\ge \tan\left(\frac{x+y+z}{3}\right) \\ \Longrightarrow\quad \frac{1}{\tan x + \tan y + \tan z} &\le \frac{1}{3\tan\left(\frac{x+y+z}{3}\right)}\,. \end{aligned}$$

Using these, we have

$$\begin{aligned} \sum_{\text{cyclic}} \frac{1}{\tan^3\frac{A}{2} + \left(\tan\frac{B}{2} + \tan\frac{C}{2}\right)^3} &\le \frac{3\cdot 4}{\left(\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}\right)^3} \\ &\le \frac{12}{27\tan^3\left(\frac{1}{3}\left(\frac{A}{2}+\frac{B}{2}+\frac{C}{2}\right)\right)} \\ &= \frac{12}{27\tan^3\left(\frac{\pi}{6}\right)} = \frac{12}{27\cdot\frac{1}{3\sqrt{3}}} = \frac{4\sqrt{3}}{3}\,. \end{aligned}$$

Equality holds in the second inequality when $A = B = C = \frac{\pi}{3}$, and in the first inequality when $\tan\frac{A}{2} = \tan\frac{B}{2} + \tan\frac{C}{2}$; clearly, these cannot both occur at once. Therefore, the given inequality must be strict.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; RICHARD B. EDEN, Ateneo de Manila University, Philippines; WALTHER

JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Most of the proposed solutions make use of Jensen's inequality. Seiffert has proven the following generalization. For $p > 1$,

$$\sum_{\text{cyclic}} \frac{1}{\tan^p \frac{A}{2} + \left(\tan \frac{B}{2} + \tan \frac{C}{2}\right)^p} < \frac{2^{p-1}}{3^{\frac{p-2}{2}}}.$$

2556★. [2000 : 304] *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

A lattice point is called *visible* (from the origin) if its coordinates are co-prime numbers. Is there any lattice point whose distance from each visible lattice point is at least 2000?

Solution by the Southwest Missouri State University Problem Solving Group, Springfield, Missouri, USA.

The answer to the question is "yes". More generally, given any n, we can find an $n \times n$ square array of non-visible points as follows:

Take n^2 distinct prime numbers, p_1, p_2, $\ldots$, p_{n^2}. By the Chinese Remainder Theorem, we can find integers a and b such that for $i = 0$, 1, $\ldots$, $n-1$ and $j = 0, 1, \ldots, n-1$,

$$a \equiv -i \pmod{p_{1+ni+j}} \quad \text{and} \quad b \equiv -j \pmod{p_{1+ni+j}}$$

But now the square array of points:

$$\{(a+i, b+j) \mid i = 0, 1, \ldots, n-1; j = 0, 1, \ldots, n-1\}$$

has the property that p_{1+ni+j} divides $\gcd(a+i, b+j)$, and hence none of these points is visible.

Editor's comment.

The proposer noted (after this problem appeared in print) that it also appeared as problem E2653 in the American Mathematical Monthly, 1978, page 599, together with the following comment:

"The same problem was proposed by Jan Mycielski and solved by M. Warmus in Colloquium Mathematicum 3 (1955) 203-205. Paul Erdős observes that the assertion follows from a result of Moser and himself in Canad. Math. Bull. 1 (1958) 5-8. Fritz Herzog and B.M. Stewart observe that it also follows from their note in this Monthly 78 (1971) 487-496. Blair Spearman notes that this problem appears as Theorem 5.29 in T.M. Apostol, Introduction to Analytic Number Theory."

2557. [2000 : 304] *Proposed by Gord Sinnamon, University of Western Ontario, London, Ontario, and Hans Heinig, McMaster University, Hamilton, Ontario.*

(a) Show that for all positive sequences $\{x_i\}$ and all integers $n > 0$,

$$\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} x_i \;\le\; 2\sum_{k=1}^{n}\left(\sum_{j=1}^{k} x_j\right)^2 x_k^{-1}\,.$$

(b)* Does the above inequality remain true without the factor 2?

(c)* [Proposed by the editors] What is the minimum constant c that can replace the factor 2 in the above inequality?

Solution to (a) *by the proposers.*

Interchanging the order of summation gives

$$\begin{aligned}\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} x_i &= \sum_{j=1}^{n}(n-j+1)\sum_{i=1}^{j} x_i \;=\; \sum_{i=1}^{n}\binom{n-i+2}{2}x_i \\ &\ge \sum_{i=1}^{n}\frac{1}{2}(n-i+1)^2 x_i\,. \qquad (1)\end{aligned}$$

This observation, together with the Cauchy-Schwarz Inequality, yields

$$\begin{aligned}\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} x_i &= \sum_{j=1}^{n}(n-j+1)\sum_{i=1}^{j} x_i \\ &= \sum_{j=1}^{n}(n-j+1)x_j^{1/2}\left(\sum_{i=1}^{j} x_i\right)x_j^{-1/2} \\ &\le \left(\sum_{j=1}^{n}(n-j+1)^2 x_j\right)^{1/2}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{j} x_i\right)^2 x_j^{-1}\right)^{1/2} \\ &\le \left(2\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} x_i\right)^{1/2}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{j} x_i\right)^2 x_j^{-1}\right)^{1/2}\end{aligned}$$

Squaring both sides and dividing by $\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} x_i$ gives the required inequality.

No other solutions were received. Therefore, parts (b) *and* (c) *remain open.*

The editor notes that inequality (1) *is, in fact, strict, slightly improving the result.*

2558. [2000 : 304] *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let Z be a half-plane bounded by a line L. Let A, B and C be any three points on L such that C lies between A and B. Denote the three semicircles in Z on AB, AC and CB as diameters by K_0, K_1 and K_2, respectively. Let F be the family of semicircles in Z with diameters on L (including all half-lines in Z perpendicular to L). Denote by f_{XY} the unique semicircle passing through the pair of distinct points X, Y in $Z \cup L$. Let P, Q, R, be three points on K_2, K_1, K_0, respectively.

If f_{AP}, f_{BQ} and f_{CR} concur at T, and the lines AP, BQ, CR concur at S, prove that f_{AP}, f_{BQ} and f_{CR} are orthogonal to K_2, K_1, K_0, respectively, and that the circle PQR is tangent to each semicircle K_j, $(j = 0, 1, 2)$.

Editor's comment.

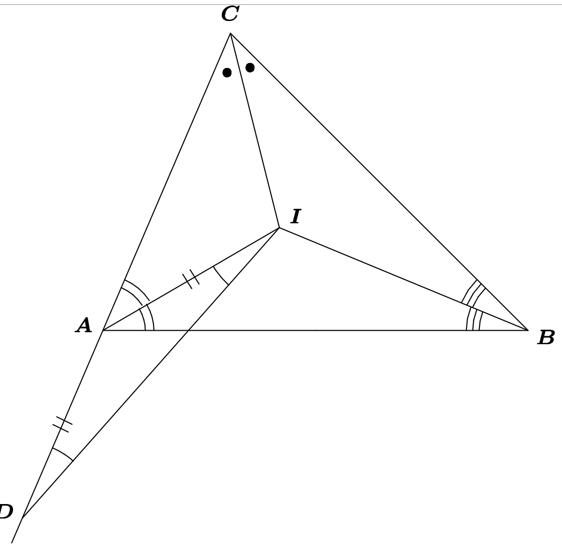
To date, we have received no solutions other than the proposer's own one, which is relatively long and complicated. We invite our readers to try to find a short (and, hopefully, simpler) solution.

2559 [2000 : 305] *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Triangle ABC has incentre I. Show that $CA + AI = CB$ if and only if $\angle CAB = 2\angle ABC$.

Solution by Toshio Seimiya, Kawasaki, Japan; Mitko Kunchev, Baba Tonka School of Mathematics, Rousse, Bulgaria; and Gottfried Perz, Pestalozzigymnasium, Graz, Austria.

Since I is the incentre of $\triangle ABC$, we have $\angle ICA = \angle ICB$, $\angle IAC = \angle IAB$ and $\angle IBC = \angle IBA$. Hence $\angle CAI = \frac{1}{2}\angle CAB$ and $\angle CBI = \frac{1}{2}\angle ABC$. Let D be the point on AC, so that A is between C and D, and $AD = AI$. Then $\angle CDI = \frac{1}{2}\angle CAI$, and therefore, $\angle CDI = \frac{1}{4}\angle CAB$.



(i) Let $CA + AI = CB$. Then $CA + AI = CA + AD = CD$, so that $CD = CB$. Since $\angle DCI = \angle BCI$, the triangles CDI and CBI are congruent. Thus $\angle CDI = \angle CBI$. Hence $\frac{1}{4}\angle CAB = \frac{1}{2}\angle ABC$. Therefore, $\angle CAB = 2\angle ABC$.

(ii) Let $\angle CAB = 2\angle ABC$. Then $\frac{1}{4}\angle CAB = \frac{1}{2}\angle ABC$, so that $\angle CDI = \angle CBI$. Since $\angle DCI = \angle BCI$, the triangles CDI and CBI are congruent, so that $CD = CB$. As $AD = AI$, we have $CD = CA + AD = CA + AI$. Therefore, $CA + AI = CB$.

From (i) and (ii), it follows that $CA + AI = CB$ if and only if $\angle CAB = 2\angle ABC$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (4 solutions); MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; HENRY LIU, Trinity College, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HENRY J. PAN, student, East York Collegiate Institute, Toronto, Ontario; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; IVAN SLAVOV, student, English Language High School, Stara Zagora, Bulgaria; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, Florida, USA; and the proposer.

2566. [2000 : 373] *Proposed by K.R.S. Sastry, Bangalore, India.*

Suppose that each of the three quadratics ax^2+bx+c, $ax^2+bx+(c+d)$ and $ax^2 + bx + (c + 2d)$ factors over the integers. Let $S = ad > 0$. Show that S represents the area of some Pythagorean triangle (integer sided right triangle).

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

It makes no difference to the problem if we use the three quadratics:

$$ax^2 + bx + (c - d)\,, \qquad ax^2 + bx + c\,, \qquad ax^2 + bx + (c + d)\,.$$

If each factors over the integers, then integers u, v, w exist such that

$$b^2 - 4a(c - d) \;=\; u^2\,, \qquad b^2 - 4ac \;=\; v^2\,, \qquad b^2 - 4a(c + d) \;=\; w^2\,.$$

Hence, $2v^2 = u^2 + w^2$ and $8ad = u^2 - w^2$. Since u and w are of the same parity, $\frac{u - w}{2}$ and $\frac{u + w}{2}$ are integers, and

$$\left(\frac{u - w}{2}\right)^2 + \left(\frac{u + w}{2}\right)^2 \;=\; v^2\,.$$

Hence, $\frac{u - w}{2}$ and $\frac{u + w}{2}$ are the sides of some Pythagorean triangle and

$$S \;=\; \frac{1}{2}\left(\frac{u - w}{2}\right)\left(\frac{u + w}{2}\right) \;=\; ad$$

is its area.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; MIHAI CIPU, IMAR, Bucharest, Romania; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; GERRY LEVERSHA, St. Paul's School, London, England; CALVIN LIN, Singapore; HENRY LIU, student, Trinity College Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; KENNETH M. WILKE, Topeka, KS, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; and the proposer.

2569. [2000 : 373] *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

Suppose that a, b, c and d, e, f are real numbers satisfying

1. the pairwise sums of a, b, c are (in some order) d, e and f; and
2. the pairwise products of d, e, f are (in some order) a, b and c.

Find all possible values of $a + b + c$.

Solution by David Loeffler, student, Cotham School, Bristol, UK.

The first condition is that $(a + b, b + c, c + a)$ is a rearrangement of (d, e, f). This implies that the three elementary symmetric functions of each list should be the same:

$$\begin{aligned} a + b + b + c + c + a &= d + e + f\,, \\ (a + b)(b + c) + (b + c)(c + a) + (c + a)(a + b) &= de + ef + fd\,, \\ (a + b)(b + c)(c + a) &= def\,. \end{aligned}$$

Likewise, the second condition that (de, ef, fd) is a rearrangement of (a, b, c) implies that

$$\begin{aligned} de + ef + fd &= a + b + c\,, \\ def(d + e + f) &= ab + bc + ca\,, \\ (def)^2 &= abc\,. \end{aligned}$$

If we write the elementary symmetric functions of a, b, c as $s_1 = a + b + c$, $s_2 = ab + bc + ca$, and $s_3 = abc$, then the above equations may be rewritten as:

$$\begin{aligned} 2s_1 &= d + e + f\,, & (1) \\ s_1^2 + s_2 &= de + ef + fd\,, & (2) \\ s_1 s_2 - s_3 &= def\,, & (3) \\ s_1 &= de + ef + fd\,, & (4) \\ s_2 &= def(d + e + f)\,, & (5) \\ s_3 &= (def)^2\,. & (6) \end{aligned}$$

The problem is now reduced to finding all possible values for s_1 in the above system of simultaneous equations.

By (2) and (4), we have $s_1^2 + s_2 = s_1$. Thus

$$s_2 = s_1(1 - s_1)\,. \tag{7}$$

Also, substituting this into (5) and comparing it with (1), we have $s_1(1 - s_1) = 2s_1 def$. Hence, either $s_1 = 0$, which we will return to later, or

$$def = \frac{1 - s_1}{2}\,. \tag{8}$$

Equations (8) and (6) imply that

$$s_3 = \left(\frac{1 - s_1}{2}\right)^2\,. \tag{9}$$

Substituting equations (9), (7), and (8) into (3) leads to

$$\begin{aligned} s_1^2(1 - s_1) - \left(\frac{1 - s_1}{2}\right)^2 &= \frac{1 - s_1}{2} \\ \text{or} \qquad (1 - s_1)(1 + s_1)(4s_1 - 3) &= 0\,. \end{aligned}$$

Thus, the only possible values of s_1 are ± 1, $\frac{3}{4}$, and 0.

It remains to show that each of these values does indeed lead to a solution. For $s_1 \neq 0$, we can calculate s_2 and s_3 for each value by (7) and (9), implying that there can be at most one possible solution in each case; for

$s_1 = 0$ there may be several solutions. In three of the cases a solution can be guessed: $s_1 = 0$ gives the trivial solution $a = b = c = d = e = f = 0$; $s_1 = 1$ gives $(a, b, c) = (1, 0, 0)$ and $(d, e, f) = (1, 1, 0)$; $s_1 = \frac{3}{4}$ leads to $(a, b, c) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and $(d, e, f) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

For $s_1 = -1$, we have, from the above, $s_2 = s_1(1 - s_1) = -2$, and $s_3 = \left(\frac{1-s_1}{2}\right)^2 = 1$. This means that a, b, and c are the roots of $x^3 + x^2 - 2x - 1 = 0$; these turn out to be

$$(a, b, c) \;=\; \left(-2\cos\frac{\pi}{7}, 2\cos\frac{2\pi}{7}, -2\cos\frac{3\pi}{7}\right) .$$

The corresponding values (d, e, f) are

$$\left(2\cos\frac{\pi}{7} - 1, -2\cos\frac{2\pi}{7} - 1, 2\cos\frac{3\pi}{7} - 1\right) .$$

Hence, each of the values ± 1, $\frac{3}{4}$, and 0 has been shown to produce (at least) one solution, so this is precisely the set of possible values of $a + b + c$.

[Remark: In fact, there are no solutions for $s_1 = 0$ other than the trivial zero solution, since $s_1 = 0$ implies that $s_2 = 0$, and does not tell us s_3. Solving the equations for a, b, and c for arbitrary $s_3 = u$ is equivalent to finding the three roots of $x^3 - u = 0$; that is, the cube roots of u. Two of these will always be complex unless u is zero.]

Also solved by ANGELO STATE PROBLEM GROUP, Angelo State University, San Angelo, TX; MICHEL BATAILLE, Rouen, France; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; and the proposer. There were three incorrect and one incomplete solutions.

YIU notes that the equation $x^3 + x^2 - 2x - 1 = 0$ is often associated with the regular heptagon.

2570. [2000 : 373] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let C be a conic with focus F and directrix d. Let A and B be the points of intersection of the conic with a line through the focus F. Let I, J and K be the feet of the perpendiculars from A, F and B to d, respectively.

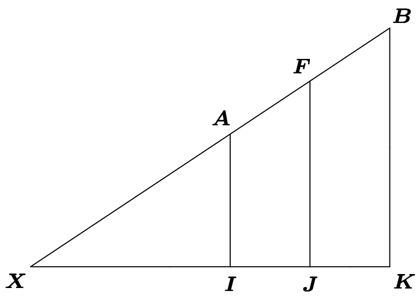
Prove that the length of FJ is the harmonic mean of the lengths of AI and BK.

Editor's comment.

All but one of the solutions assumed tacitly that the points A, B, F were all located on the same side of the directrix. Only M. Bataille observed that the result, as stated, is incorrect in the case when A, B lie on opposite sides of the directrix. As was observed by Bataille, the result can be rescued in this case if we require that distances be signed. We give the solution submitted by G. Leversha (with one small correction).

Solution by Gerry Leversha, St. Paul's School, London, England.

If AB is parallel to d, then the result is trivial, since $AI = FJ = BK$. Otherwise, suppose that AB and d meet at X.



Suppose that $XA = u$, $AF = a$, $FB = b$, $FJ = c$. Then $AI = fa$ and $BK = fb$, where f is the reciprocal of the eccentricity of the conic. By similar triangles, we have $\frac{u}{fa} = \frac{u+a}{c} = \frac{u+a+b}{fb}$.

But, since these ratios are all equal, it follows that $\frac{a}{c-fa} = \frac{b}{fb-c}$, and hence that $afb - ac = bc - fab$, so that $2fab = (a+b)c$, and hence, $\frac{1}{c} = \frac{1}{2}\left(\frac{1}{fa} + \frac{1}{fb}\right)$, which says that FJ is the harmonic mean of AI and BK.

Editor's note. As observed above, M. Bataille was the only one to note that this result breaks down in the case where A and B lie on opposite sides of the directrix. We give a slight generalization of his counterexample. Consider the hyperbola $x^2 - \frac{y^2}{e^2-1} = 1$, which has vertices $A(1,0)$, $B(-1,0)$, a focus $F(e,0)$ and directrix $x = \frac{1}{e}$. In this case, we have $I = J = K = \left(\frac{1}{e}, 0\right)$, and an easy calculation shows that $\frac{1}{FJ} = \frac{e}{e^2-1}$, whereas $\frac{1}{2}\left(\frac{1}{AI} + \frac{1}{BK}\right) = \frac{e}{e^2-1}$. Thus, the result, as stated, fails to hold in this case. If, however, we use signed distances (noting that BK then becomes negative), we obtain $\frac{1}{2}\left(\frac{1}{AI} - \frac{1}{BK}\right) = \frac{e}{e^2-1}$, which is the desired result. It is, in fact, easy to see that the proof given above by Leversha can be adapted in an obvious way to show that this is always the case.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HENRY LIU, student, Trinity College, Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, UK; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; DANIEL REISZ, Université de Bourgogne, Dijon, France; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2571. [2000 : 374] *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Suppose that a, b and c are the sides of a triangle. Prove that

$$\frac{1}{\sqrt{a}+\sqrt{b}-\sqrt{c}}+\frac{1}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{1}{\sqrt{c}+\sqrt{a}-\sqrt{b}} \geq \frac{3(\sqrt{a}+\sqrt{b}+\sqrt{c})}{a+b+c}\,.$$

Solution by Li Zhou, Polk Community College, Winter Haven, Florida, USA.

By the AM-GM inequality, we have

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 \;=\; (a+b+c)+2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right) \;\leq\; 3(a+b+c)\,.$$

So, by the AM-HM inequality, we get

$$\begin{aligned}\frac{3}{\frac{1}{\sqrt{a}+\sqrt{b}-\sqrt{c}}+\frac{1}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{1}{\sqrt{c}+\sqrt{a}-\sqrt{b}}} &\leq \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{3} \\ &\leq \frac{a+b+c}{\sqrt{a}+\sqrt{b}+\sqrt{c}}\,,\end{aligned}$$

which is equivalent to the desired inequality.

[Ed: Strictly speaking, a complete proof should first include a statement and proof of the fact that the denominators on the left side are all positive; that is, $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$ are also the sides of a triangle. However, since this is trivial to show (simply observe $(\sqrt{a}+\sqrt{b})^2 > a+b > c$, etc.), most solvers apparently assumed this fact without stating it, and the editor is willing to give the benefit of doubt to these solvers.]

Also solved by MOHAMMED AASSILA, Strasbourg, France; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); MIHÁLY BENCZE, Brasov, Romania; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIHAI CIPU, IMAR, Bucharest, Romania; RICHARD B. EDEN, Ateneo de Manila University, Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; HENRY LIU, student, Trinity College, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; VEDULA N. MURTY, Visakhapatnam, India; HENRY PAN, student, East York C.I., Toronto, Ontario; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Bencze pointed out the following immediate but interesting consequence of the given inequality:

Let P be any interior point of an equilateral triangle $\triangle ABC$. Then

$$\sum_{\text{cyclic}} \frac{1}{\sqrt{PA}+\sqrt{PB}-\sqrt{PC}} \;\geq\; \frac{3(\sqrt{PA}+\sqrt{PB}-\sqrt{PC})}{PA+PB+PC}\,.$$

Janous obtained the stronger result that for all $\lambda \in (0,1]$,

$$\sum_{\text{cyclic}} \frac{1}{a^\lambda + b^\lambda - c^\lambda} \;\geq\; \frac{9}{a^\lambda + b^\lambda + c^\lambda}\,.$$

Klamkin pointed out that the given result still holds under the weaker condition that $\sqrt{a}$, $\sqrt{b}$, *and* $\sqrt{c}$ *are the sides of a triangle.* [*Ed: See editorial comment at the end of the solution above.*] *He used Hölder's Inequality and the Power Mean Inequality to obtain the more general result, that, for* $p > 0$, x, y, $z \geq 0$ *with* $x + y + z = 1$,

$$\frac{(3x)^{p+1}}{(a+b-c)^p} + \frac{(3y)^{p+1}}{(a+b-c)^p} + \frac{(3z)^{p+1}}{(a+b-c)^p} \;\geq\; \frac{3(a+b+c)^p}{(a^2+b^2+c^2)^p}\,,$$

with equality if and only if $a = b = c$. *The proposed inequality with* (a,b,c) *replaced by* (a^2,b^2,c^2), *is the special case when* $p = 1$ *and* $x = y = z = \frac{1}{3}$.

2572. [2000 : 374] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Let a, b, c be positive real numbers. Prove that

$$a^b b^c c^a \;\leq\; \left(\frac{a+b+c}{3}\right)^{a+b+c}\,.$$

[Compare problem **2394** [1999 : 524], note by V.N. Murty on the generalization.]

I. *Independent and nearly identical solutions by Mohammed Aassila, Strasbourg, France; Mihály Bencze, Brasov, Romania; John G. Heuver, Grande Prairie Composite High School, Grande Prairie, Alberta; Joe Howard, Portales, NM, USA; Henry Liu, student, Trinity College, Cambridge, England; Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain; Panos E. Tsaoussoglou, Athens, Greece; and the proposer.*

By the weighted AM-GM inequality with weights $\dfrac{b}{a+b+c}$, $\dfrac{c}{a+b+c}$ and $\dfrac{a}{a+b+c}$, we have

$$\left(a^b b^c c^a\right)^{\frac{1}{a+b+c}} \;=\; a^{\frac{b}{a+b+c}}\, b^{\frac{c}{a+b+c}}\, c^{\frac{a}{a+b+c}} \;\leq\; \frac{ba+cb+ac}{a+b+c} \;\leq\; \frac{a+b+c}{3}$$

since $(a+b+c)^2 \geq 3(a+b+c)$ follows easily from $a^2+b^2+c^2 \geq ab+bc+ca$.

II. *Independent and virtually the same solution by Michel Bataille, Rouen, France; Mihai Cipu, IMAR, Bucharest, Romania; Murray S. Klamkin, University of Alberta, Edmonton, Alberta; Kee-Wai Lau, Hong Kong; David Loeffler, student, Cotham School, Bristol, UK; Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany; Kenneth M. Wilke, Topeka, KS, USA; and Li Zhou, Polk Community College, Winter Haven, USA.*

Since the function $\ln x$ is (strictly) concave and

$$\frac{b}{a+b+c} + \frac{c}{a+b+c} + \frac{a}{a+b+c} \;=\; 1\,,$$

Jensen's Inequality yields

$$\frac{b}{a+b+c}\ln a + \frac{c}{a+b+c}\ln b + \frac{a}{a+b+c}\ln c \;\le\; \ln\left(\frac{ba+cb+ac}{a+b+c}\right)$$

or

$$\left(a^b b^c c^a\right)^{\frac{1}{a+b+c}} \;\le\; \frac{ba+cb+ac}{a+b+c} \le \frac{a+b+c}{3}$$

as in solution I.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

There were two partially incorrect solutions, both of which used wording such as $a + a + \cdots + a$ *(b a's), thus treating b as a positive integer.*

Both Howard and Romero considered the 4–variable analogue and showed that $a^b b^c c^d d^a \le \left(\frac{a+b+c+d}{4}\right)^{a+b+c+d}$ *for positive reals a, b, c, d. Using the argument in either I or II above, we see easily that it suffices to show that*

$$\frac{ab+bc+cd+da}{a+b+c+d} \;\le\; \frac{a+b+c+d}{4}.$$

Both of them showed that this is true by noticing that $(a+b+c+d)^2 - 4(ab+bc+cd+da) = (a-b+c-d)^2 \ge 0$. *Howard asked whether the corresponding inequality still holds for $n \ge 5$ variables. Can any readers provide an answer or some references?*

2573. [2000 : 374] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Let H be the orthocentre of triangle ABC. For a point P not on the circumcircle of triangle ABC, denote by X, Y, Z the reflections of P in the sides BC, CA, and AB, respectively. Show that the areas of triangles HYZ, HZX, and HXY are in constant proportions.

A combination of similar solutions by Michel Bataille, Rouen, France; David Loeffler, student, Cotham School, Bristol, UK; and Peter Y. Woo, Biola University, La Mirada, CA, USA.

Since P is not on the circumcircle of $\triangle ABC$, the projections of P onto the lines BC, CA, AB are not collinear; further, neither are X, Y, Z. Thus XYZ is a non-degenerate triangle and we may compute the barycentric (or areal) coordinates of H with respect to it. These coordinates are proportional to the signed areas HYZ, HZX, HXY; hence, it is sufficient to show that the coordinates are independent of the position of P.

The barycentric representation of the orthocentre H with respect to $\triangle ABC$ is $H = \alpha A + \beta B + \gamma C$, where $\alpha + \beta + \gamma = 1$ and $\alpha : \beta : \gamma = \tan A : \tan B : \tan C$. [This is easily proved; it can be found in standard references such Clark Kimberling's encyclopedia of triangle centres: `cedar.evansville.edu/~ck6/tcenters/`.]

Let $P = pA + qB + rC$ for real numbers p, q, r with $p + q + r = 1$. Reflect $\triangle ABC$ across line BC so that the images of A and P are A' and X.

Let D be the foot of the altitude from A. Then $D = \dfrac{\beta B + \gamma C}{\beta + \gamma}$, and D is the mid-point of AA' so that

$$A' \;=\; 2D - A \;=\; \frac{2}{\beta+\gamma}(\beta B + \gamma C) - A\,.$$

From $P = pA + qB + rC$ we have $X = pA' + qB + rC$, or

$$X \;=\; \frac{2p}{\beta+\gamma}(\beta B + \gamma C) - pA + qB + rC\,.$$

Similarly, by reflecting $\triangle ABC$ across AC and across AB, we get

$$Y \;=\; \frac{2q}{\gamma+\alpha}(\gamma C + \alpha A) + pA - qB + rC\,,$$

and

$$Z \;=\; \frac{2r}{\alpha+\beta}(\alpha B + \beta B) + pA + qB - rC\,.$$

Then

$$\begin{aligned}
&\frac{1}{2}\Big[(\beta+\gamma)X + (\gamma+\alpha)Y + (\alpha+\beta)Z\Big] \\
&\quad= \frac{p}{2}\Big[2\beta B + 2\gamma C - (\beta+\gamma)A + (\gamma+\alpha)A + (\alpha+\beta)A\Big] \\
&\qquad\quad + \frac{q}{2}\Big[\cdots\Big] + \frac{r}{2}\Big[\cdots\Big] \\
&\quad= (\alpha A + \beta B + \gamma C)(p+q+r) \\
&\quad= \alpha A + \beta B + \gamma C \;=\; H\,.
\end{aligned}$$

This proves that H is a linear combination of X, Y and Z that is independent of p, q, r, as was required.

The argument breaks down if P is on the circumcircle — in this case the points X, Y, Z, and H are collinear. Thus, all the areas are zero and their ratios cease to be meaningful. This collinearity is essentially the result of problem 5 of the 4th Taiwan Mathematics Olympiad, see [2000 : 75]. [The proposer considers his problem to be a generalization of this result; he provided the reference J.R. Musselman, "On the Line of Images", *Amer. Math. Monthly* **45** (1938) 421–430; *ibid.*, **46** (1939) 281. The result is perhaps older than 1938 since it is problem 11 on page 145 of Nathan Altshiller Court, *College Geometry*.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria.

Both Woo and the proposer pointed out that the problem naturally belongs to affine geometry. More precisely,

> *Let K be a fixed point in the plane of $\triangle ABC$; for any point P define X, Y, Z to be the points such that $PX \| AK$, $PY \| BK$, $PZ \| CK$, and that the mid-point of PX lies on BC, the mid-point of PY lies on CA, and the mid-point of PZ lies on AB. Then the areas of HYZ, HZX, and HXY are in constant proportion except for those positions of P where one area (and therefore each) is zero; moreover, the exceptional values of P lie on a conic.*

2574. [2000 : 374] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Let P be a point in the interior of triangle ABC, whose centroid is G. Extend AP to a point X such that PX is bisected by the line BC. Similarly, extend BP to Y and CP to Z such that PY and PZ are each bisected by CA and AB, respectively. Show that the 6 points A, B, C, X, Y, Z, lie on a conic, and that the centre of the conic is the point Q dividing PG externally in the ratio $PQ : QG = 3 : -1$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

The problem belongs to affine geometry: an affine transformation maps lines to lines, ellipses to ellipses, the centre of an ellipse to the centre of its image, and it preserves ratios of distances among points on a line. Let X', Y', Z', A', B', C' be the mid-points of PX, PY, PZ, PA, PB, PC respectively. There exists an affine transformation that takes the given point P to the orthocentre of the image triangle — for example, one can fix B and C and slide A parallel to BC until $AP \perp BC$, then slide P along the new position of AP until $CP \perp AB$. To avoid introducing more notation, we may as well assume *without loss of generality* that the given point P is the orthocentre of $\triangle ABC$. Since A', B', C' are the mid-points of PA, PB, PC, the primed points all lie on the nine-point circle whose centre N lies on the Euler line PG halfway between the orthocentre P and circumcentre Q. (This result can be found in any reference that treats the nine-point circle.) With the orthocentre P as centre, the dilatation, whose ratio of magnification is 2, maps the nine-point circle to the circumcircle of $\triangle ABC$ passing through A, B, C, X, Y, Z; moreover, N is mapped to the circumcentre Q, and Q divides PG externally in the ratio $3 : -1$. [The points lie on the Euler line in the order $QGNP$, in the ratios $QG : GN : NP = 2 : 1 : 3$.]

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; and DAVID LOEFFLER, student, Cotham School, Bristol, UK.

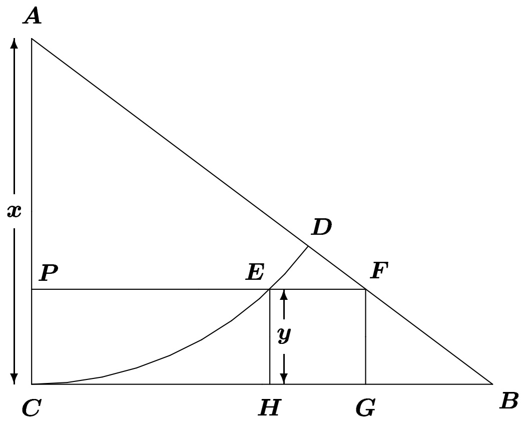
The proposer pointed out that this problem generalizes the property that the reflections of the orthocentre of a triangle in the sides lie on the circumcircle, a property that is clearly illustrated by the featured solution.

2575. [2000 : 374] *Proposed by H. Fukagawa, Kani, Gifu, Japan.*

Suppose that $\triangle ABC$ has a right angle at C. The circle, centre A and radius AC meets the hypotenuse AB at D. In the region bounded by the arc DC and the line segments BC and BD, draw a square $EFGH$ of side y, where E lies on arc DC, F lies on DB and G and H lie on BC. Assume that BC is constant and that $AC = x$ is variable.

Find $\max y$ and the corresponding value of x.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.



Let $BC = a$ and suppose that FE meets AC at P.

Since $FP \| BC$, the triangles AFP and ABC are similar, and we have

$$\frac{FP}{PA} = \frac{BC}{CA}, \qquad \text{or} \qquad \frac{FP}{x-y} = \frac{a}{x},$$

giving that $FP = \dfrac{a(x-y)}{x}$. Since $EP = FP - FE$, we have $EP = \dfrac{a(x-y)}{x} - y$.

In $\triangle AEP$, $AE = x$ and $AP = x-y$. Using the Pythagorean Theorem, we have

$$\left(\frac{a(x-y)}{x} - y\right)^2 + (x-y)^2 = x^2,$$

which we can write in the equivalent form

$$(x-y)\left(\frac{a^2(x-y)}{x^2} - \frac{2ay}{x} - 2y\right) = 0. \tag{1}$$

The expression on the left of (1) vanishes only if

$$\frac{a^2(x-y)}{x^2} - \frac{2ay}{x} - 2y = 0$$

(since, clearly, $y < x$). Solving for y, we get

$$y = \frac{a^2x}{2x^2 + 2ax + a^2} = \frac{a^2}{2x + \dfrac{a^2}{x} + 2a}.$$

Maximizing y is equivalent to minimizing $2x + \frac{a^2}{x}$. This term is the sum of two terms whose product is a constant: the sum is a minimum when $2x$ and $\frac{a^2}{x}$ are equal, from which we get

$$2x = \frac{a^2}{x}, \qquad x^2 = \frac{a^2}{2}, \qquad x = \frac{a}{\sqrt{2}},$$

where, since $x > 0$, we have discarded the negative solution.

Thus, y takes on its maximum value of $\frac{a\left(\sqrt{2}-1\right)}{2}$ when $x = \frac{a}{\sqrt{2}}$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain (second solution); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MIHAI CIPU, IMAR, Bucharest, Romania; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; and the proposer.

Janous commented that this was a marvellous problem linking various fields and ideas!!

The proposer writes that this is an elementary problem of traditional Japanese Mathematics concerning a circle, a triangle and a square.

In 1997, someone discovered this problem on the wooden ceiling inside a small old house in the countryside in Nagano prefecture. The wooden ceiling of the house is divided into about 25 parts as a lattice, and each part is a square (with side 43 cm) where flowers or birds are drawn beautifully. One of them is this geometry problem (without solution). A lover of Geometry, Yasuyuki Machida (whose birth and death years are not known), proposed this problem.

The figures on the ceiling were probably drawn in 1864 (the Edo–period) and then, until recently, no-one noticed these figures. I think that this problem is elementary and nice for high school students and lovers of geometry.

Editor's Comment. *Our readers are most likely to be familiar with* Japanese Temple Geometry Problems, *Charles Babbage Research Centre, Winnipeg, Manitoba, 1989, authored by our proposer with the cooperation of Dan Pedoe. This book contains a wealth of comparable fascinating geometry problems from similar sources. It is a must for the library of every geometer.*

2576. [2000 : 429] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

Characterize the numbers n such that $n!$ finishes (in base 2 notation) with exactly $n-1$ zeros.

I. *Solution by Li Zhou, Polk Community College, Winter Haven, Florida, USA.*

For $n \geq 1$, let $E_2(n!)$ be the greatest non-negative integer m such that $2^m | n!$. Then it is easy to see that $n!$ finishes (in base 2 notation) with exactly $n-1$ zeros if and only if $E_2(n!) = n-1$. Let $n = 2^k + r$ with $0 \leq r < 2^k$. Then $E_2(n!) = \sum_{t=1}^{k} \left\lfloor \frac{n}{2^t} \right\rfloor \leq \sum_{t=1}^{k} \frac{n}{2^t} = n\left(1 - \frac{1}{2^k}\right) \leq n-1$ with equality if and only if $r = 0$. Hence the desired numbers are $n = 2^k$, $k = 0, 1, 2, \ldots$.

II. *Solution and generalization by Pierre Bornsztein, Pontoise, France (slightly modified by the editor).*

We show more generally that if p is a prime, then $n!$ finishes (in base p notation) with exactly $n-1$ zeros if and only if **either** $n = 1$ **or** $p = 2$ and $n = 2^k$ for some positive integer k.

As in I above, let $E_p(n!)$ denote the greatest non-negative integer m such that $p^m | n!$. Then it suffices to determine all n such that $E_p(n!) = n-1$. By a well-known formula of Legendre (see reference [1]), we have $E_p(n!) = 0$ if $p > n$ and $E_p(n!) = \dfrac{n - s_p(n)}{p-1}$ if $p \leq n$ where $s_p(n)$ denotes the sum of the digits of n written in base p. It follows that if $p > n$, then $E_p(n!) = n-1$ if and only if $n = 1$ which clearly finishes in 0 zeros in base p. If $p \leq n$, then $E_p(n!) = n-1$ if and only if $n - s_p(n) = (n-1)(p-1)$ or

$$np + 1 + s_p(n) \;=\; 2n + p\,. \tag{1}$$

But if $p \geq 3$, then $np + 1 + s_p(n) > np \geq 2n + n \geq 2n + p$ and hence (1) is not satisfied. If $p = 2$, then (1) becomes $s_2(n) = 1$ which is equivalent to $n = 2^k$ for some positive integer k (since $n \geq p > 1$).

Also solved by MICHEL BATAILLE, Rouen, France; ROBERT BILINSKI, Outremont, Qubec; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIHAI CIPU, IMAR, Bucharest, Romania; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; CHARLES DIMINNIE and PAUL SWETS, Angelo State, University, San Angelo, TX, USA; RICHARD B. EDEN, Ateneo de Manila University, Philippines; KEITH EKBLAW, Walla Walla, WA, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; CALVIN ZHIWEI LIN, Singapore; HENRY LIU, Trinity College, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DAVID SINGMASTER, London, UK; SOUTHWEST MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; NICHOLAS THAM, Raffles Junior College, Singapore; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer.

Engelhaupt remarked that in general, if $n = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_r}$, *where* $0 \le a_1 < a_2 < \cdots < a_r$, *then* $n!$ *finishes in* $n - r$ *zeros since*

$$\begin{aligned} E_2(n!) &= \sum_{t=1}^{\infty} \left\lfloor \frac{n}{2^t} \right\rfloor = \sum_{i=1}^{r} \sum_{t=1}^{\infty} \left\lfloor \frac{2^{a_i}}{2^t} \right\rfloor = \sum_{i=1}^{r} (2^{a_i-1} + 2^{a_i-2} + \cdots + 1) \\ &= \sum_{i=1}^{r} (2^{a_i} - 1) = \left(\sum_{i=1}^{r} 2^{a_i} \right) - r = n - r\,. \end{aligned}$$

He gave the following example: Since $n = 216 = 2^3 + 2^4 + 2^6 + 2^7$ *has four summands,* $216!$ *finishes in* $216 - 4 = 212$ *zeros.*

Reference:

[1.] J.Robert, Elementary Number Theory — A Problem Oriented Approach, M.I.T. Press. Chapter X. Ex. 5, p. 76 and pp. 96–97.

Another maze from Isador Hafner

How can you move from the dark spot to the light spot?