MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 975 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z5 (NEW!)**. The electronic address is

NEW! mayhem-editors@cms.math.ca NEW!

The Assistant Mayhem Editor is Chris Cappadocia (University of Waterloo). The rest of the staff consists of Adrian Chan (Harvard University), Jimmy Chui (University of Toronto), Donny Cheung (University of Waterloo), and David Savitt (Harvard University).

Problem of the Month

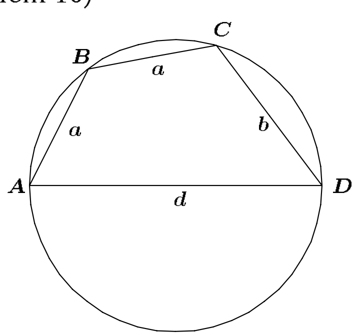
Jimmy Chui, student, University of Toronto

Problem. $ABCD$ is a cyclic quadrilateral, as shown, with side $AD = d$, where d is the diameter of the circle. $AB = a$, $BC = a$, and $CD = b$. If a, b, and d are integers, $a \neq b$,

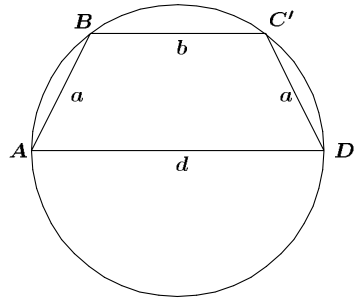
(a) prove that d cannot be a prime number.

(b) determine the *minimum* value of d.

(1999 Euclid, Problem 10)



Solution. Consider the cyclic quadrilateral $ABC'D$, where C' is the point in the interior of minor arc BD, such that $BC' = b$ and $C'D = a$. We are guaranteed a point C' exists since we are just reflecting C in the perpendicular bisector of BD.



Note that the cyclic quadrilateral $ABC'D$ is a trapezoid. Then, let $c = BD = AC'$.

By Ptolemy's Theorem, $c^2 = a^2 + bd$. (Note: This relationship can also be obtained without the construction of point C'. Take the cosine of $\angle BAD$ and the cosine of $\angle BCD$ using the Cosine Law. These cosines are negatives of each other, since the angles add up to $180°$. Adding the cosines and simplifying yields the relationship.)

Also, since AD is the diameter of a circle, we also have $\angle ABD = 90°$. Then $a^2 + c^2 = d^2$.

We know that a, b, and d are integers, therefore, let us eliminate c. We get $2a^2 = d^2 - bd = d(d - b)$.

If d is 2, then we get $a^2 = 2 - b$, and the only positive integer solution is $a = b = 1$. This contradicts $a \neq b$, so that d cannot be 2.

If d is an odd prime, then $d|a$. But that means the left side is divisible by d^2, so that the right side must be divisible by d^2. This leads to $d|(d - b)$, which is clearly impossible since b is positive.

Hence, d cannot be a prime, and so, (a) is proven.

Let us find the least possible value for d by brute force. We know d is not prime.

If $d = 4$, then from $2a^2 = d(d - b)$, we have $a^2 = 2(4 - b)$ and there is no integer solution other than $a = b = 2$.

If $d = 6$, then we have $a^2 = 3(6 - b)$ and there is no integer solution other than $a = b = 3$.

If $d = 8$, then we have $a^2 = 4(8 - b)$. There are two integer solutions: $a = b = 4$ and $a = 2$, $b = 7$. Here we have a solution with $a \neq b$.

We should check to see if this solution does indeed form a cyclic quadrilateral. We find the diagonal lengths of the quadrilateral by using the Pythagorean Theorem. From Ptolemy's Theorem, because we have an equality, the quadrilateral is cyclic with d as the diameter. (We can apply this to either $ABCD$ or $ABC'D$.)

Hence, the least possible value for d is 8.

Note: Let a quadrilateral have sides of length a_1, a_2, a_3, and a_4, *in that order*, and let it have diagonals of length d_1 and d_2. Ptolemy's Theorem states that the quadrilateral is cyclic if and only if $d_1 d_2 = a_1 a_3 + a_2 a_4$.

A related inequality is Ptolemy's Inequality. This inequality states that in *any* quadrilateral, $d_1 d_2 \leq a_1 a_3 + a_2 a_4$, with equality holding if and only if the quadrilateral is cyclic.

Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from this issue be submitted in time for issue 4 of 2002.

High School Problems

Editor: Adrian Chan, 1195 Harvard Yard Mail Center, Cambridge, MA, USA 02138-7501 <ahchan@fas.harvard.edu>

H285. Four people, A, B, C, D, are on one side of a river. To get across the river they have a rowboat, but it can only fit two people at a time. A, B, C, D, could each row across the river in the boat individually in 1, 2, 5, and 10 minutes respectively. However, when two people are on the boat, the time it takes them to row across the river is the same as the time necessary to row across for the slower of the two people. Assuming that no one can cross without the boat, and everyone is to get across, what is the minimum time for all four people to get across the river?

H286. A mouse eats his way through a $3 \times 3 \times 3$ cube of cheese by tunnelling through all of the 27 $1 \times 1 \times 1$ subcubes. If he starts at one of the corner sub-cubes and always moves onto an uneaten adjacent sub-cube, can he finish at the centre of the cube? (Assume that he can tunnel through walls but not edges or corners.)

H287. Suppose we want to construct a solid polyhedron using just n pentagons and some unknown number of hexagons (none of which need be regular), so that exactly three faces meet at every vertex on the polyhedron. For what values of n is this feasible?

H288. *Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain.*

If x, y, z are positive real numbers, show that

$$\frac{1}{\sinh(x+z)}\left(\frac{\cosh y\cosh z}{\cosh(x+y+z)}-\cosh x\right) = \frac{1}{\sinh(y+z)}\left(\frac{\cosh x\cosh z}{\cosh(x+y+z)}-\cosh y\right)$$

Advanced Problems

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A261. Let $P_k(x) = 1 + x + x^2 + \cdots + x^{k-1}$. Show that

$$\sum_{k=1}^{n}\binom{n}{k}P_k(x) = 2^{n-1}P_n\left(\frac{1+x}{2}\right)$$

for every real number x and every positive integer n.

(1998 Baltic Way)

A262. *Proposed by Mohammed Aassila, Strasbourg, France.*

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for any real number a, the sequence $f(a), f(2a), f(3a), \ldots$ converges to zero. Prove that $\lim_{x\to\infty} f(x) = 0$.

A263. Find all real numbers x for which there exists a closed convex region on the Euclidean plane with both area and perimeter equal to x.

A264. *Proposed by Mohammed Aassila, Strasbourg, France.*

Prove that for any integer a and natural number m,

$$a^m \equiv a^{m-\phi(m)} \pmod{m}.$$

(This is a generalization of Euler's Theorem.)

Challenge Board Problems

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

History of the Konhauser Problemfest

Stan Wagon (Macalester College)

Joe Konhauser, a geometer and avid problemist, came to Macalester College in 1968. He had been invited by Sy Schuster (Carleton College) to work on some geometry movies for the Mathematical Association of America, and he must have liked Minnesota, since he moved from Pennsylvania to the University of Minnesota and then Macalester shortly thereafter. He started the Problem of the Week tradition at Macalester, posing a problem every week and offering 50 cents (since raised to $1) for correct solutions. When I came to Macalester in 1990 I was very impressed by the fact that students would work on these problems just for the fun of it. Part of the reason was that Joe had a knack for posing simply stated and compelling problems, often with surprising twists.

During his career at Macalester, Joe served as editor of the Pi Mu Epsilon journal and also served on the many problems committees, including those of the USA Math Olympiad and the Putnam competition.

When Joe died in 1992, I took over the PoW program, adding an e-mail component. Joe posed problems 1–682 and I and my colleagues (notably Tom Halverson) have continued the tradition and are now into the 900s. The MAA has published a book, "Which Way Did the Bicycle Go?" (by Konhauser, D. Velleman, and me), containing the best of the first 800 problems.

In 1993, partially inspired by the Lower Michigan Math Contest, Mark Krusemeyer (Carleton College), Loren Larson (St. Olaf) and I started the Konhauser Problemfest, a three-hour contest with ten problems for teams of three students working together. The event is now in its ninth year, and the University of St. Thomas and Gustavus Adolphus College also take part. The winning team gets a cash prize and their school gets the travelling trophy, a small granite sculpture by Helaman Ferguson illustrating a proof of a geometrical fact known as the pizza theorem. Carleton has won five times, including the 2001 contest, Macalester has won twice, these two tied one year, and Gustavus Adolphus won the event in Feb. 2000.

At the URL

`http://www.macalester.edu/~mathcs/potw.html`

you can find past Konhauser problems, an archive of past Problems of the Week, and information on how to receive the Problem of the Week automatically by e-mail.

The Ninth Annual Konhauser Problemfest

Carleton College, February 24, 2001
Problems by David Savitt and Russell Mann (Harvard University)

This contest is held annually in memory of Professor Joseph Konhauser (1924-1992) of Macalester College, who posted nearly 700 Problems of the Week at Macalester over a 25-year period. Joe died in February of 1992, and the contest was started the following year.

INSTRUCTIONS: Each team must hand in all work to be graded at the same time (at the end of the three-hour period). Only one version of each problem will be accepted per team. Calculators of any sort are allowed (although they may not be all that helpful). Justifications and/or explanations are expected for all problems, but, in view of the time constraint, rigorous proofs are only required when the wording of the problem makes that clear ("show that" or "prove that"). All ten problems will be weighted equally, and partial credit will be given for substantial progress toward a solution. Good luck!

1. Last season, the Minnesota Timberwolves won five times as many games as they lost, in games in which they scored 100 or more points. On the other hand, in games in which their opponents scored 100 or more points, the Timberwolves lost 50% more games than they won. Given that there were exactly 34 games in which either the Timberwolves or their opponents scored 100 or more points, what was the Timberwolves' win-loss record in games in which *both* they and their opponents scored 100 or more points?

2. Three circles are drawn in chalk on the ground. To begin with, there is a heap of n pebbles inside one of the circles, and there are "empty heaps" (containing no pebbles) in the other two circles. Your goal is to move the entire heap of n pebbles to a different circle, using a series of moves of the following type. For any non-negative integer k, you may move exactly 2^k pebbles from one heap (call it heap A) to another (heap B), provided that heap B begins with fewer than 2^k pebbles, and that after the move, heap A ends up with fewer than 2^k pebbles. Naturally, you want to reach your goal in as few moves as possible. For what values of $n \le 100$ would you need the largest number of moves?

3. (a) Begin with a string of 10 A's, B's, and C's, for example

```
A B C C B A B C B A
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and underneath, form a new row, of length 1 shorter, as follows: between two consecutive letters that are different, you write the third letter, and between two letters that are the same, you write that same letter again. Repeat this process until you have only one letter in the new row. For example, for the string above, you will now have:

```
A B C C B A B C B A
 C A C A C C A A C
  B B B B C B A B
   B B B A A C C
    B B C A B C
     B A B C A
      C C A B
       C B C
        A A
         A
```

Prove that the letters at the corners of the resulting triangle are always either all the same or all different.

(b) For which positive integers n (besides 10) is the result from part (a) true for all strings of n A's, B's, and C's?

4. When Mark climbs a staircase, he ascends either 1, 2, or 3 stair steps with each stride, but in no particular pattern from one foot to the next. In how many ways can Mark climb a staircase of 10 steps? (Note that he must finish on the top step. Two ways are considered the same if the number of steps for each stride is the same; that is, it does not matter whether he puts his best or his worst foot forward first.) Suppose that a spill has occurred on the 6^{th} step and Mark wants to avoid it. In how many ways can he climb the staircase without stepping on the 6^{th} step?

5. Number the vertices of a cube from 1 to 8. Let A be the 8×8 matrix whose (i, j) entry is 1 if the cube has an edge between vertices i and j, and is 0 otherwise. Find the eigenvalues of A, and describe the corresponding eigenspaces.

6. Let $f(x)$ be a twice-differentiable function on the open interval $(0, 1)$ such that

$$\lim_{x \to 0+} f(x) = -\infty, \text{ and } \lim_{x \to 1^-} f(x) = +\infty.$$

Show that $f''(x)$ takes on both negative and positive values.

7. Three stationary sentries are guarding an important public square which is, in fact, square, with each side measuring 8 rods (recall that one rod equals 5.5 yards). (If any of the sentries see trouble brewing at any location on the square, the sentry closest to the trouble spot will immediately cease to be stationary and dispatch to that location. And like all good sentries, these three are continually looking in all directions for trouble to occur.) Find the maximum value of D, so that no matter how the sentries are placed, there is always some spot in the square that is at least D rods from the closest sentry.

8. The Union Atlantic Railway is planning a massive project: a rail road track joining Cambridge, Massachusetts and Northfield, Minnesota. However, the

funding for the project comes from the will of Orson Randolph Kane, the eccentric founder of the U.A.R., who has specified some strange conditions on the railway; thus the sceptical builders are unsure whether or not it is possible to build a railway subject to his unusual requirements.

Kane's will insists that there must be exactly 100 stops (each named after one of his great grand-children) between the termini, and he even dictates precisely what the distance along the track between each of these stops must be. (Unfortunately, the tables in the will *do not* list the order in which the stops are to appear along the railway.) Luckily, it is clear that Kane has put some thought into these distances; for any three distinct stops, the largest of the three distances among them is equal to the sum of the smaller two, which is an obvious necessary condition for the railway to be possible. (Also, all the given distances are shorter than the distance along a practical route from Cambridge to Northfield!)

U.A.R.'s engineers have pored over the numbers and noticed that for any four of Kane's stops, it would be possible to build a railway with these four stops and the distances between them as Kane specifies. Prove that, in fact, it is possible to complete the entire project to Kane's specifications.

9. Gail was giving a class on triangles, and she was planning to demonstrate on the blackboard that the three medians, the three angle bisectors, and the three altitudes of a triangle each meet at a point (the centroid, incentre, and orthocentre of the triangle, respectively). Unfortunately, she got a little careless in her example, and drew a certain triangle ABC with the median from vertex A, the altitude from vertex B, and the angle bisector from vertex C. Amazingly, just as she discovered her mistake, she saw that the three segments met at a point anyway! Luckily it was the end of the period, so no one had a chance to comment on her mistake. In recalling her good fortune later that day, she could only remember that the side across from vertex C was 13 inches in length, that the other two sides also measured an integral number of inches, and that none of the lengths were the same. What were the other two lengths?

10. An infinite sequence of digits "1" and "2" is determined uniquely by the following properties:

> (i) The sequence is built up by stringing together pieces of the form "12" and "112".
>
> (ii) If we replace each "12" piece with a "1" and each "112" piece with a "2", then we get the original sequence back.

(a) Write down the first dozen digits in the sequence. At which place will the 100^{th} "1" occur? What is the 1000^{th} digit?

(b) Let A_n be the number of "1"s among the first n digits of the sequence. Given that the ratio A_n/n approaches a limit, find that limit.

(c) (Tiebreaker) Show that the limit from part (b) actually exists.

The Volume of a Tetrahedron and Areas of its Faces

C.-S. Lin

A classical result states that if a tetrahedron $ABCD$ is rectangular at the vertex B, so that all the edges meeting at B are mutually perpendicular (See Figure 1), then the relation $b^2 = a^2 + c^2 + d^2$ is called the S–dimensional version of Pythagorean Theorem, where x denotes the area of the face opposite to the vertex X of the tetrahedron $ABCD$. The classical Heron's Formula of plane geometry says that for a triangle having sides of length a, b and c, and area K, we have

$$K = \frac{1}{4}\left(2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)\right)^{\frac{1}{2}}, \text{ or}$$

$$K = (s(s-a)(s-b)(s-c))^{\frac{1}{2}} \text{ for } s = \tfrac{1}{2}(a+b+c).$$

Although a well known classical result indicates that the volume of a tetrahedron can be expressed in terms of its six edges, and the formula may be found, for example, in [2, p.13 and 1731], I presented in [1] a relation which I called the S–dimensional analogue of Heron's Formula. Indeed, if V denotes the volume of an H–tetrahedron (attributed to Heron) $ABCD$; that is, it can be formed by gluing together two tetrahedra $ACDE$ and $BCDE$ both rectangular at the vertex E, and on a common congruent face $\triangle CDE$ (see figure 2) so that $\triangle ABC \perp \triangle ABD$, then

$$(*) \qquad V^2 = \frac{2cd\left(4(s-a)(s-b)(s-c)(s-d) - c^2d^2 - 2abcd\right)^{\frac{1}{2}}}{9(c^2+d^2)^{\frac{1}{2}}}$$

for $s = (a+b+c+d)$, and the small letters denote areas of faces as described before.

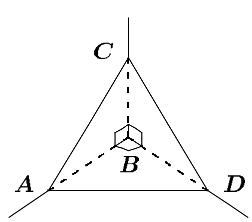

Figure 1

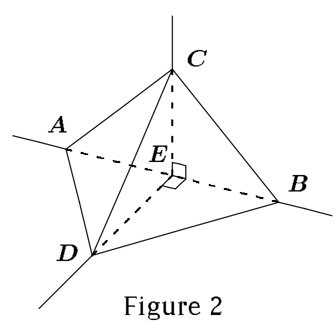

Figure 2

The identity $(*)$ leads naturally to the following question: Can we establish a formula which expresses the volume of an arbitrary tetrahedron in terms of its four faces? In this article I shall construct two formulas for an L–tetrahedron (See definition below). Each one shows the volume of an L–tetrahedron in terms of areas of its faces and some constant.

We begin with a definition first. A tetrahedron is said to be L–shaped, and called an L–tetrahedron if any two of its four faces are perpendicular. In Figure 3 or 4, we have arranged a tetrahedron $ABCD$ located in the first octant for which $\triangle ABC$ is on the xz–plane and $\triangle ABD$ on the xy–plane, so that $\triangle ABC \perp \triangle ABD$. More precisely, we assume that the vertex B is the origin such that $\overline{CE} \perp \overline{AB}$, $\overline{DF} \perp \overline{AB}$, and $\angle BED = \alpha$ with $0 < \alpha < \frac{\pi}{2}$ as in Figure 3, or $\frac{\pi}{2} < \alpha < \pi$ as in Figure 4. If $\alpha = \frac{\pi}{2}$, then $\overline{BF} = \overline{BE}$ and the L–tetrahedron becomes an H–tetrahedron as shown in Figure 2.

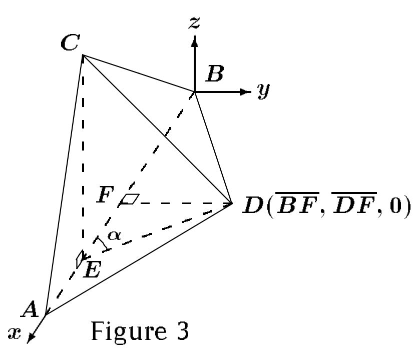

Figure 3

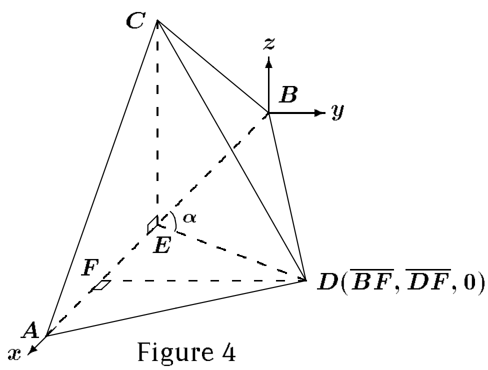

Figure 4

In what follows we shall consider an L–tetrahedron only. Of course it is possible that some L–tetrahedrons may be geometrically deformed, and are not analogously shown in Figure 3 or 4. For instance, the vertices C and D could be located anywhere on the xz–plane, and xy–plane, respectively, but not on the x–axis. However, the derivation of the volume (of a tetrahedron) formula, or the area (of a triangle) formula does not depend upon the location of the object concerned. For those people who enjoy doing long computations and simplifications the next result is a good one.

Theorem 1. Let V denote the volume of an L–tetrahedron $ABCD$ as in Figure 3 or 4, and let $r = \overline{CE}$, $q = \overline{BE}$, $u = \overline{BF}$, $m = \overline{BA}$, and $p = \overline{DF}$. If a, b, c, and d stand for areas of the four faces as described before, then we have the following relations.

(1) $V^2 = \frac{1}{9}cdrp$

(2) $a^2 = \frac{1}{4}\Big((p^2 + r^2)q^2 + p^2r^2 + (u-q)^2(r^2+q^2)$
$\quad -2r^2q(p^2 + (u-q)^2)^{\frac{1}{2}}\cos\alpha - q^2(p^2+(u-q)^2)\cos^2\alpha\Big)$

(3) $b^2 = \frac{1}{4}\big((p^2 + r^2)(m-q)^2 + p^2r^2 + (u-q)^2(r^2 + (m-q)^2)$
$\quad +2r^2(m-q)(p^2 + (u-q)^2)^{\frac{1}{2}}\cos\alpha$
$\quad -(m-q)^2(p^2 + (u-q)^2)\cos^2\alpha)$

(4) $c^2 = \frac{1}{4}p^2m^2$

(5) $d^2 = \frac{1}{4}r^2m^2$

(6) $\cos\alpha = \frac{q-u}{\sqrt{p^2+(q-u)^2}}$

Proof:

(1) Because the volume of a tetrahedron is one-third of the product of its base area with its height, the result follows.

(2) Clearly, $\overline{BC} = (q^2 + r^2)^{\frac{1}{2}}$ and $\overline{CD} = \left(r^2 + p^2 + (u-q)^2\right)^{\frac{1}{2}}$. From $\triangle BDE$ we have $\overline{DB} = \left(q^2 + p^2 + (u-q)^2 - 2q(p^2 + (u-q)^2)^{\frac{1}{2}}\cos\alpha\right)^{\frac{1}{2}}$ by the Cosine Law. In order to find a^2 we use Heron's Formula:

$$a^2 = \frac{1}{16}\left(2\left(\overline{BC}^2\overline{CD}^2 + \overline{CD}^2\overline{DB}^2 + \overline{DB}^2\overline{BC}^2\right) - \left(\overline{BC}^4 + \overline{CD}^4 + \overline{DB}^4\right)\right).$$

Long substitutions and simplifications show the identity (2), and we shall omit the details.

(3) Note that $\overline{CA} = \left(r^2 + (m-q)^2\right)^{\frac{1}{2}}$, and by $\triangle AED$ we get

$$\overline{AD} = \left(p^2 + (u-q)^2 + (m-q)^2 + 2(m-q)(p^2 + (u-q)^2)^{\frac{1}{2}}\cos\alpha\right)^{\frac{1}{2}},$$

since $\cos\angle AED = -\cos\alpha$. We also notice as in (2) that

$$b^2 = \frac{1}{16}\left(2\left(\overline{CA}^2\overline{CD}^2 + \overline{CA}^2\overline{AD}^2 + \overline{AD}^2\overline{CD}^2\right) - \left(\overline{CA}^4 + \overline{AD}^4 + \overline{CD}^4\right)\right).$$

Thus, the desired equation may be similarly obtained as equation (2).

The relations (4) and (5) are obvious by assumptions.

(6) $\cos\alpha = -\cos\angle AED = \dfrac{q-u}{\overline{ED}} = \dfrac{q-u}{\sqrt{p^2 + (q-u)^2}}$ if $\frac{\pi}{2} \le \alpha < \pi$,

and

$$\cos\alpha = \frac{q-u}{\overline{ED}} = \frac{q-u}{\sqrt{p^2 + (q-u)^2}} \quad \text{if } 0 < \alpha \le \tfrac{\pi}{2}.$$

We are now ready to state and prove the main result.

Theorem 2. Let V denote the volume of an L-tetrahedron $ABCD$ as in Figure 3 or 4, and let $r = \overline{CE}$, $q = \overline{BE}$, $u = \overline{BF}$, $m = \overline{BA}$, and $p = \overline{DF}$. If a, b, c, and d stand for areas of the four faces as described before, then we have the following relations.

(I) If $q-u=k_1p$ for some constant k_1, then

$$V^4=\frac{c^2d^2\left(2\left(a^2c^2+a^2d^2+a^2b^2+b^2c^2+b^2d^2-c^2d^2\right)-\left(a^4+b^4+c^4+d^4\right)\right)}{81\left(c^2+k_1^2c^2+d^2\right)}.$$

(II) If $u=k_2q$ for some constant $k_2\geq 0$, then

$$V^4 = \frac{c^2d^2\left(4a^2\left(c^2+d^2k_2\right)^2-\left(c^2+d^2k_2^2\right)\left(a^2+c^2+d^2-b^2\right)^2\right)}{81\left(c^2+d^2k_2\right)^2}.$$

Notice that both constants k_1 and k_2 determine different shapes of an L–tetrahedron.

Proof. (I) Since $q-u=k_1p$, and $\cos\alpha=\frac{q-u}{\sqrt{p^2+(q-u)^2}}$, by (6) in Theorem 1, we find $\cos\alpha=\frac{k_1}{\sqrt{1+k_1^2}}$. Thus, identities (2) and (3) in Theorem 1 become

(2′) $a^2=\frac{1}{4}\left(\left(p^2+r^2\right)q^2+p^2r^2+k_1^2p^2r^2-2k_1r^2pq\right)$,

(3′) $b^2=\frac{1}{4}\left(\left(p^2+r^2\right)(m-q)^2+p^2r^2+p^2k_1^2r^2+2k_1r^2(m-q)p\right)$.

Now , the whole idea is to eliminate p, q, r, and m from the five equations (1), (2′), (3′), (4), and (5) in Theorem 1, and to express V in terms of a, b, c, d, and k_1. We shall start with the next identity which follows from (1), (4) and (5).

(7) $V^4=\frac{16c^4d^4}{81m^4}$.

Subtracting equation (3′) from equation (2′), we have

$4(a^2-b^2)=(p^2+r^2)(2mq-m^2)-2k_1r^2mp$,

whence we get

$a^2-b^2=\frac{1}{m^2}[(c^2+d^2)(2mq-m^2)-4k_1cd^2]$

by (4) and (5), so that

(8) $q=\frac{1}{2m(c^2+d^2)}[m^2(a^2+c^2+d^2-b^2)+4k_1cd^2]$.

Using identities (2′), (4) and (5) we get

$m^4a^2 \;=\; m^2q^2(c^2+d^2)+4c^2d^2(1+k_1^2)-4k_1cd^2mq\,.$

Replacing the last q above by (8), and expressing q^2 in terms of the others, we get

(9) $q^2=\frac{m^2}{c^2+d^2}\Big(a^2-\frac{4c^2d^2(1+k_1^2)}{m^4}$
$\qquad+\frac{2k_1cd^2}{m^4(c^2+d^2)}\left(m^2(a^2+c^2+d^2-b^2)+4k_1cd^2\right)\Big)$.

By eliminating q from equations (8) and (9) we find that

$$
\begin{aligned}
4m^4\left(c^2+d^2\right)\Bigg(a^2-\frac{4c^2d^2\left(1+k_1^2\right)}{m^4} \\
+\frac{2k_1cd^2}{m^4\left(c^2+d^2\right)}\left(m^2\left(a^2+c^2+d^2-b^2\right)+4k_1cd^2\right)\Bigg) \\
=\left(m^2\left(a^2+c^2+d^2-b^2\right)+4k_1cd^2\right)^2 .
\end{aligned}
$$

From the above, we express m^4 in terms of a, b, c, d, and k_1 as follows.

(10) $m^4=\dfrac{16c^2d^2\left(c^2+k_1^2c^2+d^2\right)}{2\left(a^2b^2+b^2c^2+b^2d^2+a^2c^2+a^2d^2-c^2d^2\right)-\left(a^4+b^4+c^4+d^4\right)}$.

Finally, substituting m^4 in equation (7) yields the required formula (I).

(II) First we remark that the process of the proof of this formula is exactly the same as in (I) above. From the condition $u=k_2q$, identities (2) and (3) become

(2″) $a^2=\frac{1}{4}\left(\left(p^2+r^2\right)q^2+p^2r^2+q^2\left(k_2-1\right)^2r^2+2r^2q^2\left(k_2-1\right)\right)$,

(3″) $b^2=\frac{1}{4}\left(\left(p^2+r^2\right)(m-q)^2+p^2r^2+q^2\left(k_2-1\right)^2r^2-2r^2(m-q)q\left(k_2-1\right)\right)$.

Subtracting equation (3″) from equation (2″), we get

$4\left(a^2-b^2\right)=\left(p^2+r^2\right)m(2q-m)+2r^2mq\left(k_2-1\right)$,

and using identities (4) and (5), we obtain

$$
\begin{aligned}
a^2-b^2 &= \frac{1}{m}\left(c^2+d^2\right)(2q-m)+\frac{1}{m}2d^2q\left(k_2-1\right) \\
&= \frac{1}{m}2q\left(c^2+d^2k_2\right)-\left(c^2+d^2\right)
\end{aligned}
$$

Consequently,

(8′) $q=\dfrac{m(a^2+c^2+d^2-b^2)}{2(c^2+d^2k_2)}$.

With the aids of identities (4) and (5), (2″) becomes

$$
\begin{aligned}
m^4a^2 &= q^2m^2(c^2+d^2)+4c^2d^2+q^2(k_2-1)^2d^2m^2+2q^2(k_2-1)d^2m^2 \\
&= q^2(m^2c^2+k_2^2d^2m^2)+4c^2d^2 ,
\end{aligned}
$$

and hence

(9′) $q^2=\dfrac{m^4a^2-4c^2d^2}{m^2(c^2+d^2k_2^2)}$.

The next step is immediate from (8′) and (9′).

$$
m^4(c^2+d^2k_2^2)(a^2+c^2+d^2-b^2)^2 = 4(c^2+d^2k_2)^2(m^4a^2-4c^2d^2) ,
$$

and further,

$$(10') \; m^4 = \frac{16c^2d^2(c^2 + d^2k_2)^2}{4a^2(c^2 + d^2k_2)^2 - (c^2 + d^2k_2^2)(a^2 + c^2 + d^2 - b^2)^2}.$$

Substituting m^4 in (7) yields the desired formula (II), and the proof of the theorem is completed.

Remarks. Since the shape of an L–tetrahedron changes as the constant k_i, $i = 1, 2$, varies, let us consider two special cases in Theorem 2.

(A) If $k_1 = 0$ in (I), or if $k_2 = 1$ in (II); that is, the case when $\alpha = \frac{\pi}{2}$, we have, for $k_1 = 0$, that

$$V^4 = \frac{c^2d^2[2(a^2b^2+b^2c^2+b^2d^2+a^2c^2+a^2d^2-c^2d^2)-(a^4+b^4+c^4+d^4)]}{81(c^2+d^2)},$$

and, for $k_2 = 1$, that $V^4 = \frac{c^2d^2[4a^2(c^2+d^2)^2-(c^2+d^2)(a^2+c^2+d^2-b^2)^2]}{81(c^2+d^2)}$.

Rewriting the above two identities by using the expression $s = \frac{1}{2}(a+b+c+d)$ one should have the equation $(*)$ for an H–tetrahedron, and we shall omit the details.

(B) It is immediate, from Figure 3 or 4, that if $u = q = 0$, then we get a new tetrahedron rectangular at the vertex B as shown in Figure 1, which has volume

$$V = \frac{1}{6}(\overline{BA} \cdot \overline{BC} \cdot \overline{BD}) = \frac{1}{3}\sqrt{2acd}.$$

Since $u = q = 0$ implies $k_1 = 0$ in (I), and $k_2 \geq 0$ in (II), we have

$$\left(\frac{1}{3}\sqrt{2acd}\right)^4 = \frac{c^2d^2\left(2(a^2c^22+a^2d^2+a^2b^2+b^2c^2+b^2d^2-c^2d^2)-(a^4+b^4+c^4+d^4)\right)}{81(c^2+d^2)}$$

and $\left(\frac{1}{3}\sqrt{2acd}\right)^4 = \frac{c^2d^2\left(4a^2(c^2+d^2k_2)^2-(c^2+d^2k_2^2)(a^2+c^2+d^2-b^2)^2\right)}{81(c^2+d^2k_2)^2}$,

respectively. By simplifying both relations we arrive uniquely at $b^2 = a^2 + c^2 + d^2$. This is indeed an alternative proof of the 3–dimensional version of the Pythagorean Theorem.

References

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