On the closed form of power series

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Abstract

The Taylor series expansion of the function

$$f(x) = \frac{(x-1)^m}{m!}\log(1-x)$$

around the origin is used to evaluate $\displaystyle\sum_{k=1}^{\infty}\frac{x^{k+m}}{k(k+1)\dots(k+m)}$.

The problem of finding the closed form of a given power series is familiar to students of calculus. For example, the use of the Taylor series expansion of $\cos x$ about $x = 0$ allows one to obtain the elegant formula

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k} = \cos x\,.$$

What else can be done with such methods? In this note we show how Taylor series can be used to find a closed form for the power series

$$\sum_{k=1}^{\infty}\frac{x^{k+m}}{k(k+1)\cdots(k+m)}\,. \tag{1}$$

Let us do some formal calculation. Thus we change the order of operations like summation, differentiation and integration without verification. Moreover, denote the series (1) by $g(x)$, regardless of its domain of definition. By $m+1$ successive differentiations, we have $g^{(m+1)}(x) = \displaystyle\sum_{k=1}^{\infty}x^{k-1} = \frac{1}{1-x}$. Thus, by $m+1$ successive integrations, $g(x) = p(x) + q(x)\log(1-x)$, where p and q are polynomials of degree at most m. More elaborate calculation shows that $q(x) = -\dfrac{(x-1)^m}{m!}$. Our main concern thus is to find p. Therefore, let us start with the C^∞ function

$$f(x) = \frac{(x-1)^m}{m!}\log(1-x)\,, \qquad m \ge 1\,, \quad x \in (-\infty, 1)\,.$$

Since $\displaystyle\lim_{x\to 1-}\frac{(x-1)^m}{m!}\log(1-x) = 0$, we can define $f(1) = 0$. With this assumption, f is also left continuous at $x = 1$. To proceed further we should find the Taylor series expansion of f around $x = 0$. Hence we evaluate the $f^{(k)}(0)$ for $k \ge 0$.

Lemma 1. For each $x \in (-\infty, 1)$ and for each k, $1 \leq k \leq m$,

$$f^{(k)}(x) \;=\; \frac{(x-1)^{m-k}}{(m-k)!}\left(\log(1-x) + \sum_{\ell=m-k+1}^{m} \frac{1}{\ell}\right).$$

Proof. We use induction. Since

$$\begin{aligned} f'(x) &= \frac{(x-1)^{m-1}}{(m-1)!}\log(1-x) + \frac{(x-1)^m}{m!}\frac{-1}{1-x} \\ &= \frac{(x-1)^{m-1}}{(m-1)!}\left(\log(1-x) + \frac{1}{m}\right), \end{aligned}$$

the formula holds for $k = 1$. Suppose that it is true for $k < m$. Hence

$$\begin{aligned} f^{(k+1)}(x) &= \frac{(m-k)(x-1)^{m-k-1}}{(m-k)!}\left(\log(1-x) + \sum_{\ell=m-k+1}^{m} \frac{1}{\ell}\right) \\ &\quad + \frac{(x-1)^{m-k}}{(m-k)!}\frac{-1}{1-x} \\ &= \frac{(x-1)^{m-k-1}}{(m-k-1)!}\left(\log(1-x) + \sum_{\ell=m-k+1}^{m} \frac{1}{\ell} + \frac{1}{m-k}\right) \\ &= \frac{(x-1)^{m-k-1}}{(m-k-1)!}\left(\log(1-x) + \sum_{\ell=m-k}^{m} \frac{1}{\ell}\right). \end{aligned}$$

Thus it is also true for $k+1$. Therefore, the formula holds for each k with $1 \leq k \leq m$. ■

In particular, by putting $x = 0$ in the formula of Lemma 1, we have

$$f^{(k)}(0) \;=\; \frac{(-1)^{m-k}\sum_{\ell=m-k+1}^{m}\frac{1}{\ell}}{(m-k)!} \tag{2}$$

for each k with $1 \leq k \leq m$.

Lemma 2. For each $x \in (-\infty, 1)$ and for each $k \geq 1$, $f^{(m+k)}(x) = -\dfrac{(k-1)!}{(1-x)^k}$.

Proof. Again, we use induction. By Lemma 1, $f^{(m)}(x) = \log(1-x) + \sum\limits_{\ell=1}^{m} \dfrac{1}{\ell}$. Thus, $f^{(m+1)}(x) = \dfrac{-1}{(1-x)}$, showing that the formula holds for $k = 1$. Now suppose that the formula is true for k. We then have that $f^{(m+k+1)}(x) = -\dfrac{(k-1)!k}{(1-x)^{k+1}} = -\dfrac{k!}{(1-x)^{k+1}}$, showing our claim for $k+1$. Therefore, the formula holds for each $k \geq 1$. ■

In particular, put $x = 0$ in the formula of Lemma 2, to get

$$f^{(m+k)}(0) \;=\; -(k-1)! \tag{3}$$

for each $k \geq 1$.

Note: In the following, we use the simple identity

$$\frac{1}{k(k+1)\dots(k+m)} \;=\; \frac{(k-1)!}{(m+k)!}\,.$$

Theorem 3. For each $x \in (-1, 1)$

$$f(x) \;=\; \sum_{k=1}^{m}\left(\frac{(-1)^{m-k}\sum_{\ell=m-k+1}^{m}\frac{1}{\ell}}{(m-k)!k!}\right)x^k - \sum_{k=1}^{\infty}\frac{(k-1)!}{(m+k)!}x^{m+k}\,.$$

Proof. According to (2), (3) and by Taylor's Theorem [1], for each $x \in (-1, 1)$,

$$f(x) \;=\; \sum_{k=1}^{m}\left(\frac{(-1)^{m-k}\sum_{\ell=m-k+1}^{m}\frac{1}{\ell}}{(m-k)!k!}\right)x^k - \sum_{k=1}^{N}\frac{(k-1)!}{(m+k)!}x^{m+k} + R_N(x)\,,$$

where $R_N(x) = \dfrac{N!}{(m+N+1)!}\zeta_x^{m+N+1}$ for some $\zeta_x \in (-1, 1)$. Thus

$$\begin{aligned} |R_N(x)| = \frac{N!}{(m+N+1)!}|\zeta_x|^{m+N+1} &\leq \frac{N!}{(m+N+1)!} \\ = \frac{1}{(N+1)(N+2)\dots(N+m+1)} &\leq \frac{1}{N^{m+1}}\,. \end{aligned}$$

Hence for each $x \in (-1, 1)$, $\lim_{N\to\infty} R_N(x) = 0$. ■

We are now in a position to state the main result.

Main Theorem 4. Let $m \geq 1$. Then for each $x \in [-1, 1]$

$$\sum_{k=1}^{\infty}\frac{x^{m+k}}{k(k+1)\dots(k+m)} \;=\; \sum_{k=1}^{m}\left(\frac{(-1)^{m-k}\sum_{\ell=m-k+1}^{m}\frac{1}{\ell}}{(m-k)!k!}\right)x^k - \frac{(x-1)^m}{m!}\log(1-x)\,.$$

Remark: We are especially interested in justifying the preceding formula for $x = 1$ and for $x = -1$. As a matter of fact, Theorem 3 guarantees our formula for $x \in (-1, 1)$.

Proof. By Theorem 3, for each $x \in (-1, 1)$

$$\sum_{k=1}^{\infty}\frac{x^{m+k}}{k(k+1)\dots(k+m)} \;=\; \sum_{k=1}^{m}\left(\frac{(-1)^{m-k}\sum_{\ell=m-k+1}^{m}\frac{1}{\ell}}{(m-k)!k!}\right)x^k - f(x)\,. \tag{4}$$

Both sides represent continuous functions, at least on $(-1,1)$. The right side is also continuous at $x=-1$ and $x=1$. Since for each $x\in[-1,1]$

$$\left|\frac{x^{m+k}}{k(k+1)\dots(k+m)}\right| \le \frac{1}{k(k+1)\dots(k+m)} \le \frac{1}{k^{m+1}} \le \frac{1}{k^2},$$

by the Weierstrass M–test, the partial sums $\sum_{k=1}^{N}\frac{x^{m+k}}{k(k+1)\dots(k+m)}$ converge uniformly on $[-1,1]$. The limit of a uniformly convergent sequence of continuous functions is continuous [2]. Therefore the left side of equation (4) is also continuous on $[-1,1]$. Since equality holds for each $x\in(-1,1)$, it also holds for $x=-1$ and $x=1$. ■

Corollary 5 Let $m\ge 1$. Then

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)\dots(k+m)} = \frac{2^m\log 2}{m!} - \sum_{k=1}^{m}\frac{\sum_{\ell=m-k+1}^{m}\frac{1}{\ell}}{(m-k)!k!},$$

and $$\sum_{k=1}^{\infty}\frac{1}{k(k+1)\dots(k+m)} = \sum_{k=1}^{m}\frac{(-1)^{m-k}\sum_{\ell=m-k+1}^{m}\frac{1}{\ell}}{(m-k)!k!}.$$

Proof. Put $x=-1$ in (4). Thus

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^{m+k}}{k(k+1)\dots(k+m)} &= \sum_{k=1}^{m}\left(\frac{(-1)^{m-k}\sum_{\ell=m-k+1}^{m}\frac{1}{\ell}}{(m-k)!k!}\right)(-1)^k - f(-1)\\ &= \sum_{k=1}^{m}\left(\frac{(-1)^{m}\sum_{\ell=m-k+1}^{m}\frac{1}{\ell}}{(m-k)!k!}\right) - \frac{(-2)^m}{m!}\log 2.\end{aligned}$$

Hence, $$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)\dots(k+m)} = \frac{2^m\log 2}{m!} - \sum_{k=1}^{m}\frac{\sum_{\ell=m-k+1}^{m}\frac{1}{\ell}}{(m-k)!k!}.$$

Finally, put $x=1$ in equation (4). Since $f(1)=0$, then

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)\dots(k+m)} = \sum_{k=1}^{m}\frac{(-1)^{m-k}\sum_{\ell=m-k+1}^{m}\frac{1}{\ell}}{(m-k)!k!}.$$ ■

References

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