THE ACADEMY CORNER

No. 44

Bruce Shawyer

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In this issue, we present the Memorial University Fall 2001 Undergraduate Mathematics Competition, written on 19 September 2001. Locally, this paper resulted in a good spread amongst the participants. Now, how do you rate? We invite university and high school students to send in their interesting solutions.

Memorial University of Newfoundland Undergraduate Mathematics Competition Fall 2001

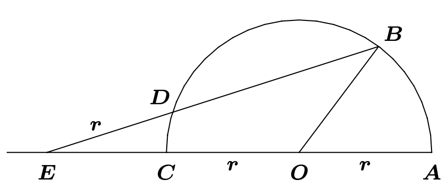
1. Find all solutions to $\Big|\big||x|-4\big|-3\Big| = 2$.

2. Let n, j, k be positive integers such that $n \geq j \geq k$. From a group of n people, we would like to choose a committee of j members. Included in the committee is an executive subcommittee of k members. Show that the number of possible ways of choosing the committee and subcommittee is the same if we choose first the j committee members and then choose the k executives from them, or if we choose first the k members of the executive subcommittee and then choose the remaining $j-k$ members of the committee.

3. Show that $\frac{1}{2} \leq \sin^4\theta + \cos^4\theta \leq 1$, where $\theta \in \mathbb{R}$.

4. In the given diagram, O is the centre of a circle of radius r and AC is a diameter of the circle. It is assumed that B, D and E are as pictured – namely, E is on AC outside the circle, D is on the circle, $DE = r$ and DE intersects the circle at B where $B \neq D$.

   Prove that $\angle BEA = \frac{1}{3}\angle BOA$.



5. Show that the number $2^{55}+1$ is

   (a) divisible by 3;

   (b) divisible by 11;

   (c) not divisible by 31.

6. Find the maximum distance and the minimum distance from the origin for all points on the curve $x^4+y^4=1$.

Here is an extra problem that will test your ingenuity.

Problem.

Suppose that s is the semiperimeter and that R is the circumradius of $\triangle ABC$. Suppose that r_A, r_B and r_C are the radii of the escribed circles to $\triangle ABC$ opposite angles A, B and C, respectively.

Prove that:

1. $r_A r_B + r_B r_C + r_C r_A \;=\; s^2$,
2. $[ABC] \;=\; \dfrac{r_A r_B r_C}{s}$.

   Here, as usual, $[ABC]$ means the area of $\triangle ABC$.

If the distances between the centres of the escribed circles are α, β and γ, and $\sigma \;=\; \dfrac{\alpha+\beta+\gamma}{2}$, prove that

$$8R \;=\; \frac{\alpha\beta\gamma}{\sqrt{\sigma(\sigma-\alpha)(\sigma-\beta)(\sigma-\gamma)}}\,.$$

THE OLYMPIAD CORNER

No. 217

R.E. Woodrow

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We begin this number with selection questions for the Armenian team for IMO99. Thanks go to Ed Barbeau, Canadian Team Leader to the IMO at Bucharest, Romania for collecting them for the *Corner*.

SELECTION QUESTIONS FOR THE ARMENIAN TEAM FOR IMO99

Geometry

June 5, 1999

1. Let O be the centre of the circumcircle of the acute triangle ABC. The lines CO, AO, and BO intersect for the second time the circumcircles of the triangles AOB, BOC and AOC at C_1, A_1 and B_1 respectively.

Prove that
$$\frac{AA_1}{OA_1}+\frac{BB_1}{OB_1}+\frac{CC_1}{OC_1} \geq 4.5\,.$$

2. Let the escribed circle (opposite $\angle A$) of the triangle ABC ($\angle A$, $\angle B$, $\angle C < 120^\circ$) with centre O be tangent to the sides of the triangle AB, BC and CA at points C_1, A_1 and B_1 respectively. Denote the mid-points of the segments AO, BO and CO by A_2, B_2 and C_2 respectively.

Prove that lines A_1A_2, B_1B_2 and C_1C_2 intersect at the same point.

3. B_1, B_2, $\ldots$, B_{10} are points inside the tetrahedron $A_1A_2A_3A_4$, such that none of the lines B_iB_j ($i \neq j$) intersects any of the edges of the tetrahedron $A_1A_2A_3A_4$.

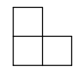
Prove that there exist such points A_i, A_j, A_k, B_l, B_m, B_n, such that no one of the tetrahedrons $A_iA_jA_kB_l$, $A_iA_jA_kB_m$ and $A_iA_jA_kB_n$ contains any other.

Algebra and Discrete Mathematics

July 5, 1999

4. Each of seven grasshoppers situated on the circumference of a circle can jump simultaneously over exactly two other grasshoppers. Is it possible that after several jumps a given pair of neighbours will interchange their places, while others will occupy their initial places?

5. Any 9 squares are removed from the 40 white squares of a 9×9 chess-like painted board. Prove that the remaining board is impossible to cover using 24 pieces of the kind as shown in the figure.



6. Solve the equation

$$\frac{1}{x^2}+\frac{1}{(4-\sqrt{3}\,x)^2} = 1\,.$$

Number Theory

October 5, 1999

7. It is known that all members of the infinite sequence $a-b$, a^2-b^2, a^3-b^3, $\ldots$, are natural numbers. Prove that a and b are integers.

8. Prove that if m and n are natural numbers, such that the number $2^{mn}-1$ is divisible by the number $(2^m-1)(2^n-1)$, then the number $2(3^{mn}-1)$ is divisible by $(3^m-1)(3^n-1)$.

9. Find all natural numbers k for which the sequence $x_n = \frac{S(n)}{S(kn)}$, $(n=1, 2, \ldots)$ will be bounded. Here $S(a)$ denotes the sum of the digits of the natural number a.

Next we give the 11^{th} form problems of the Russian Mathematical Olympiad 1999. Thanks again go to Ed Barbeau for collecting them when he was Canadian Team Leader at the IMO at Bucharest.

RUSSIAN MATHEMATICAL OLYMPIAD 1999

11^{th} Form

First Day

1. [*O. Podlipsky*] Do there exist 19 different positive integers that sum to 1999 and such that the sum of the decimal digits of each is the same?

2. [*S. Berlov*] A function $f:\mathbb{Q}\to\mathbb{Z}$ is considered. Prove that there exist two rational numbers a and b such that

$$\frac{f(a)+f(b)}{2} \le f\left(\frac{a+b}{2}\right)\,.$$

3. [*M. Sonkin*] The incircle of quadrilateral $ABCD$ touches the sides DA, AB, BC, CD in K, L, M, N respectively. Let S_1, S_2, S_3, S_4 be the incircles of triangles AKL, BLM, CMN, DKN respectively. Let l_1, l_2, l_3, l_4 be the common external tangents to the pairs S_1 and S_2, S_2 and S_3, S_3

and S_4, S_4 and S_1, different from the sides of quadrilateral $ABCD$. Prove that l_1, l_2, l_3, l_4 intersect in the vertices of a rhombus.

4. [*S. Tokarev*] There are an infinite checkerboard and n^2 checkers in some $n \times n$ cell square in it. If, in a 1×3 row or in a column, two cells a and b, with b the middle one, are occupied and the empty cell is c, then, after a move, the cells a and b become empty and the checker from a sits on the cell c, and the checker from the cell b is removed from the board. Prove that a position when no moves can be made will appear within $\lfloor \frac{n^2}{3} \rfloor$ moves.

Second Day

5. [*S. Berlov*] There are four positive integers that satisfy the condition: the square of the sum of any two of them is divisible by the sum of the two remaining numbers. Show that at least three of these numbers are equal.

6. [*V. Dol'nikov*] Prove that three convex polygons on the plane cannot be intersected by a line if and only if each of the polygons can be separated from the two remaining polygons by a line (that is, there is a line such that this polygon and the two others lie on different halfplanes).

7. [*D. Tereshin*] The plane α passing through the vertex A of tetrahedron $ABCD$ is tangent to the circumsphere of the tetrahedron. Prove that the angles between the lines of intersection of α with the planes ABC, ACD, and ABD are equal if and only if $AB \cdot CD = AC \cdot BD = AD \cdot BC$.

8. [*D. Karpov*] There is a microscheme with 2000 contacts and one wire between any two of them. Vasia and Petia in turn cut the wires. Vasia (who starts) can cut exactly one wire in one move and Petia can cut two or three wires in a move. The one to cut the last wire loses the game. Who will be the winner?

Next we turn to problems of the Hungary-Israel Mathematical Competition 1999. Thanks again go to Ed Barbeau, Canadian Team Leader at the IMO at Bucharest for collecting them for ***CRUX with MAYHEM***.

HUNGARY-ISRAEL MATHEMATICAL COMPETITION 1999

1. Let $f(x)$ be a polynomial whose degree is at least 2. Define the sequence $g_i(x)$ by: $g_1(x) = f(x)$ and $g_{n+1}(x) = f(g_n(x))$ for $n = 1, 2, \ldots$. Let r_n be the average of the roots of $g_n(x)$. It is given that $r_{19} = 99$. Find r_{99}.

2. A set of $2n+1$ lines in a plane is drawn. No two of them are parallel, and no three pass through one point. Every three of these lines form a non-right triangle. Determine the maximal number of acute angled triangles that can be formed.

3. Find all the functions f from the set of rational numbers to the set of real numbers such that for all rational x, y

$$f(x+y) \;=\; f(x)f(y) - f(xy) + 1\,.$$

4. Let c be a positive integer. Define the following sequence:

$$a_1 \;=\; c, \quad a_{n+1} \;=\; ca_n + \sqrt{(c^2-1)(a_n^2-1)}\,, \quad n = 1, 2, \dots\,.$$

Prove that all the terms a_n are positive integers.

5. The function

$$f(x,y,z) \;=\; \frac{x^2+y^2+z^2}{x+y+z}$$

is defined for every x, y, z such that $x+y+z \neq 0$. Find a point (x_0, y_0, z_0) such that $0 < x_0^2 + y_0^2 + z_0^2 < \frac{1}{1999}$ and $1.999 < f(x_0, y_0, z_0) < 2$.

6. An exam consists of four multiple choice questions, where each question has three choices. A certain group of examinees took the exam. It turned out that for any subset of three examinees, there was at least one question to which their choices covered all three possibilities. What is the maximal number of examinees in this group?

Next we give the Final Round problems of the 12$^{\text{th}}$ Korean Mathematical Olympiad. My thanks go to Ed Barbeau for collecting them for us.

12$^{\text{th}}$ KOREAN MATHEMATICAL OLYMPIAD
Final Round
April 17, 1999 – 4.5 hours

1. Let R, r be the circumradius, and the inradius of $\triangle ABC$, respectively, and let R', r' be the circumradius and inradius of $\triangle A'B'C'$, respectively. Prove that if $\angle C = \angle C'$ and $Rr' = R'r$, then the two triangles are similar.

2. Suppose $f(x)$ is a function satisfying $|f(m+n) - f(m)| \;\leq\; \frac{n}{m}$ for all rational numbers n and m. Show that for all natural numbers k

$$\sum_{i=1}^{k} |f(2^k) - f(2^i)| \;\leq\; \frac{k(k-1)}{2}\,.$$

3. Find all positive integers n such that $2^n - 1$ is a multiple of 3 and $\frac{2^n-1}{3}$ is a divisor of $4m^2 + 1$ for some integer m.

April 18, 1999 – 4.5 hours

4. Suppose that for any real x ($|x| \neq 1$), a function $f(x)$ satisfies

$$f\left(\frac{x-3}{x+1}\right) + f\left(\frac{3+x}{1-x}\right) = x\,.$$

Find all possible $f(x)$.

5. Consider a permutation $a_1a_2a_3a_4a_5a_6$ of the 6 numbers $\{1, 2, 3, 4, 5, 6\}$ which can be transformed to $1\ 2\ 3\ 4\ 5\ 6$ by transposing two numbers exactly 4 times (not less than 4 times). Find the number of such permutations.

6. Let $a_1, a_2, \ldots, a_{1999}$ be non-negative real numbers satisfying the following two conditions:

(i) $a_1 + a_2 + \cdots + a_{1999} = 2$

(ii) $a_1a_2 + a_2a_3 + \cdots + a_{1998}a_{1999} + a_{1999}a_1 = 1\,.$

Let $S = a_1^2 + a_2^2 + \cdots + a_{1999}^2$. Find the maximum and the minimum values of S.

As a final problem set for your puzzling pleasure this issue, we give the problems of the Grosman Memorial Mathematical Olympiad 1999, which took place in Israel in 1999. Thanks go to Ed Barbeau, Canadian Team Leader to the IMO at Bucharest for collecting them.

GROSMAN MEMORIAL MATHEMATICAL OLYMPIAD 1999

1. For every 16 positive integers $n, a_1, a_2, \ldots, a_{15}$ we define

$$T(n, a_1, a_2, \ldots, a_{15}) = (a_1^n + a_2^n + \cdots + a_{15}^n)a_1a_2\ldots a_{15}\,.$$

Find the smallest n for which $T(n, a_1, a_2, \ldots, a_{15})$ is divisible by 15 for every choice of $a_1, a_2, \ldots, a_{15}$.

2. Find the smallest integer n for which $0 < \sqrt[4]{n} - \lfloor\sqrt[4]{n}\rfloor < 10^{-5}$.

Remark. $\lfloor x\rfloor$ denotes the integral value of x; that is, the largest integer which does not exceed x.

3. For every triangle ABC, denote by $D(ABC)$, the triangle whose vertices are the tangency points of the incircle of ABC (touching the sides of the triangle). The given triangle ABC is not equilateral.

(a) Prove that $D(ABC)$ is also not equilateral.

(b) Find in the sequence of triangles $T_1 = \triangle ABC$, $T_{k+1} = D(T_k)$, $k = 1, 2, \ldots$ a triangle whose largest angle α satisfies the inequality $0 < \alpha - 60^\circ < 0.0001$.

4. Consider a polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ with integer coefficients a, b, c, d. Prove that if $f(x)$ has exactly one real root then $f(x)$ can be factored into terms with rational coefficients.

5. An infinite sequence of distinct real numbers is given. Prove that it contains a subsequence of 1999 terms which is either monotonically increasing or monotonically decreasing.

Remark. The sequence of numbers $a_1, \ldots, a_n$ is said to be monotone increasing if $a_1 < a_2 < \cdots < a_n$ and monotone decreasing if $a_1 > a_2 > \cdots > a_n$.

6. Six points A, B, C, D, E, F are given in space. The quadrilaterals $ABDE$, $BCEF$, $CDFA$ are parallelograms. Prove that the six mid-points of the sides AB, BC, CD, DE, EF, FA are coplanar.

In late summer, I received a package of solutions from Athanasios Kalakos of Athens, Greece. These included solutions dedicated to Murray Klamkin on the occasion of his 80$^{\text{th}}$ Birthday. These were to problems 2 and 3 of the Klamkin Quickies given in the April 2001 *Corner* [2001 : 166] and for which we published solutions in the September 2001 number [2001 : 299–300], which went to press before we received Kalakos's contributions. We look forward to using the remaining solutions in future numbers of the *Corner*.

Now, we turn to solutions from the readers to the Ukrainian Mathematical Olympiad, given [1999 : 389–390].

1. (8th grade) A regular polygon with 1996 vertices is given. What minimal number of vertices can we delete so that we do not have four remaining vertices which form: (a) a square? (b) a rectangle?

Solution by Pierre Bornsztein, Courdimanche, France.

(a) Let $A_1, A_2, \ldots, A_{1996}$ be the vertices, in that order, of the regular polygon. There are exactly 499 squares whose vertices are vertices of the polygon, which are $A_iA_{i+499}A_{i+998}A_{i+1497}$ (subscripts are modulo 1996), for $i = 1, 2, \ldots, 499$.

Then, we have to delete at least one of the vertices of each of these squares in order not to have a remaining square. Conversely, if we delete one of the vertices of the 499 squares, there is no remaining square.

Thus, the minimal number to delete is 499.

(b) A rectangle is formed by the vertices of two diameters of the circumcircle. There are 998 diameters whose vertices are vertices of the polygon, which are A_iA_{i+998} for $i = 1, \ldots, 998$.

Then, there will be no remaining rectangles if and only if we delete at least one vertex of all these 998 diameters, except one eventually.

Thus, the minimal number is 997.

2. (9th grade) Ivan has made the models of all triangles with integer lengths of sides and perimeters 1993 cm. Peter has made the models of all triangles with integer lengths of sides and perimeters 1996 cm. Who has more models?

Solutions by Pierre Bornsztein, Courdimanche, France; and by Christopher J. Bradley, Clifton College, Bristol, UK. We use Bradley's write-up.

Ivan and Peter have the same number of models.

In order to categorize the number of models with sides a, b, c set $2a = m+n$, $2b = n+l$, $2c = l+m$, the inverse transformation of which is $l = b+c-a$, $m = c+a-b$, $n = a+b-c$ so that the triangle inequalities $b + c > a$, $c + a > b$, $a + b > c$ are equivalent to $l > 0$, $m > 0$, $n > 0$, and l, m, n are also integers. Furthermore, the perimeter $a+b+c = l+m+n$. To find the number of models with $p = a+b+c$, one has to count the number of distinct sets of three positive integers summing to p, with the proviso that all of $m + n$, $n + l$, $l + m$ are even. The consequence of this is that l, m, n are *either* all even *or* all odd. Since $p = l + m + n$, it follows that if p is odd they must be all odd and if p is even, they must be all even.

The claim now is that the models of Ivan and the models of Peter are in one-to-one correspondence under the mapping

$$l_P = l_I + 1\,, \quad m_P = m_I + 1\,, \quad n_P = n_I + 1\,. \qquad (1)$$

For a start, if l_I, m_I, n_I are all odd (as they must be to sum to 1993). then (1) shows l_P, m_P, n_P are all even, as they must be to sum to 1996, and conversely. Also, $l_P + m_P + n_P = l_I + m_I + n_I + 3$, as is required since $1996 = 1993 + 3$.

Assuming without loss of generality that $l \geq m \geq n$ for counting *distinct* shaped models, then lists of the possible shapes can be made, exhibiting the bijection. It is sufficient to list the extremes; other cases following naturally in between:

l_I	m_I	n_I		l_P	m_P	n_P
1991	1	1		1992	2	2
⋮	⋮	⋮		⋮	⋮	⋮
665	665	663		666	666	664

The bijection showing the correspondence may be succinctly expressed by

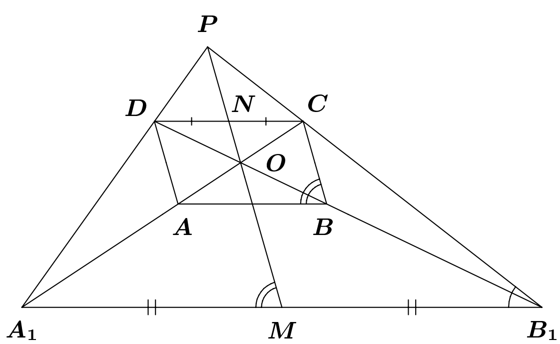
$$\begin{aligned} & l_I, m_I, 1993 - l_I - m_I \\ & \quad \longleftrightarrow l_I + 1, m_I + 1, 1994 - l_I - m_I \\ & \qquad = \ l_P, m_P, 1996 - l_P - m_P \end{aligned}$$

for $l_I \geq m_I$ and $l_I + 2m_I \geq 1993$.

5. (10th grade) Let O be the centre of the parallelogram $ABCD$, $\angle AOB > \pi/2$. We take the points A_1, B_1 on the half lines OA, OB respectively so that $A_1B_1 \parallel AB$ and $\angle A_1B_1C = \angle ABC/2$.

Prove that $A_1D \perp B_1C$.

Solution by Toshio Seimiya, Kawasaki, Japan.



Let P be the intersection of A_1D and B_1C. Since $DC \parallel AB \parallel A_1B_1$, we get $DC \parallel A_1B_1$, so that

$$\frac{PD}{DA_1} = \frac{PC}{CB_1}. \tag{1}$$

Let M and N be the intersections of PO with A_1B_1 and DC respectively. By Ceva's Theorem we have

$$\frac{A_1M}{MB_1} \cdot \frac{B_1C}{CP} \cdot \frac{PD}{DA_1} = 1.$$

Thus, we have from (1)

$$\frac{A_1M}{MB_1} = 1; \quad \text{that is,} \quad A_1M = MB_1.$$

Since $DC \parallel A_1B_1$, we have

$$\frac{DN}{A_1M} = \frac{PN}{PM} = \frac{NC}{MB_1}, \quad \text{giving} \quad DN = NC.$$

Because $DN = NC$ and $DO = OB$, we have $NO \parallel CB$; that is $PM \parallel CB$.

Since $A_1B_1 \parallel AB$ and $PM \parallel CB$, we get

$$\angle A_1MP = \angle ABC = 2\angle A_1B_1C = 2\angle A_1B_1P.$$

Thus, $\angle MPB_1 = \angle A_1MP - \angle A_1B_1P = \angle A_1B_1P$. Further, we obtain that $A_1M = B_1M = PM$. Thus, $\angle A_1PB_1 = 90°$. This implies that $A_1D \perp B_1C$.

6. (11th grade) The sequence $\{a_n\}$, $n \geq 0$, is such that $a_0 = 1$, $a_{499} = 0$ and for $n \geq 1$, $a_{n+1} = 2a_1a_n - a_{n-1}$.

(a) Prove that $|a_1| \leq 1$.
(b) Find a_{1996}.

Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Courdimanche, France. We give Bataille's write-up.

(a) Suppose on the contrary that $|a_1| > 1$.

The relation $a_{n+1} = 2a_1a_n - a_{n-1}$ for $n \geq 1$ leads to the characteristic equation $r^2 = 2a_1r - 1$. This equation has two distinct real solutions:

$$r_1 = a_1 + \sqrt{a_1^2 - 1} \quad \text{and} \quad r_2 = a_1 - \sqrt{a_1^2 - 1}.$$

Hence, a_n is given by $a_n = ur_1^n + vr_2^n$, where $u + v = a_0 = 1$ and $ur_1 + vr_2 = a_1$. Solving, we get $u = v = \frac{1}{2}$ so that

$$a_n = \frac{r_1^n + r_2^n}{2} \quad \text{for all } n \geq 0.$$

Since $r_1 > 0$, $r_2 > 0$, this last result contradicts $a_{499} = 0$. Therefore, $|a_1| \leq 1$.

(b) $|a_1| \leq 1$ enables one to set $a_1 = \cos\theta$. Then, an easy induction shows that $a_n = \cos(n\theta)$ for all $n \geq 0$. In particular $a_{499} = \cos(499\theta) = 0$. From $1996 = 4 \times 499$ and the formula $\cos 4t = 8\cos^4 t - 8\cos^2 t + 1$ (obtained for instance by expressing a_4 in terms of $a_1 = \cos(t)$), we get:

$$a_{1996} = 8\cos^4(499\theta) - 8\cos^2(499\theta) + 1 = 1.$$

7. (11th grade) Does a function $f : \mathbb{R} \to \mathbb{R}$ exist which is not a polynomial and such that for all real x

$$(x-1)f(x+1) - (x+1)f(x-1) = 4x(x^2-1)?$$

Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Courdimanche, France. We give Bataille's solution.

Yes! Take any function $f : x \mapsto x^3 + xk(x)$ where $k : \mathbb{R} \to \mathbb{R}$ is a bounded, non-constant, 2–periodic function (for example, $k(x) = \sin(\pi x)$, $k(x) = x - \lfloor x \rfloor, \ldots$). For such an f and for all real numbers x:

$$\begin{aligned}
&(x-1)f(x+1) - (x+1)f(x-1) \\
&\quad = \quad (x-1)(x+1)^3 - (x+1)(x-1)^3 + (x^2-1)(k(x+1) - k(x-1)) \\
&\quad = \quad 4x(x^2-1) + 0 \quad \text{(because of the periodicity of } k\text{)} \\
&\quad = \quad 4x(x^2-1)\,.
\end{aligned}$$

Moreover f is clearly not a polynomial: if it were, it would be a multiple of x (since $f(0) = 0$) and $k(x)$ would be a polynomial as well, which is impossible since k is non-constant and bounded.

8. (11th grade) Let M be the number of all positive integers which have n digits 1, n digits 2 and no other digits in their decimal representations. Let N be the number of all n–digit positive integers with only digits 1, 2, 3, 4 in the representation where the number of 1's equals the number of 2's. Prove that $M = N$.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Christopher J. Bradley, Clifton College, Bristol, UK. We give Bataille's solution.

The statement of the problem defines two sets of positive integers of cardinality M and N; let $\mathcal{M}$ and $\mathcal{N}$ be these two sets. We show that $M = N$ by establishing a bijection f between $\mathcal{M}$ and $\mathcal{N}$.

Any $x \in \mathcal{M}$ can be considered as a succession of n pairs of digits chosen among the pairs 11, 12, 21, 22. Taking these pairs in order from left to right, replace any pair 11 by a single 1, any pair 12 by a 3, any pair 21 by a 4 and any pair 22 by a single 2 to form an n–digit positive integer y.

Since the pairs 12 and 21 provide as many 1's as 2's to the digits of x, there must be as many pairs 11 as pairs 22 occurring in the process above. It follows that $y \in \mathcal{N}$ and that we have thus defined a function $f : x \mapsto y$ from $\mathcal{M}$ to $\mathcal{N}$.

Now, taking the digits of any $y \in \mathcal{N}$ in succession from left to right and replacing any 1 by 11, any 2 by 22, any 3 by 12 and any 4 by 21, we form a well-defined positive integer which belongs to $\mathcal{M}$ (since the representation of y counts as many 1's as 2's). We have thus defined $g : \mathcal{N} \mapsto \mathcal{M}$ and clearly $g \circ f =$ identity of $\mathcal{M}$ and $f \circ g =$ identity of $\mathcal{N}$. This shows that f and g are (reciprocal) bijections and completes the proof.

Note. To form an element of $\mathcal{M}$, we just have to choose n places (for the 1's) among $2n$ possible places. Hence, $M = \binom{2n}{n}$.

To form an element of $\mathcal{N}$, once the number k of 1's and 2's has been fixed (with $0 \le k \le \lfloor \frac{n}{2} \rfloor$), it remains to choose

(a) k places among n (for the 1's) and k places among $n-k$ for the 2's)

(b) between 2 or 3 for each of the $n-2k$ remaining places.

Hence,

$$N = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} \times \binom{n-k}{k} \times 2^{n-2k}\,.$$

Thus, by showing $M = N$ above, we have also proved the following identity:

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2k} \binom{n}{k} \binom{n-k}{k} \;=\; \binom{2n}{n} \qquad (\text{for } n = 1, 2, \dots)\,.$$

We next look at readers' solutions to problems of the XII Italian Mathematical Olympiad 1996 given [1999 : 390–391].

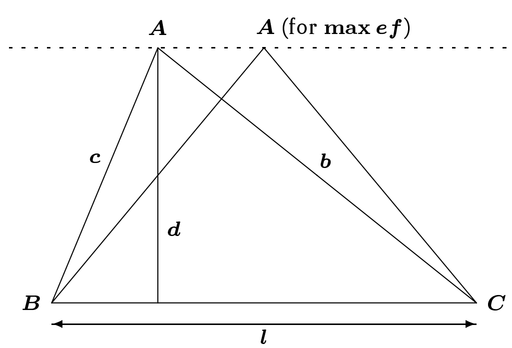
1. Among the triangles with an assigned side l and with given area S, determine all those for which the product of the three altitudes is maximum.

Solutions by Pierre Bornsztein, Courdimanche, France; by Christopher J. Bradley, Clifton College, Bristol, UK; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Bradley's solution.

Let the altitudes be d, e, f so that

$$d \;=\; \frac{2S}{a}\,, \quad e \;=\; \frac{2S}{b}\,, \quad f \;=\; \frac{2S}{c}\,.$$

Let $a = l$, so that with S fixed, so is d. One maximizes def when ef is maximum. Now $ef = \frac{4S^2}{bc}$, so that maximum ef occurs for minimum bc.



Now S is fixed, so that A lies on the dotted line parallel to BC through a point A perpendicular distance d from BC. Also, $S = \frac{1}{2}bc\sin A$ is constant, so that bc is minimized when $\angle A$ is maximized. This is true when $AB = AC$, and the triangle is isosceles.

2. Prove that the equation $a^2 + b^2 = c^2 + 3$ has infinitely many integer solutions (a, b, c).

Solutions by Pierre Bornsztein, Courdimanche, France; by Christopher J. Bradley, Clifton College, Bristol, UK; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea; by D.J. Smeenk, Zaltbommel, the Netherlands; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Lee's solution and remarks.

We show more generally that "the equation $a^2 + b^2 = c^2 + n$ has infinitely many integer solutions (a, b, c) for any $n \in \mathbb{Z}$."

Case 1. n is even ($n = 2k$ for some $k \in \mathbb{Z}$)

$$n = 2k = (2t(t+1)-k)^2 + (2t+1)^2 - (2t(t+1)-k+1)^2 \quad \text{for all } t \in \mathbb{Z}\,.$$

Case 2. n is odd ($n = 2k - 1$ for some $k \in \mathbb{Z}$)

$$n = 2k-1 = ((1-2kt^2)k)^2 + (2kt)^2 - ((1-2kt^2)k-1)^2 \quad \text{for all } t \in \mathbb{Z}\,.$$

Remark. In [1], Noam Elkies and Irving Kaplansky proposed the following problem. "Show that any integer can be expressed as a sum of two squares and a cube. Note that the integer being represented and the cube are both allowed to be negative". Andrew Adler gave the following proof.

$$\begin{aligned}
2x + 1 &= (x^3 - 3x^2 + x)^2 + (x^2 - x - 1)^2 - (x^2 - 2x)^3 \\
4x + 2 &= (2x^3 - 2x^2 - x)^2 + (2x^3 - 4x^2 - x + 1)^2 - (2x^2 - 2x - 1)^3 \\
8x + 4 &= (x^3 + x + 2)^2 + (x^2 - 2x - 1)^2 - (x^2 + 1)^3 \\
16x + 8 &= (2x^3 - 8x^2 + 4x + 2)^2 + (2x^3 - 4x^2 - 2)^2 - (2x^2 - 4x)^3 \\
16x &= (x^3 + 7x - 2)^2 + (x^2 + 2x + 11)^2 - (x^2 + 5)^3
\end{aligned}$$

Reference

1. "Problems and Solutions", *Amer. Math. Monthly* 104 (1997), 574.

4. Given an alphabet with three letters a, b, c, find the number of words of n letters which contain an even number of a's.

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France. We give Bornsztein's solution.

There are $\frac{3^n+1}{2}$ such words.

Let n be a positive integer. For each integer p such that $0 \le 2p \le n$, the number of words with exactly $2p$ letters "a" is $\binom{n}{2p} 2^{n-2p}$.

Summing over p shows that the desired number is

$$f(n) = \sum_{p \ge 0} \binom{n}{2p} 2^{n-2p} \quad \text{(with the usual convention: } \binom{n}{p} = 0 \text{ for } n < p)\,.$$

From the binomial formula, we have:

$$3^n \;=\; \sum_{p\geq 0}\binom{n}{p}2^{n-p} \;=\; f(n)+\sum_{p\geq 0}\binom{n}{2p+1}2^{n-(2p+1)} \qquad (1)$$

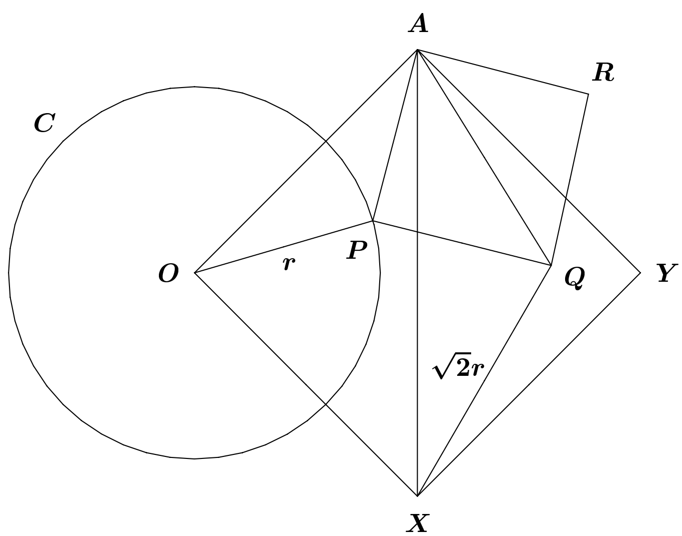
and

$$1^n \;=\; \sum_{p\geq 0}\binom{n}{p}(-1)^p 2^{n-p} \;=\; f(n)-\sum_{p\geq 0}\binom{n}{2p+1}2^{n(2p+1)}\,. \qquad (2)$$

From (1) and (2), we get $f(n) \;=\; \dfrac{3^n+1}{2}$, as claimed.

5. Let a circle C and a point A exterior to C be given. For each point P on C construct the square $APQR$, with anticlockwise ordering of the letters A, P, Q, R. Find the locus of the point Q when P runs over C.

Solutions by Pierre Bornsztein, Courdimanche, France; by Christopher J. Bradley, Clifton College, Bristol, UK; and by Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution.



Let O and r be the centre and radius of the circle C. Construct square $AOXY$ with anticlockwise ordering of the letters A, O, X, Y.

$\triangle APQ$ and $\triangle AOX$ are both right-angled isosceles triangles, and they are directly similar, so

$$\triangle AOP \;\sim\; \triangle AXQ\,.$$

Thus, we have

$$OP : XQ \;=\; AP : AQ \;=\; 1 : \sqrt{2}\,.$$

Therefore, $XQ = \sqrt{2}\,OP = \sqrt{2}\,r$.

Hence, when P runs over C, point Q describes the circle with centre X and radius $\sqrt{2}\,r$.

Therefore, the locus of Q is the circle with centre X and radius $\sqrt{2}\,r$.

Now we turn to solutions to problems of the South African Mathematics Olympiad, Third Round, September 1995 [1999 : 391–392].

SECTION A [1999 : 391].

1. Prove that there are no integers m and n such that

$$419m^2 + 95mn + 2000n^2 \;=\; 1995\,.$$

Solutions by Pierre Bornsztein, Courdimanche, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by D.J. Smeenk, Zaltbommel, the Netherlands; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution by Bradley, which represents the common theme.

Suppose integers m, n exist such that

$$419m^2 + 95mn + 2000n^2 \;=\; 1995\,.$$

Then, since 95, 2000 and 1995 are all divisible by 5, it follows that $5|m^2 \Longrightarrow 5|m$. Now put $m = 5M$ and we have

$$(419)(25)M^2 + (95)(25)Mn + 2000n^2 \;=\; 1995\,.$$

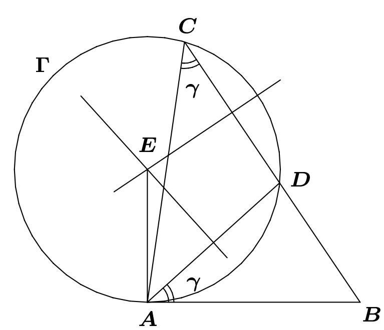
Now the left-hand side is divisible by 25 and the right-hand side is not. The contradiction establishes that no integers m and n exist satisfying the given equation.

2. ABC is a triangle with $\angle A > \angle C$, and D is the point on BC such that $\angle BAD = \angle ACB$. The perpendicular bisectors of AD and DC intersect in the point E. Prove that $\angle BAE = 90^\circ$.

Solutions by Pierre Bornsztein, Courdimanche, France; by Christopher J. Bradley, Clifton College, Bristol, UK; by Toshio Seimiya, Kawasaki, Japan; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give the write-up by Smeenk.

E is the circumcentre of $\triangle ADC$. Denote its circumcircle by Γ.

Since $\angle DAB = \angle ACD$ we have that AB is tangent to Γ at A, so that $\angle BAE = 90^\circ$.



3. Suppose that a_1, a_2, a_3, $\ldots$, a_n are the numbers 1, 2, 3, $\ldots$, n but written in any order. Prove that

$$(a_1 - 1)^2 + (a_2 - 2)^2 + (a_3 - 3)^2 + \cdots + (a_n - n)^2$$

is always even.

Solutions by Pierre Bornsztein, Courdimanche, France; by Christopher J. Bradley, Clifton College, Bristol, UK; by D.J. Smeenk, Zaltbommel, the Netherlands; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bornsztein's write-up.

Let $S = \sum_{i=1}^{n}(a_i - i)^2$. Then

$$S \;=\; \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i a_i \,.$$

Since $(a_1, \ldots, a_n)$ is a permutation of $(1, \ldots, n)$, we have

$$\sum_{i=1}^{n} a_i^2 \;=\; \sum_{i=1}^{n} i^2 \,.$$

Thus, $S \;=\; 2\left(\sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i a_i\right)$ is even.

4. Three circles, with radii p, q, r, and centres A, B, C respectively, touch one another externally at points D, E, F. Prove that the ratio of the areas of $\triangle DEF$ and $\triangle ABC$ equals

$$\frac{2pqr}{(p+q)(q+r)(r+p)} \,.$$

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Christopher J. Bradley, Clifton College, Bristol, UK; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Toshio Seimiya, Kawasaki, Japan. We give the solution by Seimiya.

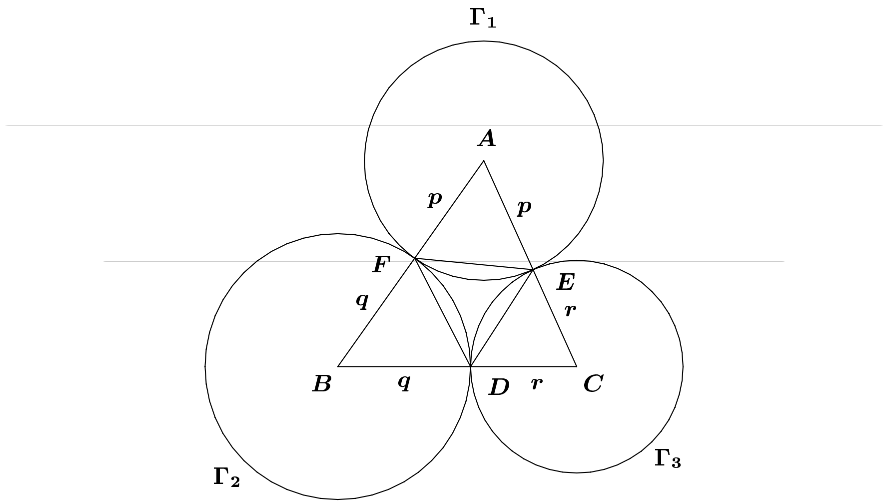
We denote circles with radii p, q, r and centres A, B, C by Γ_1, Γ_2, Γ_3 respectively. Let D, E, F be the points of tangency of Γ_2, Γ_3; Γ_3, Γ_1; and Γ_1, Γ_2 respectively.

By the assumption D, E, F are points on segments BC, CA, AB respectively.

Since $\angle FAE = \angle BAC$ we get

$$\frac{[AFE]}{[ABC]} \;=\; \frac{\frac{1}{2}AF \cdot AE \sin\angle FAE}{\frac{1}{2}AB \cdot AC \sin\angle BAC} \;=\; \frac{AF \cdot AE}{AB \cdot AC} \;=\; \frac{p^2}{(p+q)(p+r)} \,,$$

where $[PQR]$ denotes the area of triangle PQR.



Similarly we have

$$\frac{[BDF]}{[ABC]} = \frac{q^2}{(p+q)(q+r)}, \quad \text{and} \quad \frac{[CED]}{[ABC]} = \frac{r^2}{(q+r)(p+r)}.$$

Hence, we have

$$\begin{aligned}\frac{[DEF]}{[ABC]} &= 1 - \frac{[AFE]}{[ABC]} - \frac{[BDF]}{[ABC]} - \frac{[CED]}{[ABC]} \\ &= 1 - \frac{p^2}{(p+q)(p+r)} - \frac{q^2}{(p+q)(q+r)} - \frac{r^2}{(q+r)(p+r)} \\ &= \frac{(p+q)(q+r)(p+r) - p^2(q+r) - q^2(p+r) - r^2(p+q)}{(p+q)(q+r)(p+r)} \\ &= \frac{2pqr}{(p+q)(q+r)(r+p)}.\end{aligned}$$

That completes the *Corner* for this issue. Send me your nice solutions, comments, generalization, and Olympiad Contests!

BOOK REVIEWS

ALAN LAW

The Beginnings and Evolution of Algebra edited by I. Bashmakova and G. Smirnova (English translation by A. Shenitzer),
published by the Mathematical Association of America, 2000,
Dolciani Series Number 23.
ISBN 0-88385-329-9, softcover, 176 pages, $24.95 (U.S.).
Reviewed by ***Edward J. Kansa****, Embry-Riddle Aeronautical University, Oakland, CA, USA.*

This is a delightful history of algebra from the ancient Babylonians to the major developments in the 19th century, with a brief summary of the advances in the 20th century. It is not intended to be a textbook. The reader who has an introductory knowledge of algebra with applications to engineering and scientific problems can easily comprehend the first part of this book. Later portions focus on nineteenth century developments when algebra evolved into the more abstract realm of rings, groups, fields, etc.

The authors view the evolution of algebra in the following stages:

1. Ancient Babylonian numerical algebra,
2. Ancient Greek geometric algebra and the transformation into abstract theoretical science,
3. The birth of literal algebra from the beginning of the common era,
4. Algebra in the Middle Ages in the Arabic and European communities,
5. Algebraic achievements in Europe (cubic and quintic equations, calculus, indeterminate and determinate equations),
6. Algebraic developments in the 17th and 18th centuries (Descartes and Gauss, solutions with radicals, proof on unsolvability of the general quintic equation),
7. The theory of algebraic equations (group theory, Galois theory, etc.)
8. Modern Algebra (fields, commutative algebras, etc.)
9. Linear and noncommutative algebra (linear equations, matrices, symbolic algebra, algebras).

In summary, this reviewer highly recommends this book for a fascinating study of the history of algebra. The authors do not burden the reader with excessive details, but do cite the relevant literature. The volume is a valuable resource for those who are interested in the evolution of algebra throughout the ages.

On the closed form of power series

Zeynab Mashreghi and Javad Mashreghi

Abstract

The Taylor series expansion of the function

$$f(x) \;=\; \frac{(x-1)^m}{m!}\log(1-x)$$

around the origin is used to evaluate $\displaystyle\sum_{k=1}^{\infty}\frac{x^{k+m}}{k(k+1)\ldots(k+m)}$.

The problem of finding the closed form of a given power series is familiar to students of calculus. For example, the use of the Taylor series expansion of $\cos x$ about $x = 0$ allows one to obtain the elegant formula

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k} \;=\; \cos x\,.$$

What else can be done with such methods? In this note we show how Taylor series can be used to find a closed form for the power series

$$\sum_{k=1}^{\infty}\frac{x^{k+m}}{k(k+1)\cdots(k+m)}\,. \tag{1}$$

Let us do some formal calculation. Thus we change the order of operations like summation, differentiation and integration without verification. Moreover, denote the series (1) by $g(x)$, regardless of its domain of definition. By $m+1$ successive differentiations, we have $g^{(m+1)}(x) = \displaystyle\sum_{k=1}^{\infty}x^{k-1} = \frac{1}{1-x}$. Thus, by $m+1$ successive integrations, $g(x) = p(x) + q(x)\log(1-x)$, where p and q are polynomials of degree at most m. More elaborate calculation shows that $q(x) = -\dfrac{(x-1)^m}{m!}$. Our main concern thus is to find p. Therefore, let us start with the C^{∞} function

$$f(x) \;=\; \frac{(x-1)^m}{m!}\log(1-x)\,, \qquad m \ge 1\,, \quad x \in (-\infty, 1)\,.$$

Since $\displaystyle\lim_{x\to 1-}\frac{(x-1)^m}{m!}\log(1-x) \;=\; 0$, we can define $f(1) = 0$. With this assumption, f is also left continuous at $x = 1$. To proceed further we should find the Taylor series expansion of f around $x = 0$. Hence we evaluate the $f^{(k)}(0)$ for $k \ge 0$.

Lemma 1. For each $x \in (-\infty, 1)$ and for each k, $1 \le k \le m$,

$$f^{(k)}(x) \;=\; \frac{(x-1)^{m-k}}{(m-k)!}\left(\log(1-x) + \sum_{\ell=m-k+1}^{m} \frac{1}{\ell}\right).$$

Proof. We use induction. Since

$$\begin{aligned} f'(x) &= \frac{(x-1)^{m-1}}{(m-1)!}\log(1-x) + \frac{(x-1)^m}{m!}\frac{-1}{1-x} \\ &= \frac{(x-1)^{m-1}}{(m-1)!}\left(\log(1-x) + \frac{1}{m}\right), \end{aligned}$$

the formula holds for $k = 1$. Suppose that it is true for $k < m$. Hence

$$\begin{aligned} f^{(k+1)}(x) &= \frac{(m-k)(x-1)^{m-k-1}}{(m-k)!}\left(\log(1-x) + \sum_{\ell=m-k+1}^{m} \frac{1}{\ell}\right) \\ &\quad + \frac{(x-1)^{m-k}}{(m-k)!}\frac{-1}{1-x} \\ &= \frac{(x-1)^{m-k-1}}{(m-k-1)!}\left(\log(1-x) + \sum_{\ell=m-k+1}^{m} \frac{1}{\ell} + \frac{1}{m-k}\right) \\ &= \frac{(x-1)^{m-k-1}}{(m-k-1)!}\left(\log(1-x) + \sum_{\ell=m-k}^{m} \frac{1}{\ell}\right). \end{aligned}$$

Thus it is also true for $k+1$. Therefore, the formula holds for each k with $1 \le k \le m$. ∎

In particular, by putting $x = 0$ in the formula of Lemma 1, we have

$$f^{(k)}(0) \;=\; \frac{(-1)^{m-k}\sum_{\ell=m-k+1}^{m}\frac{1}{\ell}}{(m-k)!} \tag{2}$$

for each k with $1 \le k \le m$.

Lemma 2. For each $x \in (-\infty, 1)$ and for each $k \ge 1$, $f^{(m+k)}(x) = -\dfrac{(k-1)!}{(1-x)^k}$.

Proof. Again, we use induction. By Lemma 1, $f^{(m)}(x) = \log(1-x) + \sum_{\ell=1}^{m} \frac{1}{\ell}$. Thus, $f^{(m+1)}(x) = \dfrac{-1}{(1-x)}$, showing that the formula holds for $k = 1$. Now suppose that the formula is true for k. We then have that $f^{(m+k+1)}(x) = -\dfrac{(k-1)!k}{(1-x)^{k+1}} = -\dfrac{k!}{(1-x)^{k+1}}$, showing our claim for $k+1$. Therefore, the formula holds for each $k \ge 1$. ∎

In particular, put $x = 0$ in the formula of Lemma 2, to get

$$f^{(m+k)}(0) \;=\; -(k-1)! \tag{3}$$

for each $k \geq 1$.

Note: In the following, we use the simple identity

$$\frac{1}{k(k+1)\dots(k+m)} \;=\; \frac{(k-1)!}{(m+k)!}\,.$$

Theorem 3. For each $x \in (-1,1)$

$$f(x) \;=\; \sum_{k=1}^{m}\left(\frac{(-1)^{m-k}\sum_{\ell=m-k+1}^{m}\frac{1}{\ell}}{(m-k)!k!}\right)x^k - \sum_{k=1}^{\infty}\frac{(k-1)!}{(m+k)!}x^{m+k}\,.$$

Proof. According to (2), (3) and by Taylor's Theorem [1], for each $x \in (-1,1)$,

$$f(x) \;=\; \sum_{k=1}^{m}\left(\frac{(-1)^{m-k}\sum_{\ell=m-k+1}^{m}\frac{1}{\ell}}{(m-k)!k!}\right)x^k - \sum_{k=1}^{N}\frac{(k-1)!}{(m+k)!}x^{m+k} + R_N(x)\,,$$

where $R_N(x) = \dfrac{N!}{(m+N+1)!}\zeta_x^{m+N+1}$ for some $\zeta_x \in (-1,1)$. Thus

$$\begin{aligned} |R_N(x)| &= \frac{N!}{(m+N+1)!}|\zeta_x|^{m+N+1} \;\leq\; \frac{N!}{(m+N+1)!} \\ &= \frac{1}{(N+1)(N+2)\dots(N+m+1)} \;\leq\; \frac{1}{N^{m+1}}\,. \end{aligned}$$

Hence for each $x \in (-1,1)$, $\lim\limits_{N\to\infty} R_N(x) = 0$. ∎

We are now in a position to state the main result.

Main Theorem 4. Let $m \geq 1$. Then for each $x \in [-1,1]$

$$\begin{aligned} \sum_{k=1}^{\infty}\frac{x^{m+k}}{k(k+1)\dots(k+m)} \;&=\; \sum_{k=1}^{m}\left(\frac{(-1)^{m-k}\sum_{\ell=m-k+1}^{m}\frac{1}{\ell}}{(m-k)!k!}\right)x^k \\ &\quad - \frac{(x-1)^m}{m!}\log(1-x)\,. \end{aligned}$$

Remark: We are especially interested in justifying the preceding formula for $x = 1$ and for $x = -1$. As a matter of fact, Theorem 3 guarantees our formula for $x \in (-1,1)$.

Proof. By Theorem 3, for each $x \in (-1,1)$

$$\sum_{k=1}^{\infty}\frac{x^{m+k}}{k(k+1)\dots(k+m)} \;=\; \sum_{k=1}^{m}\left(\frac{(-1)^{m-k}\sum_{\ell=m-k+1}^{m}\frac{1}{\ell}}{(m-k)!k!}\right)x^k - f(x)\,. \tag{4}$$

Both sides represent continuous functions, at least on $(-1,1)$. The right side is also continuous at $x=-1$ and $x=1$. Since for each $x \in [-1,1]$

$$\left|\frac{x^{m+k}}{k(k+1)\dots(k+m)}\right| \le \frac{1}{k(k+1)\dots(k+m)} \le \frac{1}{k^{m+1}} \le \frac{1}{k^2},$$

by the Weierstrass M–test, the partial sums $\sum_{k=1}^{N} \frac{x^{m+k}}{k(k+1)\dots(k+m)}$ converge uniformly on $[-1,1]$. The limit of a uniformly convergent sequence of continuous functions is continuous [2]. Therefore the left side of equation (4) is also continuous on $[-1,1]$. Since equality holds for each $x \in (-1,1)$, it also holds for $x=-1$ and $x=1$. ■

Corollary 5 Let $m \ge 1$. Then

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)\dots(k+m)} = \frac{2^m \log 2}{m!} - \sum_{k=1}^{m} \frac{\sum_{\ell=m-k+1}^{m} \frac{1}{\ell}}{(m-k)!k!},$$

and
$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)\dots(k+m)} = \sum_{k=1}^{m} \frac{(-1)^{m-k} \sum_{\ell=m-k+1}^{m} \frac{1}{\ell}}{(m-k)!k!}.$$

Proof. Put $x=-1$ in (4). Thus

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{(-1)^{m+k}}{k(k+1)\dots(k+m)} &= \sum_{k=1}^{m} \left(\frac{(-1)^{m-k} \sum_{\ell=m-k+1}^{m} \frac{1}{\ell}}{(m-k)!k!}\right)(-1)^k - f(-1) \\ &= \sum_{k=1}^{m} \left(\frac{(-1)^{m} \sum_{\ell=m-k+1}^{m} \frac{1}{\ell}}{(m-k)!k!}\right) - \frac{(-2)^m}{m!} \log 2 .\end{aligned}$$

Hence, $$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)\dots(k+m)} = \frac{2^m \log 2}{m!} - \sum_{k=1}^{m} \frac{\sum_{\ell=m-k+1}^{m} \frac{1}{\ell}}{(m-k)!k!}.$$

Finally, put $x=1$ in equation (4). Since $f(1)=0$, then

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)\dots(k+m)} = \sum_{k=1}^{m} \frac{(-1)^{m-k} \sum_{\ell=m-k+1}^{m} \frac{1}{\ell}}{(m-k)!k!}. \qquad ■$$

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2. Walter Rudin, *Principles of Mathematical Analysis*, 3rd Edition, McGraw-Hill Book Company, 1976.

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THE SKOLIAD CORNER

No. 57

Shawn Godin

Solutions may be sent to Shawn Godin, Cairine Wilson S.S., 975 Orleans Blvd., Orleans, ON, CANADA, K1C 2Z5, or emailed to

mayhem-editors@cms.math.ca

Please include on any correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 March 2002*. Look for prizes for solutions in the new year.

Our three entries this issue come from the BC mathematics competitions. My thanks go to Jim Totten of the University College of the Cariboo and Clint Lee of Okanagan University College for forwarding the material to me.

BRITISH COLUMBIA COLLEGES

Junior High School Mathematics Contest, 2001

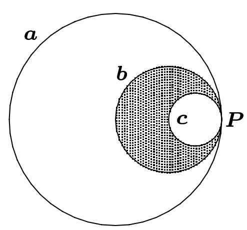
Final Round – Part A

Friday May 4, 2001

1. The integer 9 is a perfect square that is both two greater than a prime number, 7, and two less than a prime number, 11. Another such perfect square is:

   (a) 25 (b) 49 (c) 81 (d) 121 (e) 169

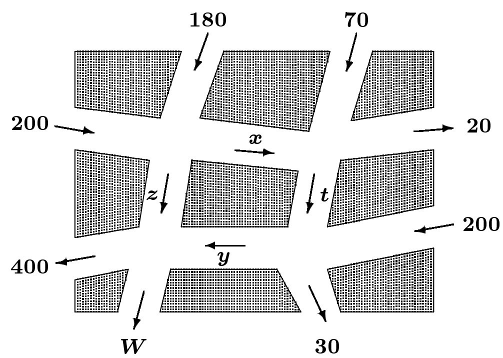
2. Three circles, a, b, and c, are tangent to each other at point P, as shown.



   The center of b is on c and the center of a is on b. The ratio of the area of the shaded region to the total area of the unshaded regions enclosed by the circles is:

   (a) $3:13$ (b) $1:3$ (c) $1:4$ (d) $2:9$ (e) $1:25$

3. Here is a diagram of part of the downtown in a medium sized town in the interior of British Columbia. The arrows indicate one-way streets. The numbers or letters by the arrows represent the number of cars that travel along that portion of the street during a typical week day.



Assuming that no car stops or parks and that no cars were there at the beginning of the day, the value of the variable W is:

(a) 30 (b) 200 (c) 250 (d) 350 (e) 600

4. The corners of a square of side x are cut off so that a regular octagon remains. The length of each side of the resulting octagon is:

(a) $\frac{\sqrt{2}}{2}x$ (b) $2x\left(2+\sqrt{2}\right)$ (c) $\frac{x}{\sqrt{2}-1}$
(d) $x\left(\sqrt{2}-1\right)$ (e) $x\left(\sqrt{2}+1\right)$

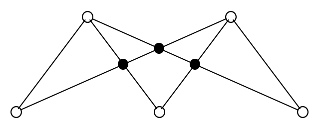
5. The value of $\left(0.0\overline{1}\right)^{-1}+1$ is: (The line over the digit 1 means that it is repeated indefinitely.)

(a) $\frac{1}{91}$ (b) $\frac{90}{91}$ (c) $\frac{91}{90}$ (d) 10 (e) 91

6. The people living on Sesame Street all decide to buy new house numbers from the same store, and they purchase the digits for their house numbers in the order of their addresses: $1, 2, 3, \ldots$. If the store has 100 of each digit, then the first address which cannot be displayed occurs at house number:

(a) 100 (b) 101 (c) 162 (d) 163 (e) 199

7. Given p dots on the top row and q dots on the bottom row, draw line segments connecting each top dot to each bottom dot. (In the diagram below, the dots referred to are the small open circles.) The dots must be arranged such that no three line segments intersect at a common point (except at the ends). The line segments connecting the dots intersect at several points. (In the diagram below, the points of intersection of the line segments are the small filled circles.) For example, when $p = 2$ and $q = 3$ there are three intersection points, as shown below.



When $p = 3$ and $q = 4$ the number of intersections is:

(a) 7 (b) 12 (c) 18 (d) 21 (e) 27

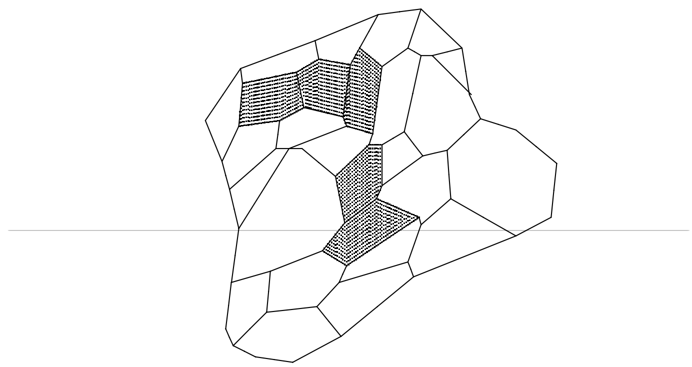
8. At one time, the population of Petticoat Junction was a perfect square. Later, with an increase of 100, the population was 1 greater than a perfect square. Now, with an additional increase of 100, the population is again a perfect square. The original population was a multiple of:

(a) 3 (b) 7 (c) 9 (d) 11 (e) 17

9. The cashier at a local movie house took in a total of \$100 from 100 people. If the rates were \$3 per adult, \$2 per teenager and 25 cents per child, then the smallest number of adults possible was:

(a) 0 (b) 2 (c) 5 (d) 13 (e) 20

10. The island of Aresia has 27 states each of which belongs to one of two factions, the white faction and the grey faction, who are sworn enemies. The United Nations wishes to bring peace to Aresia by converting one state at a time to the opposite faction; that is, converting one state from white to grey or from grey to white, so that eventually all states belong to the same faction. In doing this they must guarantee that no single state is completely surrounded by states of the **opposite faction**. Note that a coastal state can never be completely surrounded, and that it may be necessary to convert a state from one faction to the other at one stage and then convert it back to its original faction later. A map of the state of Aresia is shown.



The five shaded states belong to the grey faction, and all of the unshaded states belong to the white faction. The minimum number of conversions necessary to completely pacify Aresia is:

(a) **5** (b) **7** (c) **9** (d) **10** (e) **15**

BRITISH COLUMBIA COLLEGES

Junior High School Mathematics Contest, 2001

Final Round – Part B

Friday May 4, 2001

1. Find the smallest 3–digit integer which leaves a non-zero remainder when divided by any of 2, 3, 4, 5, or 6 but not when divided by 7.

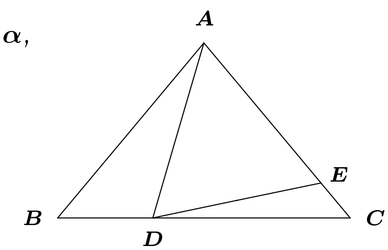
2. Assume that the land within two kilometres of the South Pole is flat. There are points in this region where you can travel one kilometre south, travel one kilometre east along *one* circuit of a latitude, and finally travel one kilometre north, and thus arrive at the point where you started. How far is such a point from the South Pole?

3. Café de la Pêche offers three fruit bowls:
   - Bowl A has two apples and one banana;
   - Bowl B has four apples, two bananas, and three pears;
   - Bowl C has two apples, one banana, and three pears.

   Your doctor tells you to eat exactly 16 apples, 8 bananas and 6 pears each day. How many of each type of bowl should you buy so there is no fruit left over? Find all possible answers. (The numbers of bowls must be non-negative integers.)

4. In the triangle shown, $\angle BAD = \alpha$, $\overline{AB} = \overline{AC}$ and $\overline{AD} = \overline{AE}$.

   Find $\angle CDE$ in terms of α.



5. In the multiplication below each of the letters stands for a distinct digit. Find all values of $JEEP$.

$$\begin{array}{r} JEEP \\ \times\, JEEP \\ \hline BEEBEEP \end{array}$$

BRITISH COLUMBIA COLLEGES

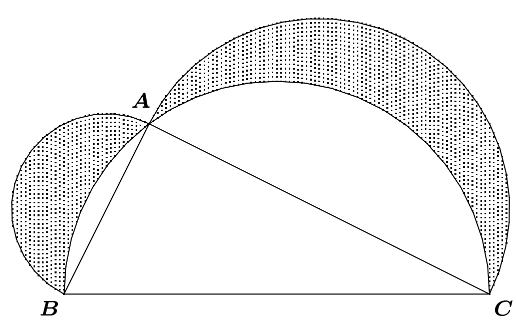
Senior High School Mathematics Contest, 2001

Final Round – Part B

Friday May 4, 2001

1. See question #4 above.

2. A semicircle BAC is mounted on the side BC of the triangle ABC. Semicircles are also mounted outwardly on the sides BA and AC, as shown in the diagram. The shaded crescents represent the area inside the smaller semicircles and outside the semicircle BAC. Show that the total shaded area equals the area of the triangle ABC.



3. Five schools competed in the finals of the British Columbia High School Track Meet. They were Cranbrook, Duchess Park, Nanaimo, Okanagan Mission, and Selkirk. The five events in the finals were: the high jump, shot put, 100–metre dash, pole vault and 4–by–100 relay. In each event the school placing first received five points; the one placing second, four points; the one placing third, three points; and so on. Thus, the one placing last received one point. At the end of the competition, the points of each school were totalled, and the totals determined the final ranking.

   (a) Cranbrook won with a total of 24 points.

   (b) Sally Sedgwick of Selkirk won the high jump hands down (and feet up), while Sven Sorenson, also of Selkirk, came in third in the pole vault.

   (c) Nanaimo had the same number of points in at least four of the five events.

   Each school had exactly one entry in each event. Assuming there were no ties and the schools ended up being ranked in the same order as the alphabetical order of their names, in what position did Doug Dolan of Duchess Park rank in the high jump?

4. A box contains tickets of two different colours: blue and green. There are 3 blue tickets. If two tickets are to be drawn together at random from the box, the probability that there is one ticket of each colour is exactly $\frac{1}{2}$. How many green tickets are in the box? Give all possible solutions.

5. In (a), (b), and (c) below the symbols m, h, t, and u can represent any integer from 0 to 9 inclusive.

   (a) If $h - t + u$ is divisible by 11, prove that $100h + 10t + u$ is divisible by 11.

   (b) If $h + u = m + t$, prove that $1000m + 100h + 10t + u$ is divisible by 11.

   (c) Is it possible for $1000m + 100h + 10t + u$ to be divisible by 11 if $h + u \neq m + t$? Explain.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 977 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z7 (NEW!)**. The electronic address is
NEW! mayhem-editors@cms.math.ca NEW!

The Assistant Mayhem Editor is Chris Cappadocia (University of Waterloo). The rest of the staff consists of Adrian Chan (Harvard University), Jimmy Chui (University of Toronto), Donny Cheung (University of Waterloo), and David Savitt (Harvard University).

Mayhem Problems

Proposals and solutions may be sent to MATHEMATICAL MAYHEM, c/o Faculty of Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, ON, N2L 3G1, or emailed to
mayhem-editors@cms.math.ca

Please include on any correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 March 2002*. Look for prizes for solutions in the new year.

M15. *Proposed by Richard Hoshino, Dalhousie University, Halifax, Nova Scotia.*

Let $ABCDEF$ be a regular hexagon with area S. Let T be the area of the hexagonal region common to both $\triangle ACE$ and $\triangle BDF$. Determine $\frac{S}{T}$.

M16. *Proposed by the Mayhem staff.*

Can an 8×9 checkerboard be completely covered by twelve 1×6 rectangles?

M17. *Proposed by Andy Liu, University of Alberta, Edmonton, Alberta.*

Seven theoretical and eleven experimental physicists were working in a low-temperature laboratory. Each day, a different physicist is in charge, and the cycle repeats when everyone has had a turn. One of the privileges of being in charge is the control of the thermostat, initially set at 0°C. When a theoretical physicist is in charge, she raises the temperature by 1.1°C. When an experimental physicist is in charge, he lowers it by 0.7°C. What is the probability that in an eighteen-day cycle, the temperature is below 0° for exactly nine days?

M18. *Proposed by Richard Hoshino, Dalhousie University, Halifax, Nova Scotia.*

For each integer n, let n^* be the integer n written backwards. For example if $n = 1234$ then $n^* = 4321$. We say that a four-digit integer n is ***magical*** if both $n + n^*$ and $n - n^*$ are (positive) palindromes. For example, 2001 is magical. If n is magical, determine all possible values of $n - n^*$.

M19. *Proposed by the Mayhem staff.*

On the magical island of Xurc, there lives a giant Ecurb. Ecurb has an unlimited supply of special coins that are worth one million dollars each. Ecurb allows people to go into his castle and take as many of these coins as they like, but, they must give some up in order to cross the bridges to leave his island. At each of the five bridges Ecurb demands that you give $\frac{99}{100}$ of a coin more than $\frac{99}{100}$ of the coins in your possession. Coins cannot be cut or broken in any way. If the demand cannot be met Ecurb takes all of your coins and eats one of your feet. How many coins do you have to start with in order to make it off the island with exactly one coin (and both feet)?

M20. *Proposed by Richard Hoshino, Dalhousie University, Halifax, Nova Scotia.*

Suppose that A, B and C are positive integers in arithmetic progression with $A^\circ < B^\circ < C^\circ < 180^\circ$.

If $\sin A^\circ + \sin B^\circ = \sin C^\circ$ and $\cos A^\circ - \cos B^\circ = \cos^\circ C$, determine the triplet (A, B, C).

M21. *Proposed by the Mayhem staff.*

Find all positive integers a, b, c, d, and e which satisfy

$$a! \;=\; b! + c! + d! + e!\,.$$

Problem of the Month

Jimmy Chui, student, University of Toronto

Problem. The sum of the series $\frac{25}{72} + \frac{25}{90} + \frac{25}{110} + \frac{25}{132} + \cdots + \frac{25}{9900}$ is

(a) 2.25 (b) 2.5 (c) 2.8125 (d) 2.875 (e) $3.044\dot{1}\dot{9}$

(1996 Cayley, Problem 24)

Solution. In this problem, we want to find a pattern, and hope that we can find a method to simplify the problem, and in effect, answer it.

The numerators are all equal to 25, and the denominators are $72 = 8 \cdot 9$, $90 = 9 \cdot 10$, $110 = 10 \cdot 11$, $132 = 11 \cdot 12$, $\cdots$, $9900 = 99 \cdot 100$.

If we try adding up the first two fractions, we get

$$\frac{25}{72}+\frac{25}{90} \;=\; 25\cdot\left(\frac{10}{720}+\frac{8}{720}\right) \;=\; 25\cdot\frac{18}{720} \;=\; 25\cdot\frac{1}{40}\,.$$

If we try adding up the first three fractions, we have

$$\frac{25}{40}+\frac{25}{110} \;=\; 25\cdot\left(\frac{11}{440}+\frac{4}{440}\right) \;=\; 25\cdot\frac{15}{440} \;=\; 25\cdot\frac{3}{88}\,.$$

There is a pattern emerging! The first fraction itself is $25\cdot\frac{1}{72}=25\cdot\frac{1}{8\cdot 9}$.

The first two fractions have a sum of $25\cdot\frac{1}{40}=25\cdot\frac{2}{8\cdot 10}$.

The first three fractions have a sum of $25\cdot\frac{3}{88}=25\cdot\frac{3}{8\cdot 11}$.

By examining this pattern, we can guess that the answer for the entire sum is

$$25\cdot\frac{92}{8\cdot 100} \;=\; \frac{92}{8\cdot 4} \;=\; \frac{23}{8} \;=\; 2.875\,.$$

Hence, with pretty good confidence, we can say that the answer is (d)!

A more mathematical argument would show why the pattern works. For problems of this type (adding terms that exhibit a pattern), one possible approach is to look for some sort of telescoping series.

If we recognize that $\frac{1}{8\cdot 9}=\frac{1}{8}-\frac{1}{9}$, $\frac{1}{9\cdot 10}=\frac{1}{9}-\frac{1}{10}$, $\dots$, $\frac{1}{99\cdot 100}=\frac{1}{99}-\frac{1}{100}$, then the expression is evaluated very quickly. Indeed, we have

$$\begin{aligned}\frac{25}{72}+\frac{25}{90}+\cdots+\frac{25}{9900} &= 25\cdot\left\{\left(\frac{1}{8}-\frac{1}{9}\right)+\left(\frac{1}{9}-\frac{1}{10}\right)+\cdots+\left(\frac{1}{99}-\frac{1}{100}\right)\right\}\\ &= 25\cdot\left(\frac{1}{8}-\frac{1}{100}\right)\\ &= 25\cdot\frac{92}{8\cdot 100}\,,\end{aligned}$$

which is identical to what we guessed above, and evaluates to 2.875.

Polya's Paragon

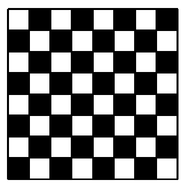
Shawn Godin

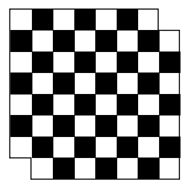
"It is amazing how a splash of colour makes anything look fresh and new," my wife says to me, handing me a can of paint. Walls of a bedroom, a bench on the porch, or the deck stand out with their new clean look. In many cases a splash of colour can shed some light on a mathematical problem as well.

As an example, let us look at a well-known problem that has appeared in many books:

A checkerboard is an 8×8 grid of squares. If a domino covers exactly 2 squares, we can easily cover the 64 squares with 32 dominoes. Suppose that two diagonally opposite squares are removed, can the new board be covered by 31 dominoes?

If we colour the checkerboard in the standard fashion, as in the diagram below, we notice that the opposite corners are the same colour. Thus, the board started with 32 black squares and 32 white squares, but ended with 30 black squares and 32 white squares. Since a domino covers a black and a white square we can fit only 30 dominoes on the board, and 2 white squares will be left uncovered.

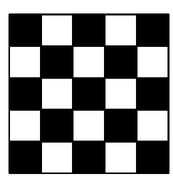




Consider the following related problem.

A class of 25 students is arranged in 5 rows and 5 columns. If a student can move to a desk directly in front, behind, to the left or to the right of his own desk, can all students move and occupy a new desk?

It would seem that you should be able to do it, but if we colour the desks in the same pattern as the checkerboard we notice that there are 13 black desks and 12 white desks. The allowed moves take a person to a desk of a different colour. Thus, we will not be able to place all of students that started in black desks, and our task is impossible.



This method of colouring is closely linked to the idea of **parity**, the evenness or oddness of a number. In the last example, if we numbered the desks 1 through 25, as below, we see that a person starting in an even numbered desk must move to an odd numbered desk and vice versa. Since we started with 12 even and 13 odd numbered desks one person would be left out seatless or unable to change.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

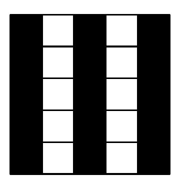
One of the problems with colouring proofs is that they are generally good for proving that something is impossible, but they do not prove that something is possible although, in some cases, it may give us a hint.

Consider again the class of 25 students in 5 rows and 5 columns. Imagine this time that they must move to a neighbour diagonally. Using the previous colouring we see that we remain on the same colour (or alternately, if we had an even numbered seat, we move to another even numbered seat, and similarly for odd numbered seats). You may be convinced at this time that the task is thus possible. Go ahead and try to do it, I will just wait here . . . .

Back so soon? What is that, you have had no luck? That is too bad. Sit down, and let us take another look. Suppose we decided to be really artistic and paint our classroom desks in four different colours. I will number the colours 1, 2, 3 and 4 because they will not let me print this section of Mayhem in multiple colours! The things a true *artist* has to deal with!

1	2	1	2	1
4	3	4	3	4
1	2	1	2	1
4	3	4	3	4
1	2	1	2	1

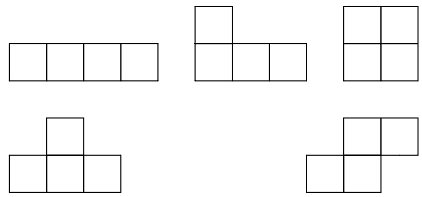
We see here that anything that started on a 1 (which I would have coloured teal) ends up on a 3 (chartreuse) and vice versa. Similarly the number 2's (magenta) swap places with the number 4's (periwinkle). The even numbers pose no problems; there are 6 of each which seems OK. But there are nine teal coloured squares (1's) and only 4 chartreuse (3's). Thus we see that this too is an impossibility. (For a less artistic colouring you can consider the one below).



Here are some problems to ponder:

1. Can an 8×8 board be covered with twenty one 3×1 rectangles and one 1×1 rectangle? If so, find a tiling.

2. A tetromino is four squares glued together along the edges. There are 5 tetrominoes pictured below. Can a 4×5 board be covered by one complete set of tetrominoes? If so, find a tiling.

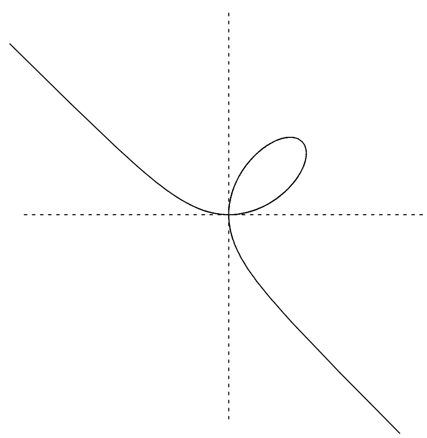


3. A rectangular floor is covered by 3×3 and 1×9 tiles. One of the tiles got smashed, but there is a tile of the other type available. Show that the floor cannot be covered by rearranging the tiles.

Challenge Board Solutions

David Savitt

C95. Prove that the curve $x^3 + y^3 = 3xy$, has a horizontal tangent at the origin. (This curve is known as the Folium of Descartes.)



Solution.

Consider the map P from the (X, Y)–plane to the (x, y)–plane given by sending (X, Y) to (X, XY).

Lemma. If $Y = f(X)$ is a C^1 function and $f(0) = 0$, then the image of the graph of f under the map P has a horizontal tangent at the origin.

Proof. For each point $(X, f(X))$ of the graph of f in the (X, Y)–plane, the image $P(X, f(X))$ in the (x, y)–plane is the point $(X, Xf(X))$. Hence the image under P of the graph of f is the graph $y = xf(x)$. This function passes through the origin, and since f is C^1, we may check that this graph has a horizontal tangent at the origin by differentiating: indeed, $y' = xf'(x) + f(x)$, and since $f(0) = 0$ we evidently have $y'(0) = 0$.

Now we can solve the given problem. Consider the function $X = g(Y) = 3Y/(1+Y^3)$. It is continuously differentiable and one-to-one in a neighbourhood of the origin, and $g'(0) \neq 0$, so by the Inverse Function Theorem, we may write $Y = f(X)$ with f a C^1 function in a neighbourhood of the origin, and $f(0) = 0$. By the lemma, the image of the graph $(X, f(X))$ under P in a neighbourhood of the origin — and so the image of the graph of $(g(Y), Y)$ under P as well — has a horizontal tangent at the origin.

But the points of the graph of $X = g(Y)$ satisfy $X(1+Y^3) = 3Y$. Multiplying through by X^2, we get $X^3 + (XY)^3 = 3X(XY)$, so that we see that the image of this graph under P satisfies $x^3 + y^3 = 3xy$. Hence the latter has a horizontal tangent at the origin.

C96. Recall that a *bipartite graph* is a graph whose vertices may be divided into two non-empty disjoint sets (call them L and R, for left and right) so that all of the edges of the graph connect a vertex in L to a vertex in R. In other words, no two vertices in L are joined by an edge, and similarly for R. Let G be a bipartite graph with 27 edges and in which L and R each contain exactly 9 vertices. Show that we can find three vertices l_0, l_1, $l_2 \in L$ and three vertices r_0, r_1, $r_2 \in R$ such that at least six of the nine potential edges l_0r_0, l_0r_1, l_0r_2, l_1r_0, l_1r_1, l_1r_2, l_2r_0, l_2r_1, l_2r_2 are indeed edges of G.

Solution.

Assume a counterexample. We will steadily eliminate possibilities for the list of degrees of the vertices. (Recall that the degree of a vertex is the number of edges touching that vertex.) As this list becomes more and more restricted, the problem becomes more and more rigid, until it is easy to see that a counterexample cannot exist. We will refer to the vertices l_m as left-hand vertices, and to the r_i as right-hand vertices.

We begin by eliminating the possibility that a vertex has large degree. For example, suppose l_1 has degree exactly 5, with edges from l_1 to each of $r_1, \dots, r_5$. If some other l_m has edges to two vertices r_i and r_j with $1 \leq i, j \leq 5$, then no other l_n could have an edge to any r_k with $1 \leq k \leq 5$ or else there would be six edges between l_1, l_m, l_n, r_i, r_j, and r_k. Hence there are at most 8 edges from $l_2, \dots, l_9$ to $r_1, \dots, r_5$, and the total of the degrees of r_1 through r_5 is at most 13. It follows that $r_6, \dots, r_9$ have degrees summing to at least 14, and therefore the three of those with largest degree have degrees summing to at least 11. But these 11 edges connect to the eight vertices l_2 through l_9 and not to l_1, so by the Pigeonhole Principle, at least 6 of these 11 edges connect to a subset of only three left-hand vertices. Thus, we have found the expected three left-hand and three right-hand vertices with 6 edges between them. Therefore, there is no vertex of degree exactly 5. One can use (easier versions of) the same argument to see as well that no vertex may have degree larger than 5.

Next, we show that there cannot be two left-hand vertices of degree 4. Certainly there can be at most two: if there were three left-hand vertices of degree 4, then these three vertices would have total degree 12, and so by the same pigeonhole argument as above, there would be three right-hand vertices with 6 edges to these three left-hand vertices. Therefore, if there are two left-hand vertices of degree 4, there must also be at least five left-hand vertices of degree exactly 3. Suppose first that l_1 connects to $r_1, \dots, r_4$ and l_2 connects to $r_5, \dots, r_8$. If some other l_m connected to three of $r_1, \dots, r_8$, then those three right-hand vertices, together with l_1, l_2, and l_m, would provide the sought-for subgraph. Therefore each of the five l_m of degree exactly 3 must have an edge to r_9. This is a contradiction, since r_9 would then have degree at least 5. Hence the neighbourhoods of l_1 and l_2 must not be disjoint. (Recall that the neighbourhood of a vertex is the set of vertices to which it is joined by an edge.)

However, if l_1 connects to $r_1, \dots, r_4$ and l_2 connects to $r_4, \dots, r_7$, this leaves 19 edges emanating from $l_3, \dots, l_9$, at most 9 of which may connect to r_8 and r_9. (Otherwise, since neither joins to l_1 or l_2, the Pigeonhole Principle would guarantee that the neighbourhoods of r_8 and r_9 have intersection of size at least three, giving the expected subgraph with 6 edges.) This leaves at least 10 edges joining $l_3, \dots, l_9$ to $r_1, \dots, r_7$. Evidently (more than) one l_m joins to two vertices r_i and r_j, with $3 \leq m \leq 9$ and $1 \leq i, j \leq 7$. Then there are six edges between l_1, l_2, l_m and r_i, r_j, r_4 (or, if either $i = 4$ or $j = 4$, between l_1, l_2, l_m and r_i, r_j, r_k with k distinct from i and j and $1 \leq k \leq 7$). This shows that the neighbourhoods of l_1 and l_2 must have at least two vertices in common. Yet if l_1 connects to $r_1, \dots, r_4$ and l_2 connects to $r_3, \dots, r_6$, it is easy to see that more than one r_i, $i = 1$, 2, 5, 6, must connect to a vertex l_m besides l_1 and l_2 (for example, because otherwise there would be three vertices of degree 2, requiring either a vertex of degree 5 or three of degree 4). Then there are six edges between l_1, l_2, l_m and r_3, r_4, r_i. Since plainly the neighbourhoods of l_1 and l_2 cannot have three vertices in common, this concludes the demonstration that there are not two left-hand vertices of degree exactly 4.

This leaves only two possibilities for the list of degrees of the left-hand (and, similarly, the right-hand) vertices: 4, 3, 3, 3, 3, 3, 3, 3, 2 and 3, 3, 3, 3, 3, 3, 3, 3, 3. Either way, there are at least seven left-hand vertices of degree exactly 3, at most four of which can connect to a vertex of degree 4, and at most two of which can connect to a vertex of degree 2. Therefore, there exists a vertex of degree 3 which connects to three vertices of degree 3; without loss of generality, suppose that l_1 connects to r_1, r_2, r_3, all of degree exactly 3. If, say, r_1 and r_2 both connected to l_m, with $m \neq 1$, then choosing any other l_n connected to r_1, r_2, or r_3 gives 6 edges between l_1, l_m, l_n and r_1, r_2, r_3. Without loss of generality, we can therefore suppose that r_1 connects to l_1, l_2, l_3, that r_2 connects to l_1, l_4, l_5, and that r_3 connects to l_1, l_6, l_7. If some other r_i were connected to two l_m and l_n out of $l_2, \dots, l_7$, then if l_m and l_n connect to r_j and r_k respectively, $1 \leq j, k \leq 3$, we would have six edges between l_1, l_m, l_n and r_i, r_j, r_k. Therefore each of $r_4, \ldots, r_9$ connects to at most one of $l_1, \ldots, l_7$, and the total of the degrees of l_1 through l_7 is at most $9 + 6 = 15$. Hence the degrees of l_8 and l_9 sum to at least 12. This is certainly a contradiction, completing our solution.

C97. Given a positive integer n, let $\overline{0}, \overline{1}, \ldots, \overline{n-1}$ denote the integers modulo n (so that $\overline{a}$ is the reduction of a modulo n). Find all positive integers n with the property that the set

$$\{\overline{a} \mid 0 < a < n/2 \text{ with } a \text{ and } n \text{ relatively prime}\}$$

is a group under multiplication.

Solution by Kiran Kedlaya, University of California-Berkeley, Berkeley, CA, USA.

If $n = 2k+1$ is odd and $n \geq 5$, then 2 and k are relatively prime to n and lie below $n/2$, while the reduction mod n of $2 \cdot k$ is $\overline{n-1}$. Therefore, the only odd n with the desired property is $n = 3$.

Henceforth, assume n is even and greater than 2. If n satisfies the desired condition, then every integer in the interval $(n/2, n)$ which is coprime to n must in fact be a prime, and therefore must divide $\binom{n}{n/2}$. It follows that

$$2^n \;\geq\; \binom{n}{n/2} \;\geq\; (n/2)^{\phi(n)/2}\,,$$

where ϕ denotes Euler's ϕ–function: $\phi(n)$ is the number of integers between 1 and n which are relatively prime to n. Now

$$\phi(n) \;=\; n\prod_{i=1}^{k}\left(1 - \frac{1}{p_k}\right)\,,$$

where $p_1, \ldots, p_k$ are the distinct primes dividing n. This product is bounded below by

$$\left(1-\frac{1}{2}\right)\cdot\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k+1}\right) \;=\; \frac{1}{k+1}\,,$$

and combining this with the previous inequality yields

$$2^n \;\geq\; (n/2)^{n/2(k+1)}$$

or

$$2k+3 \;\geq\; \log_2(n)\,.$$

To bound k from above in terms of n, we have the inequality $n \geq 2^k$. This will not be of any help, but similarly we always have $n \geq 2 \cdot 3^{k-1}$, and $n \geq 2 \cdot 3 \cdot 5^{k-2}$, and so on. For example, $n \geq 2 \cdot 3 \cdots 13 \cdot 17^{k-6}$, and therefore,

$$k \;\leq\; 6 + \log_{17}(n/30030)\,.$$

Hence,

$$15 + 2\log_{17}(n/30030) \;\geq\; \log_2 n$$

which amounts to $7.7220\ldots \geq 0.7367\ldots \cdot \log_e(n)$, yielding

$$n \;\leq\; 35625\,.$$

With this tractable upper-bound on n, we can apply a computer search. For example, we have already observed that if n has the desired property, then every integer between $n/2$ and n which is coprime to n must be prime. Thus, if $\pi(x)$ denotes the number of primes less than or equal to x, we must have $\pi(n) - \pi(n/2) = \phi(n)/2$ for $n > 2$. For integers in the range under consideration, these functions are quickly computed, and a one-line program in any mathematical computation package (such as Maple, Mathematica, etc.) yields the possibilities $n = 4$, 6, 8, 12, 18, 20, 24, 30. Of these, both $n = 18$ ($7 \cdot 7 = 49 \equiv 13$) and $n = 30$ ($11 \cdot 13 = 143 \equiv 23$) fail, but the remainder succeed. Hence, the complete list of integers which satisfy the conditions of the problem is $n = 2$, 3, 4, 6, 8, 12, 20, 24.

C98. Find all pairs of integers (x, y) which satisfy the equation

$$x^2 - 34y^2 = -1\,.$$

Solution.

The equation has no integer solutions. We will examine the equation $x^2 - 34y^2 = \pm 1$, and we will make a great deal of use of the factorization

$$(x + \sqrt{34}y)(x - \sqrt{34}y) \;=\; \pm 1\,.$$

Notice that if (x, y) is a solution to the equation, then so are $(\pm x, \pm y)$. Suppose that x, $y > 0$. Then $x + y\sqrt{34} > 1$, so

$$|x - y\sqrt{34}| \;=\; 1/|x + y\sqrt{34}| \;<\; 1\,.$$

It follows that if $x^2 - 34y^2 = \pm 1$, then $x + \sqrt{34}y$ is greater than 1 if and only if x and y are both positive (and less than -1 if and only if they are both negative, and between -1 and 1 if and only if they have mixed signs).

It follows that if $x^2 - 34y^2 = \pm 1$ has any solution with $x + \sqrt{34}y > 1$, there must be such a solution (X, Y) with $X + \sqrt{34}Y > 1$ minimal. For example, $35^2 - 34 \cdot 6^2 = 1$, and $35 + 6 \cdot \sqrt{34} < 70$. Therefore, any solution (x, y) with smaller $x + \sqrt{34}y > 1$ would have to have $1 \leq x < 70$ and $1 \leq y \leq 70/\sqrt{34} < 13$; since there are only a finite number of possibilities, a minimal solution must exist. It is easy to verify that, in fact, $\epsilon = 35 + 6 \cdot \sqrt{34}$ is minimal. Note crucially that $35^2 - 34 \cdot 6^2 = 1$, not -1.

Now if (x, y) is any solution with x, $y > 0$, if $(x, y) \neq (35, 6)$, we have $x + y\sqrt{34} > \epsilon$. Considering the product

$$\begin{aligned}(x + y\sqrt{34})\epsilon^{-1} \;&=\; (x + y\sqrt{34})(35 - 6\sqrt{34}) \\ &=\; (35x - 204y) + (35y - 6x)\sqrt{34} > 1\end{aligned}$$

we find that $(35x - 204y, 35y - 6x)$ must be a smaller solution with $35x - 204y, 35y - 6x > 0$. Repeating this process, it follows eventually that $(x + y\sqrt{34})\epsilon^{-n} = \epsilon$; that is, $x + y\sqrt{34} = (35 + 6\sqrt{34})^{n+1} = \epsilon^{n+1}$. But then $x - y\sqrt{34} = (35 - 6\sqrt{34})^{n+1} = \epsilon^{-(n+1)}$, so $x^2 - 34y^2 = \epsilon^{n+1}\epsilon^{-(n+1)} = 1$. It follows, easily, that the equation $x^2 - 34y^2 = \pm 1$ has solutions only such that $x^2 - 34y^2 = 1$.

Remark 1.

This equation is a special case of a well-studied class of equations known as Pell's equation: $x^2 - Dy^2 = \pm 1$. The reason this special case is of particular interest is that while in fact $x^2 - 34y^2 = -1$ has no integer solutions, it *does* have rational solutions, for example $(3/5)^2 - 34(1/5)^2 = -1$. The existence of this rational solution, combined with a little bit of work for powers of 5, implies that $x^2 - 34y^2 = -1$ has a solution modulo n for any integer n. Thus we have found an example of an equation with an integer solution mod n for any n, but no integer solution. (Incidentally, while Pell may have studied this equation, it appears that Euler mis-attributed to Pell a solution due to William Brouncker. Thus, the equation is likely ill-named!)

Remark 2.

The argument we have given shows in general that for Pell's equation $x^2 - Dy^2 = \pm 1$, if there exists at least one non-trivial solution, then there is a positive solution (X, Y) so that $\epsilon = X + Y\sqrt{D} > 1$ is minimal, and every solution (x, y) of the equation is given via $x + \sqrt{D}y = \pm\epsilon^n$ for integer n. Moreover, the argument shows that a solution with $x^2 - Dy^2 = -1$ exists only when $X^2 - DY^2 = -1$; in that case, the solutions $x^2 - Dy^2 = -1$ correspond precisely to odd powers of ϵ.

Remark 3.

In fact for any D, Pell's equation always has a non-trivial solution. For example, one way to find the solution $35^2 - 34 \cdot 6^2 = 1$ is as follows: the continued fraction expansion of $\sqrt{34}$ is $[5, 1, 4, 1, 10, 1, 4, 1, 10, \dots] = [5, \overline{1, 4, 1, 10}]$, and the minimal solution ϵ corresponds to the convergent

$$[5, 1, 4, 1] \;=\; 5 + \cfrac{1}{1 + \frac{1}{4 + \frac{1}{1}}} \;=\; 35/6\,.$$

The fact that the solution yields $+1$ instead of -1 corresponds to the fact that the number of terms in this convergent is even. It is, so far as I know, an unsolved problem to characterize those D for which the continued fraction of $\sqrt{D}$ has odd length; that is, for which $x^2 - Dy^2 = -1$ has a solution.

C99. Find all collections of polynomials p_{11}, p_{12}, p_{21}, p_{22} with complex coefficients satisfying the relation

$$\begin{pmatrix} p_{11}(XY) & p_{12}(XY) \\ p_{21}(XY) & p_{22}(XY) \end{pmatrix} \;=\; \begin{pmatrix} p_{11}(X) & p_{12}(X) \\ p_{21}(X) & p_{22}(X) \end{pmatrix} \cdot \begin{pmatrix} p_{11}(Y) & p_{12}(Y) \\ p_{21}(Y) & p_{22}(Y) \end{pmatrix}\,.$$

Solution.
We begin with the observation that if

$$f(XY) \;=\; \sum_{i=1}^{n} g_i(X)h_i(Y)\,,$$

then the polynomial f has at most n non-zero terms. To see this, first note that for any polynomial in two variables $q(X,Y) = \sum_{j,k} a_{j,k}X^jY^k$ we may consider the matrix of coefficients $(a_{j,k})$. For the products $g_i(X)h_i(Y)$, this matrix has rank 1, and so the matrix of coefficients of the polynomial $\sum_{i=1}^{n} g_i(X)h_i(Y)$ has rank at most n. However, the rank of the matrix associated to $f(XY)$ is exactly equal to the number of non-zero terms of f, and so this number is at most n.

For example, suppose $f(XY) = f(X)f(Y)$. Applying our observation, either $f = 0$ or f is a monomial $f(X) = cX^n$, and in the latter case we see $c(XY)^n = cX^ncY^n$ and so $c = 1$. Hence $f(X) = 0$ or $f(X) = X^n$ for some non-negative integer n.

Let $P(X)$ denote our two-by-two matrix of polynomials

$$\begin{pmatrix} p_{11}(X) & p_{12}(X) \\ p_{21}(X) & p_{22}(X) \end{pmatrix}\,.$$

Then $P(X)P(Y) = P(XY)$ implies that $P(1)^2 = P(1)$. This leaves three possibilities for the minimal polynomial for $P(1)$: the minimal polynomial is either $P(1) = 0$, $P(1)(P(1) - I) = 0$, or $P(1) - I = 0$. In the middle case, $P(1)$ has eigenvalues 0 and 1, and therefore, there exists a matrix A with complex entries such that

$$P(1) \;=\; A^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A\,.$$

We now consider the three cases in turn.

Case 1. If $P(1) = 0$, then $P(X) = P(X)P(1) = 0$. Hence P is identically 0.

Case 2. If $P(1)$ satisfies

$$P(1) \;=\; A^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A\,,$$

we consider the conjugate matrix $Q(X) = AP(X)A^{-1}$. Then it is still the case that $Q(XY) = Q(X)Q(Y)$, and now

$$Q(1) \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\,.$$

But then

$$Q(X) \;=\; Q(1)Q(X) \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \;=\; \begin{pmatrix} q_{11} & q_{12} \\ 0 & 0 \end{pmatrix}$$

and similarly

$$Q(X) \;=\; Q(X)Q(1) \;=\; \begin{pmatrix} q_{11} & 0 \\ q_{21} & 0 \end{pmatrix}$$

from which it follows that

$$Q(X) \;=\; \begin{pmatrix} q_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

and $q_{11}(XY) = q_{11}(X)q_{11}(Y)$. By our earlier observation, it follows that $q_{11}(X) = X^m$ for some non-negative m, and in summary we have obtained:

$$P(X) \;=\; A^{-1} \begin{pmatrix} X^m & 0 \\ 0 & 0 \end{pmatrix} A$$

for a complex matrix A.

Case 3. If we have $P(1) = I$, our solution follows the method of Case 2, but is more complicated. In this case $p_{ij}(XY) = p_{i1}(X)p_{1j}(Y) + p_{i2}(X)p_{2j}(Y)$, and by our initial observation, it follows that each p_{ij} has at most two terms. However, we can say even more: for an invertible complex matrix A, the conjugate $Q(X) = A^{-1}P(X)A$ still satisfies $Q(XY) = Q(X)Q(Y)$, so the entries of $A^{-1}P(X)A$ must also have at most two terms each. Writing down the entries of $A^{-1}P(X)A$ explicitly in terms of the entries of A, it is easy to see that among the various terms of $p_{11}, p_{12}, p_{21}, p_{22}$ there can be terms of at most two different degrees—otherwise, it would be possible to arrange a choice of A so that $A^{-1}P(X)A$ had an entry which had three terms.

Each p_{ij} may therefore be written $p_{ij} = a_{ij}X^m + b_{ij}X^n$ for some common pair of distinct integers m and n. Using $P(1) = I$, we see moreover that $P(X)$ can be written

$$P(X) \;=\; \begin{pmatrix} aX^m + (1-a)X^n & b(X^m - X^n) \\ c(X^m - X^n) & dX^m + (1-d)X^n \end{pmatrix},$$

which we rewrite as

$$P(X) \;=\; X^n I + (X^m - X^n)M$$

for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Expanding the condition $P(XY) = P(X)P(Y)$ in terms of the above expression, we quickly obtain

$$(X^m - X^n)(Y^m - Y^n)M^2 \;=\; (X^m - X^n)(Y^m - Y^n)M\,,$$

and thus, $M^2 = M$. As earlier, it follows that either $M = 0$, $M = I$, or $M = A^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A$. In these three cases we see, respectively, that $P(X) = X^n I$, $P(X) = X^m I$, or

$$P(X) \;=\; A^{-1}\left(X^n I + (X^m - X^n)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) A \;=\; A^{-1}\begin{pmatrix} X^m & 0 \\ 0 & X^n \end{pmatrix} A\,.$$

In summary, we have shown that $P(X)$ must be of the form:

$$P(X) \;=\; 0\,,$$

$$P(X) \;=\; A^{-1}\begin{pmatrix} X^m & 0 \\ 0 & 0 \end{pmatrix} A\,,$$

or

$$P(X) \;=\; A^{-1}\begin{pmatrix} X^m & 0 \\ 0 & X^n \end{pmatrix} A\,,$$

for some invertible complex matrix A.

C100. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Let x_1, x_2, $\ldots$, x_n be positive real numbers, let $S = \sum_{k=1}^{n} x_k$, and suppose that $(n-1)x_k < S$ for all k. Prove that

$$\prod_{k=1}^{n}\left(S-(n-1)x_k\right) \;\le\; \prod_{k=1}^{n} x_k\,.$$

When does equality occur?

Solution.

(Solved by Michel Bataille, Rouen, France, and David Loeffler, student, Trinity College, Cambridge, UK)

Observe that

$$\sum_{k=1}^{n}(S-(n-1)x_k) \;=\; nS-(n-1)\sum_{k=1}^{n} x_k = S\,,$$

and therefore,

$$\sum_{k\neq j}(S-(n-1)x_k) \;=\; S-(S-(n-1)x_j) \;=\; (n-1)x_j\,.$$

Noting that each term in the sum is positive, it follows from the AM–GM inequality that

$$x_j \;\ge\; \prod_{k\neq j}(S-(n-1)x_k)^{1/(n-1)}\,.$$

Multiplying together these inequalities for $j = 1, \ldots, n$, yields the desired inequality.

To determine when equality holds, note from our use of the AM–GM inequality that we must have $x_j = S-(n-1)x_k$ for all $k \neq j$. This implies that any selection of $n-1$ of the x_k's must all be equal, and so for $n > 2$, equality holds if and only if the x_k are all equal. Moreover, one checks easily that equality always holds if $n = 1$ or $n = 2$.

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (⋆) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, ***please send your proposals and solutions on signed and separate standard*** $8\frac{1}{2}$"×11" ***or*** A4 ***sheets of paper***. *These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 May 2002**. *They may also be sent by email to* **crux-editors@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX). *Graphics files should be in* LaTeX *format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2676. *Proposed by Vedula N. Murty, Dover, PA, USA.*

Let A, B and C be the angles of a triangle. Show that

$$(\sin A + \sin B + \sin C)^2 \;\leq\; 6\,(1 + \cos A \cos B \cos C)\;.$$

When does equality occur?

2677. *Proposed by P. Ivady, Budapest, Hungary*

For $0 < x < \frac{\pi}{2}$, show that $\dfrac{\pi^2 - x^2}{\pi^2 + x^2} \;<\; \cos\left(\dfrac{x}{\sqrt{3}}\right)$.

2678. *Proposed by David Chow, student, Clifton College, Bristol, England.*

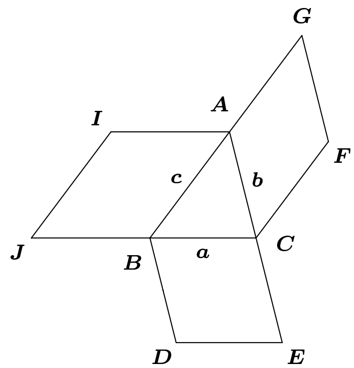
Prove that $\triangle ABC$ is isosceles if and only if

$$a\left(a^2 - b^2\right)\sin B + b\left(b^2 - c^2\right)\sin C + c\left(c^2 - a^2\right)\sin A \;=\; 0\,.$$

2679. *Proposed by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.*

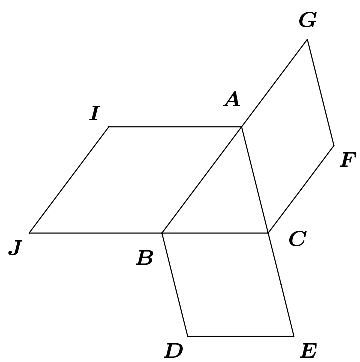
Find all solutions of $\sin x + \sin 2x = \sin 4x$.

2680. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*



Given $\triangle ABC$, construct parallelograms $ABJI$, $BCED$ and $CAGF$ outside the triangle such that $AI = \sqrt{ca}$, $BD = \sqrt{ab}$ and $CF = \sqrt{bc}$. Show that AD, BF and CI are concurrent.

2681. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*



Given $\triangle ABC$, construct rhombi $ABJI$, $BCED$ and $CAGF$ outside the triangle. Show that AD, BF and CI are concurrent.

2682. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

The sequence of functions, $\{J(n) = J(n,w)\}$, $n = 0, 1, \ldots$, is defined as follows:

$$J(0) = a\,, \qquad J(1) = w + b\,,$$

$$J(n+1) = \frac{J(n)\;(J(n)\;(w\,J(n) - 1) - J(n-1))}{J(n-1)\;(w\,J(n) + 1) + J(n)} \qquad \text{for } n > 0.$$

(a) Show that, if $a = 0$, then the sequence consists of polynomials.

(b) Show that there exists a pair (a, b) of non-zero integers such that all the $J(n)$ are polynomials with integer coefficients.

2683. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Find the value of $\lim\limits_{n\to\infty}\left(\frac{1}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{n+1}{2k+1}\binom{n+1}{2k+1}\right)$.

2684. *Proposed by Mohammed Aassila, Strasbourg, France.*

Does there exist an infinitely differentiable function $f:\mathbb{R}\to\mathbb{R}$ such that for every rational number p, the n^{th} derivative $f^{(n)}(p)$ is a rational number whenever n is even, and is an irrational number whenever n is odd?

2685. *Proposed by Mohammed Aassila, Strasbourg, France.*

(a) Let $\mathcal{C}$ be a bounded, closed and convex domain in the plane. Construct a parallelogram $\mathcal{P}$ contained in $\mathcal{C}$ such that $\mathcal{A}(\mathcal{P})\ \geq\ \frac{1}{2}\mathcal{A}(\mathcal{C})$, where $\mathcal{A}$ denotes area.

(b)* Prove that if, further, $\mathcal{C}$ is centrally symmetric, then one can construct a parallelogram $\mathcal{P}$ such that $A(\mathcal{P})\geq\frac{2}{\pi}A(\mathcal{C}\mathcal{P})$.

2686★. *Proposed by Mohammed Aassila, Strasbourg, France.*

Let $\mathcal{C}$ be a bounded, closed and convex domain in space. Construct a parallelepiped $\mathcal{P}$ contained in $\mathcal{C}$ such that $\mathcal{V}(\mathcal{P})\ \geq\ \frac{4}{9}\mathcal{V}(\mathcal{C})$, where $\mathcal{V}$ denotes volume.

2687. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Determine the locus of points (x,y) (in the real plane) for which the equation in z, $x\,z^3+y\,z^2+1=0$, has two complex roots of modulus twice the modulus of its real root.

2688. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Suppose that P is an arbitrary point inside cyclic quadrilateral $ABCD$. Let K, L, M and N be the projections of P onto AB, BC, CD and DA respectively.

Show that $AB\cdot PM+CD\cdot PK=BC\cdot PN+DA\cdot PL$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2555. [2000 : 304] *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

In any triangle ABC, show that

$$\sum_{\text{cyclic}} \frac{1}{\tan^3\frac{A}{2} + \left(\tan\frac{B}{2} + \tan\frac{C}{2}\right)^3} < \frac{4\sqrt{3}}{3}\,.$$

Solution by Henry Liu, Trinity College, Cambridge, England.

The function x^3 is convex on $(0, +\infty)$. Thus, by Jensen's inequality,

$$\begin{aligned} \frac{x^3+y^3}{2} &\geq \left(\frac{x+y}{2}\right)^3 \\ \Longrightarrow \quad \frac{1}{x^3+y^3} &\leq \frac{4}{(x+y)^3}\,. \end{aligned}$$

Similarly, the function $\tan x$ is convex on $\left(0, \frac{\pi}{2}\right)$, so that

$$\begin{aligned} \frac{\tan x + \tan y + \tan z}{3} &\geq \tan\left(\frac{x+y+z}{3}\right) \\ \Longrightarrow \quad \frac{1}{\tan x + \tan y + \tan z} &\leq \frac{1}{3\tan\left(\frac{x+y+z}{3}\right)}\,. \end{aligned}$$

Using these, we have

$$\begin{aligned} \sum_{\text{cyclic}} \frac{1}{\tan^3\frac{A}{2} + \left(\tan\frac{B}{2} + \tan\frac{C}{2}\right)^3} &\leq \frac{3\cdot 4}{\left(\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}\right)^3} \\ &\leq \frac{12}{27\tan^3\left(\frac{1}{3}\left(\frac{A}{2}+\frac{B}{2}+\frac{C}{2}\right)\right)} \\ &= \frac{12}{27\tan^3\left(\frac{\pi}{6}\right)} = \frac{12}{27\cdot\frac{1}{3\sqrt{3}}} = \frac{4\sqrt{3}}{3}\,. \end{aligned}$$

Equality holds in the second inequality when $A = B = C = \frac{\pi}{3}$, and in the first inequality when $\tan\frac{A}{2} = \tan\frac{B}{2} + \tan\frac{C}{2}$; clearly, these cannot both occur at once. Therefore, the given inequality must be strict.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; RICHARD B. EDEN, Ateneo de Manila University, Philippines; WALTHER

JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Most of the proposed solutions make use of Jensen's inequality. Seiffert has proven the following generalization. For $p > 1$,

$$\sum_{\text{cyclic}} \frac{1}{\tan^p \frac{A}{2} + \left(\tan \frac{B}{2} + \tan \frac{C}{2}\right)^p} \; < \; \frac{2^{p-1}}{3^{\frac{p-2}{2}}} \,.$$

2556*. [2000 : 304] *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

A lattice point is called *visible* (from the origin) if its coordinates are co-prime numbers. Is there any lattice point whose distance from each visible lattice point is at least 2000?

Solution by the Southwest Missouri State University Problem Solving Group, Springfield, Missouri, USA.

The answer to the question is "yes". More generally, given any n, we can find an $n \times n$ square array of non-visible points as follows:

Take n^2 distinct prime numbers, p_1, p_2, ..., p_{n^2}. By the Chinese Remainder Theorem, we can find integers a and b such that for $i = 0$, 1, ..., $n-1$ and $j = 0$, 1, ..., $n-1$,

$$a \;\equiv\; -i \pmod{p_{1+ni+j}} \quad \text{and} \quad b \;\equiv\; -j \pmod{p_{1+ni+j}}$$

But now the square array of points:

$$\{(a+i, b+j) \mid i = 0, 1, \ldots, n-1; j = 0, 1, \ldots, n-1\}$$

has the property that p_{1+ni+j} divides $\gcd(a+i, b+j)$, and hence none of these points is visible.

Editor's comment.

The proposer noted (after this problem appeared in print) that it also appeared as problem E2653 in the American Mathematical Monthly, 1978, page 599, together with the following comment:

"The same problem was proposed by Jan Mycielski and solved by M. Warmus in Colloquium Mathematicum 3 (1955) 203-205. Paul Erdős observes that the assertion follows from a result of Moser and himself in Canad. Math. Bull. 1 (1958) 5-8. Fritz Herzog and B.M. Stewart observe that it also follows from their note in this Monthly 78 (1971) 487-496. Blair Spearman notes that this problem appears as Theorem 5.29 in T.M. Apostol, Introduction to Analytic Number Theory."

2557. [2000 : 304] *Proposed by Gord Sinnamon, University of Western Ontario, London, Ontario, and Hans Heinig, McMaster University, Hamilton, Ontario.*

(a) Show that for all positive sequences $\{x_i\}$ and all integers $n > 0$,

$$\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} x_i \;\leq\; 2\sum_{k=1}^{n}\left(\sum_{j=1}^{k} x_j\right)^2 x_k^{-1}\,.$$

(b)* Does the above inequality remain true without the factor 2?

(c)* [Proposed by the editors] What is the minimum constant c that can replace the factor 2 in the above inequality?

Solution to (a) *by the proposers.*

Interchanging the order of summation gives

$$\begin{aligned}\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} x_i &= \sum_{j=1}^{n}(n-j+1)\sum_{i=1}^{j} x_i \;=\; \sum_{i=1}^{n}\binom{n-i+2}{2}x_i \\ &\geq \sum_{i=1}^{n}\frac{1}{2}(n-i+1)^2 x_i\,. \qquad (1)\end{aligned}$$

This observation, together with the Cauchy-Schwarz Inequality, yields

$$\begin{aligned}\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} x_i &= \sum_{j=1}^{n}(n-j+1)\sum_{i=1}^{j} x_i \\ &= \sum_{j=1}^{n}(n-j+1)x_j^{1/2}\left(\sum_{i=1}^{j} x_i\right)x_j^{-1/2} \\ &\leq \left(\sum_{j=1}^{n}(n-j+1)^2 x_j\right)^{1/2}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{j} x_i\right)^2 x_j^{-1}\right)^{1/2} \\ &\leq \left(2\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} x_i\right)^{1/2}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{j} x_i\right)^2 x_j^{-1}\right)^{1/2}\end{aligned}$$

Squaring both sides and dividing by $\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} x_i$ gives the required inequality.

No other solutions were received. Therefore, parts (b) *and* (c) *remain open.*

The editor notes that inequality (1) *is, in fact, strict, slightly improving the result.*

2558. [2000 : 304] *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let Z be a half-plane bounded by a line L. Let A, B and C be any three points on L such that C lies between A and B. Denote the three semicircles in Z on AB, AC and CB as diameters by K_0, K_1 and K_2, respectively. Let F be the family of semicircles in Z with diameters on L (including all half-lines in Z perpendicular to L). Denote by f_{XY} the unique semicircle passing through the pair of distinct points X, Y in $Z \cup L$. Let P, Q, R, be three points on K_2, K_1, K_0, respectively.

If f_{AP}, f_{BQ} and f_{CR} concur at T, and the lines AP, BQ, CR concur at S, prove that f_{AP}, f_{BQ} and f_{CR} are orthogonal to K_2, K_1, K_0, respectively, and that the circle PQR is tangent to each semicircle K_j, ($j = 0, 1, 2$).

Editor's comment.

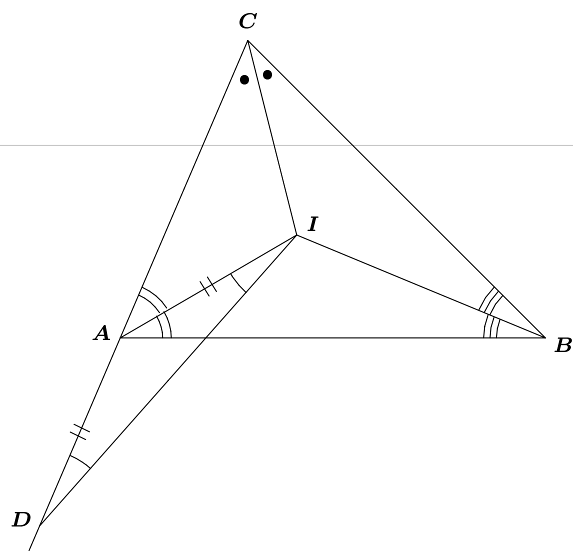
To date, we have received no solutions other than the proposer's own one, which is relatively long and complicated. We invite our readers to try to find a short (and, hopefully, simpler) solution.

2559 [2000 : 305] *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Triangle ABC has incentre I. Show that $CA + AI = CB$ if and only if $\angle CAB = 2\angle ABC$.

Solution by Toshio Seimiya, Kawasaki, Japan; Mitko Kunchev, Baba Tonka School of Mathematics, Rousse, Bulgaria; and Gottfried Perz, Pestalozzigymnasium, Graz, Austria.

Since I is the incentre of $\triangle ABC$, we have $\angle ICA = \angle ICB$, $\angle IAC = \angle IAB$ and $\angle IBC = \angle IBA$. Hence $\angle CAI = \frac{1}{2}\angle CAB$ and $\angle CBI = \frac{1}{2}\angle ABC$. Let D be the point on AC, so that A is between C and D, and $AD = AI$. Then $\angle CDI = \frac{1}{2}\angle CAI$, and therefore, $\angle CDI = \frac{1}{4}\angle CAB$.



(i) Let $CA + AI = CB$. Then $CA + AI = CA + AD = CD$, so that $CD = CB$. Since $\angle DCI = \angle BCI$, the triangles CDI and CBI are congruent. Thus $\angle CDI = \angle CBI$. Hence $\frac{1}{4}\angle CAB = \frac{1}{2}\angle ABC$. Therefore, $\angle CAB = 2\angle ABC$.

(ii) Let $\angle CAB = 2\angle ABC$. Then $\frac{1}{4}\angle CAB = \frac{1}{2}\angle ABC$, so that $\angle CDI = \angle CBI$. Since $\angle DCI = \angle BCI$, the triangles CDI and CBI are congruent, so that $CD = CB$. As $AD = AI$, we have $CD = CA + AD = CA + AI$. Therefore, $CA + AI = CB$.

From (i) and (ii), it follows that $CA + AI = CB$ if and only if $\angle CAB = 2\angle ABC$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (4 solutions); MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; HENRY LIU, Trinity College, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HENRY J. PAN, student, East York Collegiate Institute, Toronto, Ontario; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; IVAN SLAVOV, student, English Language High School, Stara Zagora, Bulgaria; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, Florida, USA; and the proposer.

2566. [2000 : 373] *Proposed by K.R.S. Sastry, Bangalore, India.*

Suppose that each of the three quadratics ax^2+bx+c, $ax^2+bx+(c+d)$ and $ax^2 + bx + (c + 2d)$ factors over the integers. Let $S = ad > 0$. Show that S represents the area of some Pythagorean triangle (integer sided right triangle).

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

It makes no difference to the problem if we use the three quadratics:

$$ax^2 + bx + (c - d)\,, \qquad ax^2 + bx + c\,, \qquad ax^2 + bx + (c + d)\,.$$

If each factors over the integers, then integers u, v, w exist such that

$$b^2 - 4a(c - d) \;=\; u^2\,, \qquad b^2 - 4ac \;=\; v^2\,, \qquad b^2 - 4a(c + d) \;=\; w^2\,.$$

Hence, $2v^2 = u^2 + w^2$ and $8ad = u^2 - w^2$. Since u and w are of the same parity, $\frac{u - w}{2}$ and $\frac{u + w}{2}$ are integers, and

$$\left(\frac{u - w}{2}\right)^2 + \left(\frac{u + w}{2}\right)^2 \;=\; v^2\,.$$

Hence, $\frac{u - w}{2}$ and $\frac{u + w}{2}$ are the sides of some Pythagorean triangle and

$$S \;=\; \frac{1}{2}\left(\frac{u - w}{2}\right)\left(\frac{u + w}{2}\right) \;=\; ad$$

is its area.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; MIHAI CIPU, IMAR, Bucharest, Romania; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; GERRY LEVERSHA, St. Paul's School, London, England; CALVIN LIN, Singapore; HENRY LIU, student, Trinity College Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; KENNETH M. WILKE, Topeka, KS, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; and the proposer.

2569. [2000 : 373] *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

Suppose that a, b, c and d, e, f are real numbers satisfying

1. the pairwise sums of a, b, c are (in some order) d, e and f; and
2. the pairwise products of d, e, f are (in some order) a, b and c.

Find all possible values of $a + b + c$.

Solution by David Loeffler, student, Cotham School, Bristol, UK.

The first condition is that $(a + b, b + c, c + a)$ is a rearrangement of (d, e, f). This implies that the three elementary symmetric functions of each list should be the same:

$$\begin{aligned} a + b + b + c + c + a &= d + e + f\,, \\ (a + b)(b + c) + (b + c)(c + a) + (c + a)(a + b) &= de + ef + fd\,, \\ (a + b)(b + c)(c + a) &= def\,. \end{aligned}$$

Likewise, the second condition that (de, ef, fd) is a rearrangement of (a, b, c) implies that

$$\begin{aligned} de + ef + fd &= a + b + c\,, \\ def(d + e + f) &= ab + bc + ca\,, \\ (def)^2 &= abc\,. \end{aligned}$$

If we write the elementary symmetric functions of a, b, c as $s_1 = a + b + c$, $s_2 = ab + bc + ca$, and $s_3 = abc$, then the above equations may be rewritten as:

$$\begin{aligned} 2s_1 &= d + e + f\,, && (1) \\ s_1^2 + s_2 &= de + ef + fd\,, && (2) \\ s_1 s_2 - s_3 &= def\,, && (3) \\ s_1 &= de + ef + fd\,, && (4) \\ s_2 &= def(d + e + f)\,, && (5) \\ s_3 &= (def)^2\,. && (6) \end{aligned}$$

The problem is now reduced to finding all possible values for s_1 in the above system of simultaneous equations.

By (2) and (4), we have $s_1^2 + s_2 = s_1$. Thus

$$s_2 = s_1(1 - s_1)\,. \qquad (7)$$

Also, substituting this into (5) and comparing it with (1), we have $s_1(1 - s_1) = 2s_1 def$. Hence, either $s_1 = 0$, which we will return to later, or

$$def = \frac{1 - s_1}{2}\,. \qquad (8)$$

Equations (8) and (6) imply that

$$s_3 = \left(\frac{1 - s_1}{2}\right)^2\,. \qquad (9)$$

Substituting equations (9), (7), and (8) into (3) leads to

$$\begin{aligned} s_1^2(1 - s_1) - \left(\frac{1 - s_1}{2}\right)^2 &= \frac{1 - s_1}{2} \\ \text{or} \qquad (1 - s_1)(1 + s_1)(4s_1 - 3) &= 0\,. \end{aligned}$$

Thus, the only possible values of s_1 are ± 1, $\frac{3}{4}$, and 0.

It remains to show that each of these values does indeed lead to a solution. For $s_1 \neq 0$, we can calculate s_2 and s_3 for each value by (7) and (9), implying that there can be at most one possible solution in each case; for

$s_1 = 0$ there may be several solutions. In three of the cases a solution can be guessed: $s_1 = 0$ gives the trivial solution $a = b = c = d = e = f = 0$; $s_1 = 1$ gives $(a, b, c) = (1, 0, 0)$ and $(d, e, f) = (1, 1, 0)$; $s_1 = \frac{3}{4}$ leads to $(a, b, c) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and $(d, e, f) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

For $s_1 = -1$, we have, from the above, $s_2 = s_1(1 - s_1) = -2$, and $s_3 = \left(\frac{1-s_1}{2}\right)^2 = 1$. This means that a, b, and c are the roots of $x^3 + x^2 - 2x - 1 = 0$; these turn out to be

$$(a, b, c) = \left(-2\cos\frac{\pi}{7}, 2\cos\frac{2\pi}{7}, -2\cos\frac{3\pi}{7}\right) .$$

The corresponding values (d, e, f) are

$$\left(2\cos\frac{\pi}{7} - 1, -2\cos\frac{2\pi}{7} - 1, 2\cos\frac{3\pi}{7} - 1\right) .$$

Hence, each of the values ± 1, $\frac{3}{4}$, and 0 has been shown to produce (at least) one solution, so this is precisely the set of possible values of $a + b + c$.

[Remark: In fact, there are no solutions for $s_1 = 0$ other than the trivial zero solution, since $s_1 = 0$ implies that $s_2 = 0$, and does not tell us s_3. Solving the equations for a, b, and c for arbitrary $s_3 = u$ is equivalent to finding the three roots of $x^3 - u = 0$; that is, the cube roots of u. Two of these will always be complex unless u is zero.]

Also solved by ANGELO STATE PROBLEM GROUP, Angelo State University, San Angelo, TX; MICHEL BATAILLE, Rouen, France; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; and the proposer. There were three incorrect and one incomplete solutions.

YIU notes that the equation $x^3 + x^2 - 2x - 1 = 0$ is often associated with the regular heptagon.

2570. [2000 : 373] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let C be a conic with focus F and directrix d. Let A and B be the points of intersection of the conic with a line through the focus F. Let I, J and K be the feet of the perpendiculars from A, F and B to d, respectively.

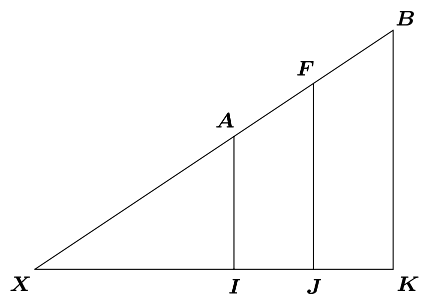
Prove that the length of FJ is the harmonic mean of the lengths of AI and BK.

Editor's comment.

All but one of the solutions assumed tacitly that the points A, B, F were all located on the same side of the directrix. Only M. Bataille observed that the result, as stated, is incorrect in the case when A, B lie on opposite sides of the directrix. As was observed by Bataille, the result can be rescued in this case if we require that distances be signed. We give the solution submitted by G. Leversha (with one small correction).

Solution by Gerry Leversha, St. Paul's School, London, England.

If AB is parallel to d, then the result is trivial, since $AI = FJ = BK$. Otherwise, suppose that AB and d meet at X.



Suppose that $XA = u$, $AF = a$, $FB = b$, $FJ = c$. Then $AI = fa$ and $BK = fb$, where f is the reciprocal of the eccentricity of the conic. By similar triangles, we have $\dfrac{u}{fa} = \dfrac{u+a}{c} = \dfrac{u+a+b}{fb}$.

But, since these ratios are all equal, it follows that $\dfrac{a}{c-fa} = \dfrac{b}{fb-c}$, and hence that $afb - ac = bc - fab$, so that $2fab = (a+b)c$, and hence, $\dfrac{1}{c} = \dfrac{1}{2}\left(\dfrac{1}{fa} + \dfrac{1}{fb}\right)$, which says that FJ is the harmonic mean of AI and BK.

Editor's note. As observed above, M. Bataille was the only one to note that this result breaks down in the case where A and B lie on opposite sides of the directrix. We give a slight generalization of his counterexample. Consider the hyperbola $x^2 - \dfrac{y^2}{e^2-1} = 1$, which has vertices $A(1,0)$, $B(-1,0)$, a focus $F(e,0)$ and directrix $x = \dfrac{1}{e}$. In this case, we have $I = J = K = \left(\frac{1}{e}, 0\right)$, and an easy calculation shows that $\dfrac{1}{FJ} = \dfrac{e}{e^2-1}$, whereas $\dfrac{1}{2}\left(\dfrac{1}{AI} + \dfrac{1}{BK}\right) = \dfrac{e}{e^2-1}$. Thus, the result, as stated, fails to hold in this case. If, however, we use signed distances (noting that BK then becomes negative), we obtain $\dfrac{1}{2}\left(\dfrac{1}{AI} - \dfrac{1}{BK}\right) = \dfrac{e}{e^2-1}$, which is the desired result. It is, in fact, easy to see that the proof given above by Leversha can be adapted in an obvious way to show that this is always the case.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HENRY LIU, student, Trinity College, Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, UK; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; DANIEL REISZ, Université de Bourgogne, Dijon, France; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2571. [2000 : 374] *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Suppose that a, b and c are the sides of a triangle. Prove that

$$\frac{1}{\sqrt{a}+\sqrt{b}-\sqrt{c}}+\frac{1}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{1}{\sqrt{c}+\sqrt{a}-\sqrt{b}} \geq \frac{3(\sqrt{a}+\sqrt{b}+\sqrt{c})}{a+b+c}.$$

Solution by Li Zhou, Polk Community College, Winter Haven, Florida, USA.

By the AM-GM inequality, we have

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 \;=\; (a+b+c)+2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right) \;\leq\; 3(a+b+c).$$

So, by the AM-HM inequality, we get

$$\frac{3}{\frac{1}{\sqrt{a}+\sqrt{b}-\sqrt{c}}+\frac{1}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{1}{\sqrt{c}+\sqrt{a}-\sqrt{b}}} \;\leq\; \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{3} \;\leq\; \frac{a+b+c}{\sqrt{a}+\sqrt{b}+\sqrt{c}},$$

which is equivalent to the desired inequality.

[Ed: Strictly speaking, a complete proof should first include a statement and proof of the fact that the denominators on the left side are all positive; that is, $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$ are also the sides of a triangle. However, since this is trivial to show (simply observe $(\sqrt{a}+\sqrt{b})^2 > a+b > c$, etc.), most solvers apparently assumed this fact without stating it, and the editor is willing to give the benefit of doubt to these solvers.]

Also solved by MOHAMMED AASSILA, Strasbourg, France; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); MIHÁLY BENCZE, Brasov, Romania; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIHAI CIPU, IMAR, Bucharest, Romania; RICHARD B. EDEN, Ateneo de Manila University, Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; HENRY LIU, student, Trinity College, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; VEDULA N. MURTY, Visakhapatnam, India; HENRY PAN, student, East York C.I., Toronto, Ontario; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Bencze pointed out the following immediate but interesting consequence of the given inequality:

Let P be any interior point of an equilateral triangle $\triangle ABC$. Then

$$\sum_{\text{cyclic}} \frac{1}{\sqrt{PA}+\sqrt{PB}-\sqrt{PC}} \;\geq\; \frac{3(\sqrt{PA}+\sqrt{PB}-\sqrt{PC})}{PA+PB+PC}.$$

Janous obtained the stronger result that for all $\lambda \in (0,1]$,

$$\sum_{\text{cyclic}} \frac{1}{a^\lambda + b^\lambda - c^\lambda} \;\geq\; \frac{9}{a^\lambda + b^\lambda + c^\lambda}\,.$$

Klamkin pointed out that the given result still holds under the weaker condition that $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$ are the sides of a triangle. [Ed: See editorial comment at the end of the solution above.] He used Hölder's Inequality and the Power Mean Inequality to obtain the more general result, that, for $p > 0$, x, y, $z \geq 0$ with $x + y + z = 1$,

$$\frac{(3x)^{p+1}}{(a+b-c)^p} + \frac{(3y)^{p+1}}{(a+b-c)^p} + \frac{(3z)^{p+1}}{(a+b-c)^p} \;\geq\; \frac{3(a+b+c)^p}{(a^2+b^2+c^2)^p}\,,$$

with equality if and only if $a = b = c$. The proposed inequality with (a,b,c) replaced by (a^2,b^2,c^2), is the special case when $p = 1$ and $x = y = z = \frac{1}{3}$.

2572. [2000 : 374] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Let a, b, c be positive real numbers. Prove that

$$a^b b^c c^a \;\leq\; \left(\frac{a+b+c}{3}\right)^{a+b+c}\,.$$

[Compare problem **2394** [1999 : 524], note by V.N. Murty on the generalization.]

I. *Independent and nearly identical solutions by Mohammed Aassila, Strasbourg, France; Mihály Bencze, Brasov, Romania; John G. Heuver, Grande Prairie Composite High School, Grande Prairie, Alberta; Joe Howard, Portales, NM, USA; Henry Liu, student, Trinity College, Cambridge, England; Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain; Panos E. Tsaoussoglou, Athens, Greece; and the proposer.*

By the weighted AM-GM inequality with weights $\dfrac{b}{a+b+c}$, $\dfrac{c}{a+b+c}$ and $\dfrac{a}{a+b+c}$, we have

$$\left(a^b b^c c^a\right)^{\frac{1}{a+b+c}} \;=\; a^{\frac{b}{a+b+c}}\, b^{\frac{c}{a+b+c}}\, c^{\frac{a}{a+b+c}} \;\leq\; \frac{ba+cb+ac}{a+b+c} \;\leq\; \frac{a+b+c}{3}$$

since $(a+b+c)^2 \geq 3(a+b+c)$ follows easily from $a^2+b^2+c^2 \geq ab+bc+ca$.

II. *Independent and virtually the same solution by Michel Bataille, Rouen, France; Mihai Cipu, IMAR, Bucharest, Romania; Murray S. Klamkin, University of Alberta, Edmonton, Alberta; Kee-Wai Lau, Hong Kong; David Loeffler, student, Cotham School, Bristol, UK; Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany; Kenneth M. Wilke, Topeka, KS, USA; and Li Zhou, Polk Community College, Winter Haven, USA.*

Since the function $\ln x$ is (strictly) concave and

$$\frac{b}{a+b+c} + \frac{c}{a+b+c} + \frac{a}{a+b+c} \;=\; 1\,,$$

Jensen's Inequality yields

$$\frac{b}{a+b+c}\ln a+\frac{c}{a+b+c}\ln b+\frac{a}{a+b+c}\ln c \;\leq\; \ln\left(\frac{ba+cb+ac}{a+b+c}\right)$$

or

$$\left(a^b b^c c^a\right)^{\frac{1}{a+b+c}} \;\leq\; \frac{ba+cb+ac}{a+b+c} \leq \frac{a+b+c}{3}$$

as in solution I.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

There were two partially incorrect solutions, both of which used wording such as $a+a+\cdots+a$ $(b\ a$*'s), thus treating* b *as a positive integer.*

Both Howard and Romero considered the 4*–variable analogue and showed that* $a^b b^c c^d d^a \;\leq\; \left(\frac{a+b+c+d}{4}\right)^{a+b+c+d}$ *for positive reals* a*,* b*,* c*,* d*. Using the argument in either* I *or* II *above, we see easily that it suffices to show that*

$$\frac{ab+bc+cd+da}{a+b+c+d} \;\leq\; \frac{a+b+c+d}{4}\,.$$

Both of them showed that this is true by noticing that $(a+b+c+d)^2-4(ab+bc+cd+da) = (a-b+c-d)^2 \geq 0$*. Howard asked whether the corresponding inequality still holds for* $n \geq 5$ *variables. Can any readers provide an answer or some references?*

2573. [2000 : 374] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Let H be the orthocentre of triangle ABC. For a point P not on the circumcircle of triangle ABC, denote by X, Y, Z the reflections of P in the sides BC, CA, and AB, respectively. Show that the areas of triangles HYZ, HZX, and HXY are in constant proportions.

A combination of similar solutions by Michel Bataille, Rouen, France; David Loeffler, student, Cotham School, Bristol, UK; and Peter Y. Woo, Biola University, La Mirada, CA, USA.

Since P is not on the circumcircle of $\triangle ABC$, the projections of P onto the lines BC, CA, AB are not collinear; further, neither are X, Y, Z. Thus XYZ is a non-degenerate triangle and we may compute the barycentric (or areal) coordinates of H with respect to it. These coordinates are proportional to the signed areas HYZ, HZX, HXY; hence, it is sufficient to show that the coordinates are independent of the position of P.

The barycentric representation of the orthocentre H with respect to $\triangle ABC$ is $H = \alpha A + \beta B + \gamma C$, where $\alpha+\beta+\gamma = 1$ and $\alpha : \beta : \gamma = \tan A : \tan B : \tan C$. [This is easily proved; it can be found in standard references such Clark Kimberling's encyclopedia of triangle centres: `cedar.evansville.edu/~ck6/tcenters/`.]

Let $P = pA+qB+rC$ for real numbers p, q, r with $p+q+r=1$. Reflect $\triangle ABC$ across line BC so that the images of A and P are A' and X.

Let D be the foot of the altitude from A. Then $D = \dfrac{\beta B + \gamma C}{\beta + \gamma}$, and D is the mid-point of AA' so that

$$A' \;=\; 2D - A \;=\; \frac{2}{\beta+\gamma}(\beta B + \gamma C) - A\,.$$

From $P = pA + qB + rC$ we have $X = pA' + qB + rC$, or

$$X \;=\; \frac{2p}{\beta+\gamma}(\beta B + \gamma C) - pA + qB + rC\,.$$

Similarly, by reflecting $\triangle ABC$ across AC and across AB, we get

$$Y \;=\; \frac{2q}{\gamma+\alpha}(\gamma C + \alpha A) + pA - qB + rC\,,$$

and

$$Z \;=\; \frac{2r}{\alpha+\beta}(\alpha B + \beta B) + pA + qB - rC\,.$$

Then

$$\begin{aligned} &\frac{1}{2}\Big[(\beta+\gamma)X + (\gamma+\alpha)Y + (\alpha+\beta)Z\Big] \\ &\quad= \frac{p}{2}\Big[2\beta B + 2\gamma C - (\beta+\gamma)A + (\gamma+\alpha)A + (\alpha+\beta)A\Big] \\ &\qquad\qquad + \frac{q}{2}\Big[\ldots\Big] + \frac{r}{2}\Big[\ldots\Big] \\ &\quad= (\alpha A + \beta B + \gamma C)(p+q+r) \\ &\quad= \alpha A + \beta B + \gamma C \;=\; H\,. \end{aligned}$$

This proves that H is a linear combination of X, Y and Z that is independent of p, q, r, as was required.

The argument breaks down if P is on the circumcircle — in this case the points X, Y, Z, and H are collinear. Thus, all the areas are zero and their ratios cease to be meaningful. This collinearity is essentially the result of problem 5 of the 4th Taiwan Mathematics Olympiad, see [2000 : 75]. [The proposer considers his problem to be a generalization of this result; he provided the reference J.R. Musselman, "On the Line of Images", *Amer. Math. Monthly* **45** (1938) 421–430; *ibid.*, **46** (1939) 281. The result is perhaps older than 1938 since it is problem 11 on page 145 of Nathan Altshiller Court, *College Geometry*.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria.

Both Woo and the proposer pointed out that the problem naturally belongs to affine geometry. More precisely,

> *Let K be a fixed point in the plane of $\triangle ABC$; for any point P define X, Y, Z to be the points such that $PX \| AK$, $PY \| BK$, $PZ \| CK$, and that the mid-point of PX lies on BC, the mid-point of PY lies on CA, and the mid-point of PZ lies on AB. Then the areas of HYZ, HZX, and HXY are in constant proportion except for those positions of P where one area (and therefore each) is zero; moreover, the exceptional values of P lie on a conic.*

2574. [2000 : 374] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Let P be a point in the interior of triangle ABC, whose centroid is G. Extend AP to a point X such that PX is bisected by the line BC. Similarly, extend BP to Y and CP to Z such that PY and PZ are each bisected by CA and AB, respectively. Show that the 6 points A, B, C, X, Y, Z, lie on a conic, and that the centre of the conic is the point Q dividing PG externally in the ratio $PQ : QG = 3 : -1$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

The problem belongs to affine geometry: an affine transformation maps lines to lines, ellipses to ellipses, the centre of an ellipse to the centre of its image, and it preserves ratios of distances among points on a line. Let X', Y', Z', A', B', C' be the mid-points of PX, PY, PZ, PA, PB, PC respectively. There exists an affine transformation that takes the given point P to the orthocentre of the image triangle — for example, one can fix B and C and slide A parallel to BC until $AP \perp BC$, then slide P along the new position of AP until $CP \perp AB$. To avoid introducing more notation, we may as well assume *without loss of generality* that the given point P is the orthocentre of $\triangle ABC$. Since A', B', C' are the mid-points of PA, PB, PC, the primed points all lie on the nine-point circle whose centre N lies on the Euler line PG halfway between the orthocentre P and circumcentre Q. (This result can be found in any reference that treats the nine-point circle.) With the orthocentre P as centre, the dilatation, whose ratio of magnification is 2, maps the nine-point circle to the circumcircle of $\triangle ABC$ passing through A, B, C, X, Y, Z; moreover, N is mapped to the circumcentre Q, and Q divides PG externally in the ratio $3 : -1$. [The points lie on the Euler line in the order $QGNP$, in the ratios $QG : GN : NP = 2 : 1 : 3$.]

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; and DAVID LOEFFLER, student, Cotham School, Bristol, UK.

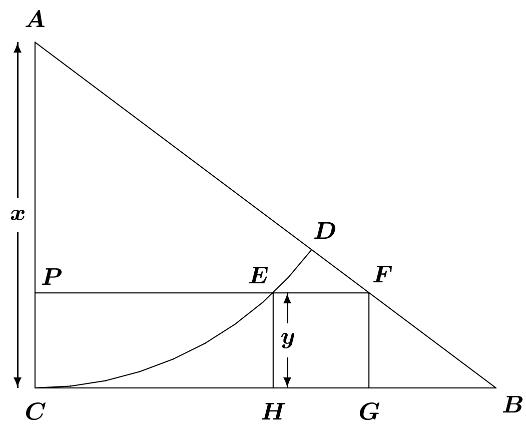
The proposer pointed out that this problem generalizes the property that the reflections of the orthocentre of a triangle in the sides lie on the circumcircle, a property that is clearly illustrated by the featured solution.

2575. [2000 : 374] *Proposed by H. Fukagawa, Kani, Gifu, Japan.*

Suppose that $\triangle ABC$ has a right angle at C. The circle, centre A and radius AC meets the hypotenuse AB at D. In the region bounded by the arc DC and the line segments BC and BD, draw a square $EFGH$ of side y, where E lies on arc DC, F lies on DB and G and H lie on BC. Assume that BC is constant and that $AC = x$ is variable.

Find $\max y$ and the corresponding value of x.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.



Let $BC = a$ and suppose that FE meets AC at P.

Since $FP \| BC$, the triangles AFP and ABC are similar, and we have

$$\frac{FP}{PA} = \frac{BC}{CA}, \qquad \text{or} \qquad \frac{FP}{x-y} = \frac{a}{x},$$

giving that $FP = \dfrac{a(x-y)}{x}$. Since $EP = FP - FE$, we have $EP = \dfrac{a(x-y)}{x} - y$.

In $\triangle AEP$, $AE = x$ and $AP = x - y$. Using the Pythagorean Theorem, we have

$$\left(\frac{a(x-y)}{x} - y\right)^2 + (x-y)^2 = x^2,$$

which we can write in the equivalent form

$$(x-y)\left(\frac{a^2(x-y)}{x^2} - \frac{2ay}{x} - 2y\right) = 0. \tag{1}$$

The expression on the left of (1) vanishes only if

$$\frac{a^2(x-y)}{x^2} - \frac{2ay}{x} - 2y = 0$$

(since, clearly, $y < x$). Solving for y, we get

$$y = \frac{a^2 x}{2x^2 + 2ax + a^2} = \frac{a^2}{2x + \dfrac{a^2}{x} + 2a}.$$

Maximizing y is equivalent to minimizing $2x + \frac{a^2}{x}$. This term is the sum of two terms whose product is a constant: the sum is a minimum when $2x$ and $\frac{a^2}{x}$ are equal, from which we get

$$2x = \frac{a^2}{x}, \qquad x^2 = \frac{a^2}{2}, \qquad x = \frac{a}{\sqrt{2}},$$

where, since $x > 0$, we have discarded the negative solution.

Thus, y takes on its maximum value of $\frac{a\left(\sqrt{2}-1\right)}{2}$ when $x = \frac{a}{\sqrt{2}}$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain (second solution); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MIHAI CIPU, IMAR, Bucharest, Romania; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; and the proposer.

Janous commented that this was a marvellous problem linking various fields and ideas!!

The proposer writes that this is an elementary problem of traditional Japanese Mathematics concerning a circle, a triangle and a square.

In 1997, someone discovered this problem on the wooden ceiling inside a small old house in the countryside in Nagano prefecture. The wooden ceiling of the house is divided into about 25 parts as a lattice, and each part is a square (with side 43 cm) where flowers or birds are drawn beautifully. One of them is this geometry problem (without solution). A lover of Geometry, Yasuyuki Machida (whose birth and death years are not known), proposed this problem.

The figures on the ceiling were probably drawn in 1864 (the Edo–period) and then, until recently, no-one noticed these figures. I think that this problem is elementary and nice for high school students and lovers of geometry.

Editor's Comment. *Our readers are most likely to be familiar with* Japanese Temple Geometry Problems, *Charles Babbage Research Centre, Winnipeg, Manitoba, 1989, authored by our proposer with the cooperation of Dan Pedoe. This book contains a wealth of comparable fascinating geometry problems from similar sources. It is a must for the library of every geometer.*

2576. [2000 : 429] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

Characterize the numbers n such that $n!$ finishes (in base 2 notation) with exactly $n - 1$ zeros.

I. *Solution by Li Zhou, Polk Community College, Winter Haven, Florida, USA.*

For $n \geq 1$, let $E_2(n!)$ be the greatest non-negative integer m such that $2^m | n!$. Then it is easy to see that $n!$ finishes (in base 2 notation) with exactly $n-1$ zeros if and only if $E_2(n!) = n-1$. Let $n = 2^k + r$ with $0 \leq r < 2^k$. Then $E_2(n!) = \sum_{t=1}^{k} \left\lfloor \frac{n}{2^t} \right\rfloor \leq \sum_{t=1}^{k} \frac{n}{2^t} = n\left(1 - \frac{1}{2^k}\right) \leq n-1$ with equality if and only if $r = 0$. Hence the desired numbers are $n = 2^k$, $k = 0, 1, 2, \ldots$.

II. *Solution and generalization by Pierre Bornsztein, Pontoise, France (slightly modified by the editor).*

We show more generally that if p is a prime, then $n!$ finishes (in base p notation) with exactly $n-1$ zeros if and only if **either** $n = 1$ **or** $p = 2$ and $n = 2^k$ for some positive integer k.

As in **I** above, let $E_p(n!)$ denote the greatest non-negative integer m such that $p^m | n!$. Then it suffices to determine all n such that $E_p(n!) = n-1$. By a well-known formula of Legendre (see reference [1]), we have $E_p(n!) = 0$ if $p > n$ and $E_p(n!) = \dfrac{n - s_p(n)}{p-1}$ if $p \leq n$ where $s_p(n)$ denotes the sum of the digits of n written in base p. It follows that if $p > n$, then $E_p(n!) = n-1$ if and only if $n = 1$ which clearly finishes in 0 zeros in base p. If $p \leq n$, then $E_p(n!) = n-1$ if and only if $n - s_p(n) = (n-1)(p-1)$ or

$$np + 1 + s_p(n) \;=\; 2n + p\,. \qquad (1)$$

But if $p \geq 3$, then $np + 1 + s_p(n) > np \geq 2n + n \geq 2n + p$ and hence (1) is not satisfied. If $p = 2$, then (1) becomes $s_2(n) = 1$ which is equivalent to $n = 2^k$ for some positive integer k (since $n \geq p > 1$).

Also solved by MICHEL BATAILLE, Rouen, France; ROBERT BILINSKI, Outremont, Qubec; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIHAI CIPU, IMAR, Bucharest, Romania; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; CHARLES DIMINNIE and PAUL SWETS, Angelo State, University, San Angelo, TX, USA; RICHARD B. EDEN, Ateneo de Manila University, Philippines; KEITH EKBLAW, Walla Walla, WA, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; CALVIN ZHIWEI LIN, Singapore; HENRY LIU, Trinity College, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DAVID SINGMASTER, London, UK; SOUTHWEST MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; NICHOLAS THAM, Raffles Junior College, Singapore; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer.

Engelhaupt remarked that in general, if $n = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_r}$, *where* $0 \le a_1 < a_2 < \cdots < a_r$, *then* $n!$ *finishes in* $n - r$ *zeros since*

$$E_2(n!) = \sum_{t=1}^{\infty} \left\lfloor \frac{n}{2^t} \right\rfloor = \sum_{i=1}^{r} \sum_{t=1}^{\infty} \left\lfloor \frac{2^{a_i}}{2^t} \right\rfloor = \sum_{i=1}^{r} (2^{a_i-1} + 2^{a_i-2} + \cdots + 1)$$
$$= \sum_{i=1}^{r} (2^{a_i} - 1) = \left(\sum_{i=1}^{r} 2^{a_i} \right) - r = n - r .$$

He gave the following example: Since $n = 216 = 2^3 + 2^4 + 2^6 + 2^7$ *has four summands,* $216!$ *finishes in* $216 - 4 = 212$ *zeros.*

Reference:

[1.] J.Robert, Elementary Number Theory — A Problem Oriented Approach, M.I.T. Press. Chapter X. Ex. 5, p. 76 and pp. 96–97.

Another maze from Isador Hafner

How can you move from the dark spot to the light spot?