THE OLYMPIAD CORNER

No. 217

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We begin this number with selection questions for the Armenian team for IMO99. Thanks go to Ed Barbeau, Canadian Team Leader to the IMO at Bucharest, Romania for collecting them for the *Corner*.

SELECTION QUESTIONS FOR THE ARMENIAN TEAM FOR IMO99

Geometry

June 5, 1999

1. Let O be the centre of the circumcircle of the acute triangle ABC. The lines CO, AO, and BO intersect for the second time the circumcircles of the triangles AOB, BOC and AOC at C_1, A_1 and B_1 respectively.

Prove that

$$\frac{AA_1}{OA_1}+\frac{BB_1}{OB_1}+\frac{CC_1}{OC_1} \geq 4.5 .$$

2. Let the escribed circle (opposite $\angle A$) of the triangle ABC ($\angle A$, $\angle B$, $\angle C < 120^\circ$) with centre O be tangent to the sides of the triangle AB, BC and CA at points C_1, A_1 and B_1 respectively. Denote the mid-points of the segments AO, BO and CO by A_2, B_2 and C_2 respectively.

Prove that lines A_1A_2, B_1B_2 and C_1C_2 intersect at the same point.

3. B_1, B_2, $\ldots$, B_{10} are points inside the tetrahedron $A_1A_2A_3A_4$, such that none of the lines B_iB_j ($i \neq j$) intersects any of the edges of the tetrahedron $A_1A_2A_3A_4$.

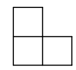
Prove that there exist such points A_i, A_j, A_k, B_l, B_m, B_n, such that no one of the tetrahedrons $A_iA_jA_kB_l$, $A_iA_jA_kB_m$ and $A_iA_jA_kB_n$ contains any other.

Algebra and Discrete Mathematics

July 5, 1999

4. Each of seven grasshoppers situated on the circumference of a circle can jump simultaneously over exactly two other grasshoppers. Is it possible that after several jumps a given pair of neighbours will interchange their places, while others will occupy their initial places?

5. Any 9 squares are removed from the 40 white squares of a 9×9 chess-like painted board. Prove that the remaining board is impossible to cover using 24 pieces of the kind as shown in the figure.



6. Solve the equation

$$\frac{1}{x^2} + \frac{1}{(4-\sqrt{3}\,x)^2} = 1\,.$$

Number Theory

October 5, 1999

7. It is known that all members of the infinite sequence $a-b$, a^2-b^2, a^3-b^3, ..., are natural numbers. Prove that a and b are integers.

8. Prove that if m and n are natural numbers, such that the number $2^{mn}-1$ is divisible by the number $(2^m-1)(2^n-1)$, then the number $2(3^{mn}-1)$ is divisible by $(3^m-1)(3^n-1)$.

9. Find all natural numbers k for which the sequence $x_n = \frac{S(n)}{S(kn)}$, $(n = 1, 2, \dots)$ will be bounded. Here $S(a)$ denotes the sum of the digits of the natural number a.

Next we give the 11^{th} form problems of the Russian Mathematical Olympiad 1999. Thanks again go to Ed Barbeau for collecting them when he was Canadian Team Leader at the IMO at Bucharest.

RUSSIAN MATHEMATICAL OLYMPIAD 1999

11^{th} Form

First Day

1. [*O. Podlipsky*] Do there exist 19 different positive integers that sum to 1999 and such that the sum of the decimal digits of each is the same?

2. [*S. Berlov*] A function $f : \mathbb{Q} \to \mathbb{Z}$ is considered. Prove that there exist two rational numbers a and b such that

$$\frac{f(a)+f(b)}{2} \le f\left(\frac{a+b}{2}\right)\,.$$

3. [*M. Sonkin*] The incircle of quadrilateral $ABCD$ touches the sides DA, AB, BC, CD in K, L, M, N respectively. Let S_1, S_2, S_3, S_4 be the incircles of triangles AKL, BLM, CMN, DKN respectively. Let l_1, l_2, l_3, l_4 be the common external tangents to the pairs S_1 and S_2, S_2 and S_3, S_3

and S_4, S_4 and S_1, different from the sides of quadrilateral $ABCD$. Prove that l_1, l_2, l_3, l_4 intersect in the vertices of a rhombus.

4. [*S. Tokarev*] There are an infinite checkerboard and n^2 checkers in some $n \times n$ cell square in it. If, in a 1×3 row or in a column, two cells a and b, with b the middle one, are occupied and the empty cell is c, then, after a move, the cells a and b become empty and the checker from a sits on the cell c, and the checker from the cell b is removed from the board. Prove that a position when no moves can be made will appear within $\lfloor \frac{n^2}{3} \rfloor$ moves.

Second Day

5. [*S. Berlov*] There are four positive integers that satisfy the condition: the square of the sum of any two of them is divisible by the sum of the two remaining numbers. Show that at least three of these numbers are equal.

6. [*V. Dol'nikov*] Prove that three convex polygons on the plane cannot be intersected by a line if and only if each of the polygons can be separated from the two remaining polygons by a line (that is, there is a line such that this polygon and the two others lie on different halfplanes).

7. [*D. Tereshin*] The plane α passing through the vertex A of tetrahedron $ABCD$ is tangent to the circumsphere of the tetrahedron. Prove that the angles between the lines of intersection of α with the planes ABC, ACD, and ABD are equal if and only if $AB \cdot CD = AC \cdot BD = AD \cdot BC$.

8. [*D. Karpov*] There is a microscheme with 2000 contacts and one wire between any two of them. Vasia and Petia in turn cut the wires. Vasia (who starts) can cut exactly one wire in one move and Petia can cut two or three wires in a move. The one to cut the last wire loses the game. Who will be the winner?

Next we turn to problems of the Hungary-Israel Mathematical Competition 1999. Thanks again go to Ed Barbeau, Canadian Team Leader at the IMO at Bucharest for collecting them for ***CRUX with MAYHEM***.

HUNGARY-ISRAEL MATHEMATICAL COMPETITION 1999

1. Let $f(x)$ be a polynomial whose degree is at least 2. Define the sequence $g_i(x)$ by: $g_1(x) = f(x)$ and $g_{n+1}(x) = f(g_n(x))$ for $n = 1, 2, \ldots$. Let r_n be the average of the roots of $g_n(x)$. It is given that $r_{19} = 99$. Find r_{99}.

2. A set of $2n+1$ lines in a plane is drawn. No two of them are parallel, and no three pass through one point. Every three of these lines form a non-right triangle. Determine the maximal number of acute angled triangles that can be formed.

3. Find all the functions f from the set of rational numbers to the set of real numbers such that for all rational x, y

$$f(x+y) = f(x)f(y) - f(xy) + 1 .$$

4. Let c be a positive integer. Define the following sequence:

$$a_1 = c, \quad a_{n+1} = ca_n + \sqrt{(c^2-1)(a_n^2-1)}, \quad n = 1, 2, \ldots .$$

Prove that all the terms a_n are positive integers.

5. The function

$$f(x,y,z) = \frac{x^2+y^2+z^2}{x+y+z}$$

is defined for every x, y, z such that $x+y+z \neq 0$. Find a point (x_0, y_0, z_0) such that $0 < x_0^2 + y_0^2 + z_0^2 < \frac{1}{1999}$ and $1.999 < f(x_0, y_0, z_0) < 2$.

6. An exam consists of four multiple choice questions, where each question has three choices. A certain group of examinees took the exam. It turned out that for any subset of three examinees, there was at least one question to which their choices covered all three possibilities. What is the maximal number of examinees in this group?

Next we give the Final Round problems of the 12^{th} Korean Mathematical Olympiad. My thanks go to Ed Barbeau for collecting them for us.

12^{th} KOREAN MATHEMATICAL OLYMPIAD
Final Round
April 17, 1999 – 4.5 hours

1. Let R, r be the circumradius, and the inradius of $\triangle ABC$, respectively, and let R', r' be the circumradius and inradius of $\triangle A'B'C'$, respectively. Prove that if $\angle C = \angle C'$ and $Rr' = R'r$, then the two triangles are similar.

2. Suppose $f(x)$ is a function satisfying $|f(m+n) - f(m)| \leq \frac{n}{m}$ for all rational numbers n and m. Show that for all natural numbers k

$$\sum_{i=1}^{k} |f(2^k) - f(2^i)| \leq \frac{k(k-1)}{2} .$$

3. Find all positive integers n such that $2^n - 1$ is a multiple of 3 and $\frac{2^n-1}{3}$ is a divisor of $4m^2 + 1$ for some integer m.

April 18, 1999 – 4.5 hours

4. Suppose that for any real x ($|x| \neq 1$), a function $f(x)$ satisfies

$$f\left(\frac{x-3}{x+1}\right) + f\left(\frac{3+x}{1-x}\right) = x .$$

Find all possible $f(x)$.

5. Consider a permutation $a_1a_2a_3a_4a_5a_6$ of the 6 numbers $\{1, 2, 3, 4, 5, 6\}$ which can be transformed to $1\ 2\ 3\ 4\ 5\ 6$ by transposing two numbers exactly 4 times (not less than 4 times). Find the number of such permutations.

6. Let $a_1, a_2, \ldots, a_{1999}$ be non-negative real numbers satisfying the following two conditions:

(i) $a_1 + a_2 + \cdots + a_{1999} = 2$

(ii) $a_1a_2 + a_2a_3 + \cdots + a_{1998}a_{1999} + a_{1999}a_1 = 1$.

Let $S = a_1^2 + a_2^2 + \cdots + a_{1999}^2$. Find the maximum and the minimum values of S.

As a final problem set for your puzzling pleasure this issue, we give the problems of the Grosman Memorial Mathematical Olympiad 1999, which took place in Israel in 1999. Thanks go to Ed Barbeau, Canadian Team Leader to the IMO at Bucharest for collecting them.

GROSMAN MEMORIAL MATHEMATICAL OLYMPIAD 1999

1. For every 16 positive integers $n, a_1, a_2, \ldots, a_{15}$ we define

$$T(n, a_1, a_2, \ldots, a_{15}) = (a_1^n + a_2^n + \cdots + a_{15}^n)a_1a_2\ldots a_{15} .$$

Find the smallest n for which $T(n, a_1, a_2, \ldots, a_{15})$ is divisible by 15 for every choice of $a_1, a_2, \ldots, a_{15}$.

2. Find the smallest integer n for which $0 < \sqrt[4]{n} - \lfloor \sqrt[4]{n} \rfloor < 10^{-5}$.

Remark. $\lfloor x \rfloor$ denotes the integral value of x; that is, the largest integer which does not exceed x.

3. For every triangle ABC, denote by $D(ABC)$, the triangle whose vertices are the tangency points of the incircle of ABC (touching the sides of the triangle). The given triangle ABC is not equilateral.

(a) Prove that $D(ABC)$ is also not equilateral.

(b) Find in the sequence of triangles $T_1 = \triangle ABC$, $T_{k+1} = D(T_k)$, $k = 1, 2, \ldots$ a triangle whose largest angle α satisfies the inequality $0 < \alpha - 60^\circ < 0.0001$.

4. Consider a polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ with integer coefficients a, b, c, d. Prove that if $f(x)$ has exactly one real root then $f(x)$ can be factored into terms with rational coefficients.

5. An infinite sequence of distinct real numbers is given. Prove that it contains a subsequence of 1999 terms which is either monotonically increasing or monotonically decreasing.

Remark. The sequence of numbers $a_1, \ldots, a_n$ is said to be monotone increasing if $a_1 < a_2 < \cdots < a_n$ and monotone decreasing if $a_1 > a_2 > \cdots > a_n$.

6. Six points A, B, C, D, E, F are given in space. The quadrilaterals $ABDE$, $BCEF$, $CDFA$ are parallelograms. Prove that the six mid-points of the sides AB, BC, CD, DE, EF, FA are coplanar.

In late summer, I received a package of solutions from Athanasios Kalakos of Athens, Greece. These included solutions dedicated to Murray Klamkin on the occasion of his 80th Birthday. These were to problems 2 and 3 of the Klamkin Quickies given in the April 2001 *Corner* [2001 : 166] and for which we published solutions in the September 2001 number [2001 : 299–300], which went to press before we received Kalakos's contributions. We look forward to using the remaining solutions in future numbers of the *Corner*.

Now, we turn to solutions from the readers to the Ukrainian Mathematical Olympiad, given [1999 : 389–390].

1. (8th grade) A regular polygon with 1996 vertices is given. What minimal number of vertices can we delete so that we do not have four remaining vertices which form: (a) a square? (b) a rectangle?

Solution by Pierre Bornsztein, Courdimanche, France.

(a) Let $A_1, A_2, \ldots, A_{1996}$ be the vertices, in that order, of the regular polygon. There are exactly 499 squares whose vertices are vertices of the polygon, which are $A_iA_{i+499}A_{i+998}A_{i+1497}$ (subscripts are modulo 1996), for $i = 1, 2, \ldots, 499$.

Then, we have to delete at least one of the vertices of each of these squares in order not to have a remaining square. Conversely, if we delete one of the vertices of the 499 squares, there is no remaining square.

Thus, the minimal number to delete is 499.

(b) A rectangle is formed by the vertices of two diameters of the circumcircle. There are 998 diameters whose vertices are vertices of the polygon, which are A_iA_{i+998} for $i = 1, \ldots, 998$.

Then, there will be no remaining rectangles if and only if we delete at least one vertex of all these 998 diameters, except one eventually.

Thus, the minimal number is 997.

2. (9th grade) Ivan has made the models of all triangles with integer lengths of sides and perimeters 1993 cm. Peter has made the models of all triangles with integer lengths of sides and perimeters 1996 cm. Who has more models?

Solutions by Pierre Bornsztein, Courdimanche, France; and by Christopher J. Bradley, Clifton College, Bristol, UK. We use Bradley's write-up.

Ivan and Peter have the same number of models.

In order to categorize the number of models with sides a, b, c set $2a = m+n$, $2b = n+l$, $2c = l+m$, the inverse transformation of which is $l = b+c-a$, $m = c+a-b$, $n = a+b-c$ so that the triangle inequalities $b+c > a$, $c+a > b$, $a+b > c$ are equivalent to $l > 0$, $m > 0$, $n > 0$, and l, m, n are also integers. Furthermore, the perimeter $a+b+c = l+m+n$. To find the number of models with $p = a+b+c$, one has to count the number of distinct sets of three positive integers summing to p, with the proviso that all of $m+n$, $n+l$, $l+m$ are even. The consequence of this is that l, m, n are *either* all even *or* all odd. Since $p = l+m+n$, it follows that if p is odd they must be all odd and if p is even, they must be all even.

The claim now is that the models of Ivan and the models of Peter are in one-to-one correspondence under the mapping

$$l_P \;=\; l_I+1\,, \quad m_P \;=\; m_I+1\,, \quad n_P \;=\; n_I+1\,. \qquad (1)$$

For a start, if l_I, m_I, n_I are all (odd as they must be to sum to 1993). then (1) shows l_P, m_P, n_P are all even, as they must be to sum to 1996, and conversely. Also, $l_P + m_P + n_P = l_I + m_I + n_I + 3$, as is required since $1996 = 1993 + 3$.

Assuming without loss of generality that $l \geq m \geq n$ for counting *distinct* shaped models, then lists of the possible shapes can be made, exhibiting the bijection. It is sufficient to list the extremes; other cases following naturally in between:

l_I	m_I	n_I		l_P	m_P	n_P
1991	1	1		1992	2	2
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
665	665	663		666	666	664

The bijection showing the correspondence may be succinctly expressed by

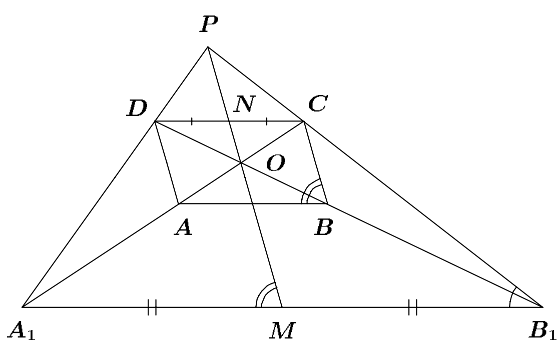
$$\begin{aligned} & l_I, m_I, 1993 - l_I - m_I \\ & \quad \longleftrightarrow l_I + 1, m_I + 1, 1994 - l_I - m_I \\ & \quad\quad = \; l_P, m_P, 1996 - l_P - m_P \end{aligned}$$

for $l_I \geq m_I$ and $l_I + 2m_I \geq 1993$.

5. (10th grade) Let O be the centre of the parallelogram $ABCD$, $\angle AOB > \pi/2$. We take the points A_1, B_1 on the half lines OA, OB respectively so that $A_1B_1 \parallel AB$ and $\angle A_1B_1C = \angle ABC/2$.

Prove that $A_1D \perp B_1C$.

Solution by Toshio Seimiya, Kawasaki, Japan.



Let P be the intersection of A_1D and B_1C. Since $DC \parallel AB \parallel A_1B_1$, we get $DC \parallel A_1B_1$, so that

$$\frac{PD}{DA_1} = \frac{PC}{CB_1}. \tag{1}$$

Let M and N be the intersections of PO with A_1B_1 and DC respectively. By Ceva's Theorem we have

$$\frac{A_1M}{MB_1} \cdot \frac{B_1C}{CP} \cdot \frac{PD}{DA_1} = 1.$$

Thus, we have from (1)

$$\frac{A_1M}{MB_1} = 1; \quad \text{that is,} \quad A_1M = MB_1.$$

Since $DC \parallel A_1B_1$, we have

$$\frac{DN}{A_1M} = \frac{PN}{PM} = \frac{NC}{MB_1}, \quad \text{giving} \quad DN = NC.$$

Because $DN = NC$ and $DO = OB$, we have $NO \parallel CB$; that is $PM \parallel CB$.

Since $A_1B_1 \parallel AB$ and $PM \parallel CB$, we get

$$\angle A_1MP = \angle ABC = 2\angle A_1B_1C = 2\angle A_1B_1P.$$

Thus, $\angle MPB_1 = \angle A_1MP - \angle A_1B_1P = \angle A_1B_1P$. Further, we obtain that $A_1M = B_1M = PM$. Thus, $\angle A_1PB_1 = 90°$. This implies that $A_1D \perp B_1C$.

6. (11th grade) The sequence $\{a_n\}$, $n \geq 0$, is such that $a_0 = 1$, $a_{499} = 0$ and for $n \geq 1$, $a_{n+1} = 2a_1a_n - a_{n-1}$.

(a) Prove that $|a_1| \leq 1$.
(b) Find a_{1996}.

Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Courdimanche, France. We give Bataille's write-up.

(a) Suppose on the contrary that $|a_1| > 1$.

The relation $a_{n+1} = 2a_1a_n - a_{n-1}$ for $n \geq 1$ leads to the characteristic equation $r^2 = 2a_1r - 1$. This equation has two distinct real solutions:

$$r_1 = a_1 + \sqrt{a_1^2 - 1} \quad \text{and} \quad r_2 = a_1 - \sqrt{a_1^2 - 1}.$$

Hence, a_n is given by $a_n = ur_1^n + vr_2^n$, where $u + v = a_0 = 1$ and $ur_1 + vr_2 = a_1$. Solving, we get $u = v = \frac{1}{2}$ so that

$$a_n = \frac{r_1^n + r_2^n}{2} \quad \text{for all } n \geq 0.$$

Since $r_1 > 0$, $r_2 > 0$, this last result contradicts $a_{499} = 0$. Therefore, $|a_1| \leq 1$.

(b) $|a_1| \leq 1$ enables one to set $a_1 = \cos\theta$. Then, an easy induction shows that $a_n = \cos(n\theta)$ for all $n \geq 0$. In particular $a_{499} = \cos(499\theta) = 0$. From $1996 = 4 \times 499$ and the formula $\cos 4t = 8\cos^4 t - 8\cos^2 t + 1$ (obtained for instance by expressing a_4 in terms of $a_1 = \cos(t)$), we get:

$$a_{1996} = 8\cos^4(499\theta) - 8\cos^2(499\theta) + 1 = 1.$$

7. (11th grade) Does a function $f : \mathbb{R} \to \mathbb{R}$ exist which is not a polynomial and such that for all real x

$$(x-1)f(x+1) - (x+1)f(x-1) = 4x(x^2-1)?$$

Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Courdimanche, France. We give Bataille's solution.

Yes! Take any function $f : x \mapsto x^3 + xk(x)$ where $k : \mathbb{R} \to \mathbb{R}$ is a bounded, non-constant, 2–periodic function (for example, $k(x) = \sin(\pi x)$, $k(x) = x - \lfloor x \rfloor, \ldots$). For such an f and for all real numbers x:

$$\begin{aligned}
&(x-1)f(x+1) - (x+1)f(x-1) \\
&\quad = \quad (x-1)(x+1)^3 - (x+1)(x-1)^3 + (x^2-1)(k(x+1) - k(x-1)) \\
&\quad = \quad 4x(x^2-1) + 0 \quad \text{(because of the periodicity of } k\text{)} \\
&\quad = \quad 4x(x^2-1)\,.
\end{aligned}$$

Moreover f is clearly not a polynomial: if it were, it would be a multiple of x (since $f(0) = 0$) and $k(x)$ would be a polynomial as well, which is impossible since k is non-constant and bounded.

8. (11th grade) Let M be the number of all positive integers which have n digits 1, n digits 2 and no other digits in their decimal representations. Let N be the number of all n–digit positive integers with only digits 1, 2, 3, 4 in the representation where the number of 1's equals the number of 2's. Prove that $M = N$.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Christopher J. Bradley, Clifton College, Bristol, UK. We give Bataille's solution.

The statement of the problem defines two sets of positive integers of cardinality M and N; let $\mathcal{M}$ and $\mathcal{N}$ be these two sets. We show that $M = N$ by establishing a bijection f between $\mathcal{M}$ and $\mathcal{N}$.

Any $x \in \mathcal{M}$ can be considered as a succession of n pairs of digits chosen among the pairs 11, 12, 21, 22. Taking these pairs in order from left to right, replace any pair 11 by a single 1, any pair 12 by a 3, any pair 21 by a 4 and any pair 22 by a single 2 to form an n–digit positive integer y.

Since the pairs 12 and 21 provide as many 1's as 2's to the digits of x, there must be as many pairs 11 as pairs 22 occurring in the process above. It follows that $y \in \mathcal{N}$ and that we have thus defined a function $f : x \mapsto y$ from $\mathcal{M}$ to $\mathcal{N}$.

Now, taking the digits of any $y \in \mathcal{N}$ in succession from left to right and replacing any 1 by 11, any 2 by 22, any 3 by 12 and any 4 by 21, we form a well-defined positive integer which belongs to $\mathcal{M}$ (since the representation of y counts as many 1's as 2's). We have thus defined $g : \mathcal{N} \mapsto \mathcal{M}$ and clearly $g \circ f =$ identity of $\mathcal{M}$ and $f \circ g =$ identity of $\mathcal{N}$. This shows that f and g are (reciprocal) bijections and completes the proof.

Note. To form an element of $\mathcal{M}$, we just have to choose n places (for the 1's) among $2n$ possible places. Hence, $M = \binom{2n}{n}$.

To form an element of $\mathcal{N}$, once the number k of 1's and 2's has been fixed (with $0 \le k \le \lfloor \frac{n}{2} \rfloor$), it remains to choose

(a) k places among n (for the 1's) and k places among $n-k$ for the 2's)

(b) between 2 or 3 for each of the $n-2k$ remaining places.

Hence,

$$N = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} \times \binom{n-k}{k} \times 2^{n-2k} .$$

Thus, by showing $M = N$ above, we have also proved the following identity:

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2k} \binom{n}{k} \binom{n-k}{k} = \binom{2n}{n} \quad (\text{for } n = 1, 2, \dots) .$$

We next look at readers' solutions to problems of the XII Italian Mathematical Olympiad 1996 given [1999 : 390–391].

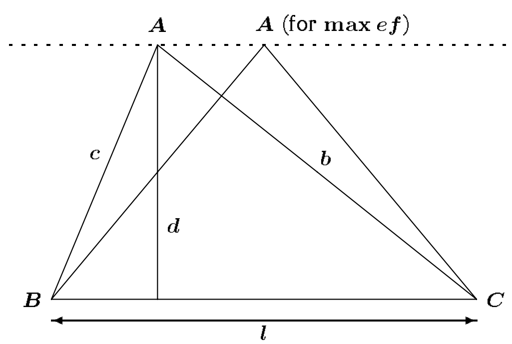
1. Among the triangles with an assigned side l and with given area S, determine all those for which the product of the three altitudes is maximum.

Solutions by Pierre Bornsztein, Courdimanche, France; by Christopher J. Bradley, Clifton College, Bristol, UK; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Bradley's solution.

Let the altitudes be d, e, f so that

$$d = \frac{2S}{a}, \quad e = \frac{2S}{b}, \quad f = \frac{2S}{c} .$$

Let $a = l$, so that with S fixed, so is d. One maximizes def when ef is maximum. Now $ef = \frac{4S^2}{bc}$, so that maximum ef occurs for minimum bc.



Now S is fixed, so that A lies on the dotted line parallel to BC through a point A perpendicular distance d from BC. Also, $S = \frac{1}{2}bc \sin A$ is constant, so that bc is minimized when $\angle A$ is maximized. This is true when $AB = AC$, and the triangle is isosceles.

2. Prove that the equation $a^2 + b^2 = c^2 + 3$ has infinitely many integer solutions (a, b, c).

Solutions by Pierre Bornsztein, Courdimanche, France; by Christopher J. Bradley, Clifton College, Bristol, UK; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea; by D.J. Smeenk, Zaltbommel, the Netherlands; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Lee's solution and remarks.

We show more generally that "the equation $a^2 + b^2 = c^2 + n$ has infinitely many integer solutions (a, b, c) for any $n \in \mathbb{Z}$."

Case 1. n is even ($n = 2k$ for some $k \in \mathbb{Z}$)

$$n = 2k = (2t(t+1)-k)^2 + (2t+1)^2 - (2t(t+1)-k+1)^2 \quad \text{for all } t \in \mathbb{Z}\,.$$

Case 2. n is odd ($n = 2k - 1$ for some $k \in \mathbb{Z}$)

$$n = 2k-1 = ((1-2kt^2)k)^2 + (2kt)^2 - ((1-2kt^2)k-1)^2 \quad \text{for all } t \in \mathbb{Z}\,.$$

Remark. In [1], Noam Elkies and Irving Kaplansky proposed the following problem. "Show that any integer can be expressed as a sum of two squares and a cube. Note that the integer being represented and the cube are both allowed to be negative". Andrew Adler gave the following proof.

$$\begin{aligned}
2x+1 &= (x^3 - 3x^2 + x)^2 + (x^2 - x - 1)^2 - (x^2 - 2x)^3 \\
4x+2 &= (2x^3 - 2x^2 - x)^2 + (2x^3 - 4x^2 - x + 1)^2 - (2x^2 - 2x - 1)^3 \\
8x+4 &= (x^3 + x + 2)^2 + (x^2 - 2x - 1)^2 - (x^2 + 1)^3 \\
16x+8 &= (2x^3 - 8x^2 + 4x + 2)^2 + (2x^3 - 4x^2 - 2)^2 - (2x^2 - 4x)^3 \\
16x &= (x^3 + 7x - 2)^2 + (x^2 + 2x + 11)^2 - (x^2 + 5)^3
\end{aligned}$$

Reference

1. "Problems and Solutions", *Amer. Math. Monthly* 104 (1997), 574.

4. Given an alphabet with three letters a, b, c, find the number of words of n letters which contain an even number of a's.

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France. We give Bornsztein's solution.

There are $\frac{3^n+1}{2}$ such words.

Let n be a positive integer. For each integer p such that $0 \le 2p \le n$, the number of words with exactly $2p$ letters "a" is $\binom{n}{2p} 2^{n-2p}$.

Summing over p shows that the desired number is

$$f(n) = \sum_{p \ge 0} \binom{n}{2p} 2^{n-2p} \quad \text{(with the usual convention: } \binom{n}{p} = 0 \text{ for } n < p)\,.$$

From the binomial formula, we have:

$$3^n = \sum_{p\geq 0}\binom{n}{p}2^{n-p} = f(n)+\sum_{p\geq 0}\binom{n}{2p+1}2^{n-(2p+1)} \qquad (1)$$

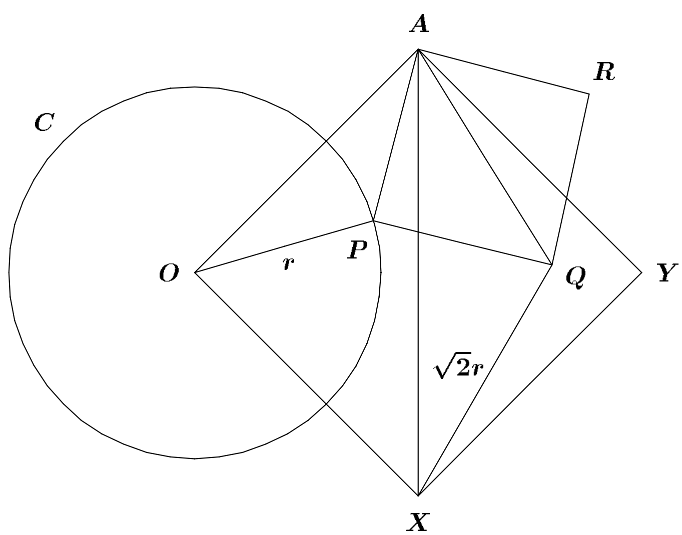
and

$$1^n = \sum_{p\geq 0}\binom{n}{p}(-1)^p 2^{n-p} = f(n)-\sum_{p\geq 0}\binom{n}{2p+1}2^{n(2p+1)}\,. \qquad (2)$$

From (1) and (2), we get $f(n) = \dfrac{3^n+1}{2}$, as claimed.

5. Let a circle C and a point A exterior to C be given. For each point P on C construct the square $APQR$, with anticlockwise ordering of the letters A, P, Q, R. Find the locus of the point Q when P runs over C.

Solutions by Pierre Bornsztein, Courdimanche, France; by Christopher J. Bradley, Clifton College, Bristol, UK; and by Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution.



Let O and r be the centre and radius of the circle C. Construct square $AOXY$ with anticlockwise ordering of the letters A, O, X, Y.

$\triangle APQ$ and $\triangle AOX$ are both right-angled isosceles triangles, and they are directly similar, so

$$\triangle AOP \sim \triangle AXQ\,.$$

Thus, we have

$$OP : XQ = AP : AQ = 1 : \sqrt{2}\,.$$

Therefore, $XQ = \sqrt{2}\,OP = \sqrt{2}\,r$.

Hence, when P runs over C, point Q describes the circle with centre X and radius $\sqrt{2}\,r$.

Therefore, the locus of Q is the circle with centre X and radius $\sqrt{2}\,r$.

Now we turn to solutions to problems of the South African Mathematics Olympiad, Third Round, September 1995 [1999 : 391–392].

SECTION A [1999 : 391].

1. Prove that there are no integers m and n such that

$$419m^2 + 95mn + 2000n^2 \;=\; 1995\,.$$

Solutions by Pierre Bornsztein, Courdimanche, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by D.J. Smeenk, Zaltbommel, the Netherlands; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution by Bradley, which represents the common theme.

Suppose integers m, n exist such that

$$419m^2 + 95mn + 2000n^2 \;=\; 1995\,.$$

Then, since 95, 2000 and 1995 are all divisible by 5, it follows that $5|m^2 \Longrightarrow 5|m$. Now put $m = 5M$ and we have

$$(419)(25)M^2 + (95)(25)Mn + 2000n^2 \;=\; 1995\,.$$

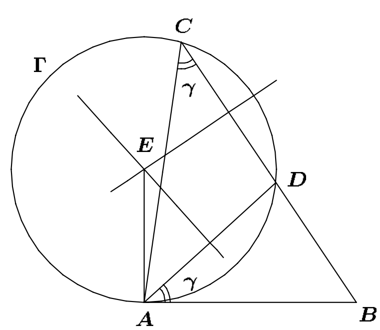
Now the left-hand side is divisible by 25 and the right-hand side is not. The contradiction establishes that no integers m and n exist satisfying the given equation.

2. ABC is a triangle with $\angle A > \angle C$, and D is the point on BC such that $\angle BAD = \angle ACB$. The perpendicular bisectors of AD and DC intersect in the point E. Prove that $\angle BAE = 90^\circ$.

Solutions by Pierre Bornsztein, Courdimanche, France; by Christopher J. Bradley, Clifton College, Bristol, UK; by Toshio Seimiya, Kawasaki, Japan; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give the write-up by Smeenk.

E is the circumcentre of $\triangle ADC$. Denote its circumcircle by Γ.

Since $\angle DAB = \angle ACD$ we have that AB is tangent to Γ at A, so that $\angle BAE = 90^\circ$.



3. Suppose that a_1, a_2, a_3, ..., a_n are the numbers 1, 2, 3, ..., n but written in any order. Prove that

$$(a_1 - 1)^2 + (a_2 - 2)^2 + (a_3 - 3)^2 + \cdots + (a_n - n)^2$$

is always even.

Solutions by Pierre Bornsztein, Courdimanche, France; by Christopher J. Bradley, Clifton College, Bristol, UK; by D.J. Smeenk, Zaltbommel, the Netherlands; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bornsztein's write-up.

Let $S = \sum_{i=1}^{n}(a_i - i)^2$. Then

$$S = \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i a_i .$$

Since $(a_1, \dots, a_n)$ is a permutation of $(1, \dots, n)$, we have

$$\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} i^2 .$$

Thus, $S = 2\left(\sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i a_i\right)$ is even.

4. Three circles, with radii p, q, r, and centres A, B, C respectively, touch one another externally at points D, E, F. Prove that the ratio of the areas of $\triangle DEF$ and $\triangle ABC$ equals

$$\frac{2pqr}{(p+q)(q+r)(r+p)} .$$

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Christopher J. Bradley, Clifton College, Bristol, UK; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Toshio Seimiya, Kawasaki, Japan. We give the solution by Seimiya.

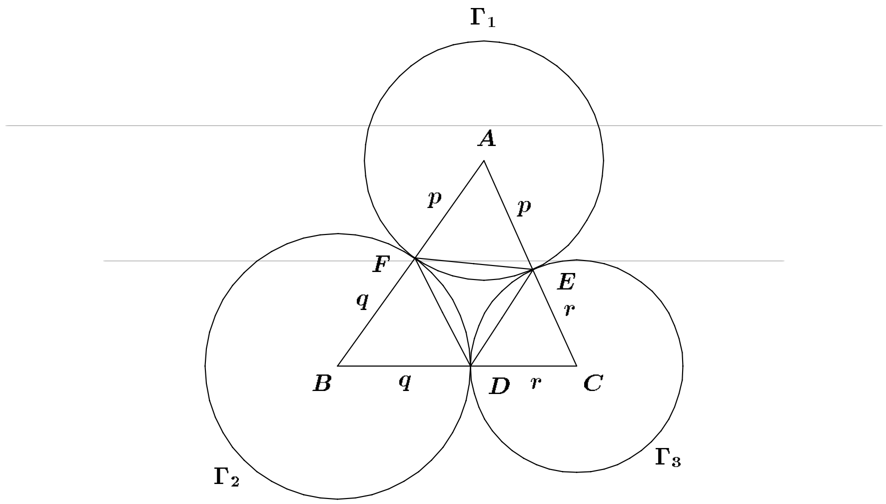
We denote circles with radii p, q, r and centres A, B, C by Γ_1, Γ_2, Γ_3 respectively. Let D, E, F be the points of tangency of Γ_2, Γ_3; Γ_3, Γ_1; and Γ_1, Γ_2 respectively.

By the assumption D, E, F are points on segments BC, CA, AB respectively.

Since $\angle FAE = \angle BAC$ we get

$$\frac{[AFE]}{[ABC]} = \frac{\frac{1}{2}AF \cdot AE \sin\angle FAE}{\frac{1}{2}AB \cdot AC \sin\angle BAC} = \frac{AF \cdot AE}{AB \cdot AC} = \frac{p^2}{(p+q)(p+r)} ,$$

where $[PQR]$ denotes the area of triangle PQR.



Similarly we have

$$\frac{[BDF]}{[ABC]} = \frac{q^2}{(p+q)(q+r)}, \quad \text{and} \quad \frac{[CED]}{[ABC]} = \frac{r^2}{(q+r)(p+r)}.$$

Hence, we have

$$\begin{aligned}
\frac{[DEF]}{[ABC]} &= 1 - \frac{[AFE]}{[ABC]} - \frac{[BDF]}{[ABC]} - \frac{[CED]}{[ABC]} \\
&= 1 - \frac{p^2}{(p+q)(p+r)} - \frac{q^2}{(p+q)(q+r)} - \frac{r^2}{(q+r)(p+r)} \\
&= \frac{(p+q)(q+r)(p+r) - p^2(q+r) - q^2(p+r) - r^2(p+q)}{(p+q)(q+r)(p+r)} \\
&= \frac{2pqr}{(p+q)(q+r)(r+p)}.
\end{aligned}$$

That completes the *Corner* for this issue. Send me your nice solutions, comments, generalization, and Olympiad Contests!