THE ACADEMY CORNER

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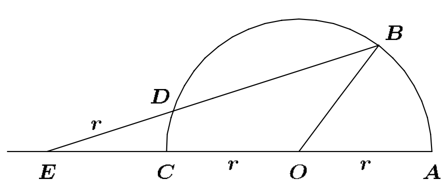
In this issue, we present the Memorial University Fall 2001 Undergraduate Mathematics Competition, written on 19 September 2001. Locally, this paper resulted in a good spread amongst the participants. Now, how do you rate? We invite university and high school students to send in their interesting solutions.

Memorial University of Newfoundland Undergraduate Mathematics Competition Fall 2001

1. Find all solutions to $\Big|\big||x| - 4\big| - 3\Big| = 2$.
2. Let n, j, k be positive integers such that $n \geq j \geq k$. From a group of n people, we would like to choose a committee of j members. Included in the committee is an executive subcommittee of k members. Show that the number of possible ways of choosing the committee and subcommittee is the same if we choose first the j committee members and then choose the k executives from them, or if we choose first the k members of the executive subcommittee and then choose the remaining $j - k$ members of the committee.
3. Show that $\dfrac{1}{2} \leq \sin^4\theta + \cos^4\theta \leq 1$, where $\theta \in \mathbb{R}$.
4. In the given diagram, O is the centre of a circle of radius r and AC is a diameter of the circle. It is assumed that B, D and E are as pictured – namely, E is on AC outside the circle, D is on the circle, $DE = r$ and DE intersects the circle at B where $B \neq D$.

   Prove that $\angle BEA = \frac{1}{3}\angle BOA$.



5. Show that the number $2^{55} + 1$ is

   (a) divisible by 3;

   (b) divisible by 11;

   (c) not divisible by 31.

6. Find the maximum distance and the minimum distance from the origin for all points on the curve $x^4 + y^4 = 1$.

Here is an extra problem that will test your ingenuity.

Problem.

Suppose that s is the semiperimeter and that R is the circumradius of $\triangle ABC$. Suppose that r_A, r_B and r_C are the radii of the escribed circles to $\triangle ABC$ opposite angles A, B and C, respectively.

Prove that:

1. $r_A r_B + r_B r_C + r_C r_A \; = \; s^2$,

2. $[ABC] \; = \; \dfrac{r_A r_B r_C}{s}$.

   Here, as usual, $[ABC]$ means the area of $\triangle ABC$.

If the distances between the centres of the escribed circles are α, β and γ, and $\sigma \; = \; \dfrac{\alpha + \beta + \gamma}{2}$, prove that

$$8R \; = \; \frac{\alpha\beta\gamma}{\sqrt{\sigma(\sigma - \alpha)(\sigma - \beta)(\sigma - \gamma)}}\,.$$