PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (⋆) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, ***please send your proposals and solutions on signed and separate standard*** $8\frac{1}{2}$"×11" ***or*** A4 ***sheets of paper****. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 May 2002**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in LaTeX format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2676. *Proposed by Vedula N. Murty, Dover, PA, USA.*

Let A, B and C be the angles of a triangle. Show that

$$(\sin A + \sin B + \sin C)^2 \;\le\; 6\,(1 + \cos A \cos B \cos C)\;.$$

When does equality occur?

2677. *Proposed by P. Ivady, Budapest, Hungary*

For $0 < x < \frac{\pi}{2}$, show that $\dfrac{\pi^2 - x^2}{\pi^2 + x^2} \;<\; \cos\left(\dfrac{x}{\sqrt{3}}\right)$.

2678. *Proposed by David Chow, student, Clifton College, Bristol, England.*

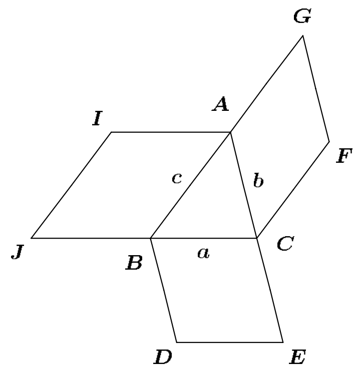
Prove that $\triangle ABC$ is isosceles if and only if

$$a\left(a^2 - b^2\right)\sin B + b\left(b^2 - c^2\right)\sin C + c\left(c^2 - a^2\right)\sin A \;=\; 0\;.$$

2679. *Proposed by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.*

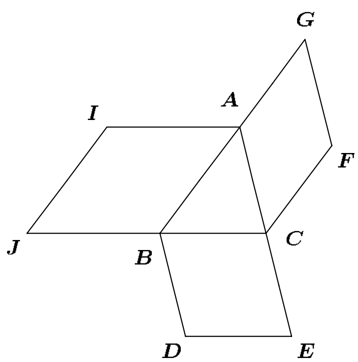
Find all solutions of $\sin x + \sin 2x = \sin 4x$.

2680. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*



Given $\triangle ABC$, construct parallelograms $ABJI$, $BCED$ and $CAGF$ outside the triangle such that $AI = \sqrt{ca}$, $BD = \sqrt{ab}$ and $CF = \sqrt{bc}$. Show that AD, BF and CI are concurrent.

2681. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*



Given $\triangle ABC$, construct rhombi $ABJI$, $BCED$ and $CAGF$ outside the triangle. Show that AD, BF and CI are concurrent.

2682. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

The sequence of functions, $\{J(n) = J(n, w)\}$, $n = 0, 1, \ldots$, is defined as follows:

$$\begin{aligned} J(0) &= a\,, \qquad J(1) \;=\; w + b\,, \\ J(n+1) &= \frac{J(n)\,\left(J(n)\,(w\,J(n) - 1) - J(n-1)\right)}{J(n-1)\,(w\,J(n) + 1) + J(n)} \qquad \text{for } n > 0. \end{aligned}$$

(a) Show that, if $a = 0$, then the sequence consists of polynomials.

(b) Show that there exists a pair (a, b) of non-zero integers such that all the $J(n)$ are polynomials with integer coefficients.

2683. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Find the value of $\lim\limits_{n\to\infty}\left(\frac{1}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{n+1}{2k+1}\binom{n+1}{2k+1}\right)$.

2684. *Proposed by Mohammed Aassila, Strasbourg, France.*

Does there exist an infinitely differentiable function $f:\mathbb{R}\to\mathbb{R}$ such that for every rational number p, the n^{th} derivative $f^{(n)}(p)$ is a rational number whenever n is even, and is an irrational number whenever n is odd?

2685. *Proposed by Mohammed Aassila, Strasbourg, France.*

(a) Let $\mathcal{C}$ be a bounded, closed and convex domain in the plane. Construct a parallelogram $\mathcal{P}$ contained in $\mathcal{C}$ such that $\mathcal{A}(\mathcal{P})\geq\frac{1}{2}\mathcal{A}(\mathcal{C})$, where $\mathcal{A}$ denotes area.

(b)* Prove that if, further, $\mathcal{C}$ is centrally symmetric, then one can construct a parallelogram $\mathcal{P}$ such that $A(\mathcal{P})\geq\frac{2}{\pi}A(\mathcal{C}\mathcal{P})$.

2686★. *Proposed by Mohammed Aassila, Strasbourg, France.*

Let $\mathcal{C}$ be a bounded, closed and convex domain in space. Construct a parallelepiped $\mathcal{P}$ contained in $\mathcal{C}$ such that $\mathcal{V}(\mathcal{P})\geq\frac{4}{9}\mathcal{V}(\mathcal{C})$, where $\mathcal{V}$ denotes volume.

2687. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Determine the locus of points (x,y) (in the real plane) for which the equation in z, $x\,z^3+y\,z^2+1=0$, has two complex roots of modulus twice the modulus of its real root.

2688. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Suppose that P is an arbitrary point inside cyclic quadrilateral $ABCD$. Let K, L, M and N be the projections of P onto AB, BC, CD and DA respectively.

Show that $AB\cdot PM+CD\cdot PK=BC\cdot PN+DA\cdot PL$.