2635. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Consider triangle ABC, and three squares $BCDE$, $CAFG$ and $ABHI$ constructed on its sides, outside the triangle. Let XYZ be the triangle enclosed by the lines EF, DI and GH.

Prove that $[XYZ] \le (4-2\sqrt{3})[ABC]$, where $[PQR]$ denotes the area of $\triangle PQR$.

2636. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that $A_1A_2\ldots A_n$ is a convex n–gon with $n \ge 5$, and that the angle at each vertex is divided into $(n-2)$ equal angles by the $(n-3)$ diagonals through that vertex. Prove that $A_1A_2\ldots A_n$ is a regular n–gon.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

There was a typographical error in the statement of Peter Woo's theorem in the second solution to 2516 — the theorem should read:

Theorem. Let CD, BE be cevians of a triangle ABC where $BD \ge CD$. Let CD intersect BE at P. Then $AD + DP > AE + EP$.

2513. [2000 : 114] *Proposed by Waldemar Pompe, University of Darmstadt, Darmstadt, Germany; dedicated to Prof. Toshio Seimiya on his 90^{th} birthday.*

A circle is tangent to the sides BC, AD of convex quadrilateral $ABCD$ in points C, D, respectively. The same circle intersects the side AB in points K and L. The lines AC and BD meet in P. Let M be the mid-point of CD. Prove that if $CL = DL$, then the points K, P, M are collinear.

I. *Solution by Michel Bataille, Rouen, France.*

We suppose first that AD is parallel to BC. In this case, M is the centre of the circle, say Γ, defined in the statement of the problem, L is the mid-point of AB, and $PC/PA = PB/PD = CB/AD$. We also assume $K \ne L$ (otherwise $ABCD$ is a rectangle and the conclusion is obvious), and we call I the intersection of lines MP and BC.

From Menelaus' Theorem applied to the transversal MP of $\triangle BCD$, and since $MC/DM = 1$ and $PD/BP = -AD/CB$, we get

$$\frac{MC}{DM}\cdot\frac{PD}{BP}\cdot\frac{IB}{CI} = -1\,,$$

so that $IB/CI = CB/AD$. Expressing the powers of B and A with respect to Γ, we obtain $BK \cdot BL = (BC)^2$ and $AK \cdot AL = (AD)^2$, which yields $KA/BK = (AD)^2/(BC)^2$ (since $BL = -AL$). Now

$$\frac{PC}{AP} \cdot \frac{KA}{BK} \cdot \frac{IB}{CI} = -\frac{CB}{AD} \cdot \frac{(AD)^2}{(BC)^2} \cdot \frac{CB}{AD} = -1,$$

and, from the converse of Menelaus' Theorem, the points P, K, I are collinear. The collinearity of P, K, M follows immediately.

As for the general case, it can be reduced to the previous one by means of a central projection transforming the circle Γ (with its interior point M) into a circle Γ' whose centre is the image M' of M. The whole figure is then transformed into that of the particular case, and the collinearity proved above implies the collinearity of M, P, K in the general case.

Note. For the use of a central projection, we refer to *Geometric Transformations III* by I. M. Yaglom, MAA (NML 21), Random House (1973), page 54, Theorem 1. See also Chapter 26 of *The Enjoyment of Mathematics*, Rademacher and Toeplitz, Princeton Univ. Press, 1957.

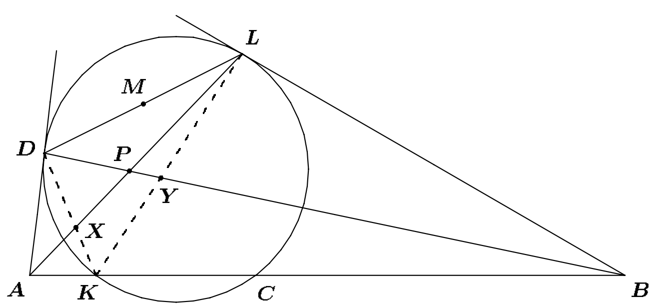
II. *Solution by the proposer*.

Let $X = DK \cap AC$, $Y = CK \cap BD$ (see the figure below). Since $CL = DL$,

$$\angle AKD = \angle BKC. \qquad (1)$$

[Since arc CL = arc DL, $\angle CKL = 180° - \angle DKL = \angle DKA$. — *Ed*.] The given circle is tangent to the sides BC and AD, so we get

$$\angle ADK = \angle DCK \qquad \text{and} \qquad \angle BCK = \angle CDK. \qquad (2)$$



According to Ceva's Theorem [and because $DM = MC$], the points K, P, M are collinear if and only if

$$\frac{KX}{XD} = \frac{KY}{YC}.$$

Denote by $[TUW]$ the area of triangle TUW. Then the above equality can be rewritten as

$$\frac{[AKC]}{[ADC]} = \frac{[BKD]}{[BCD]} \quad \text{or} \quad \frac{[AKC]}{[BKD]} = \frac{[ADC]}{[BCD]}.$$

Using the equalities (1) and (2), we see that the last equality is equivalent to

$$\frac{AK \cdot KC}{BK \cdot KD} = \frac{AD}{BC}. \tag{3}$$

We need to prove (3). In order to do this, we start with the following equality

$$\frac{[AKD]}{[DCK]} \cdot \frac{[DCK]}{[BKC]} = \frac{[AKD]}{[BKC]}.$$

Thus, using once again the equalities (1) and (2), gives

$$\frac{AD \cdot KD}{DC \cdot KC} \cdot \frac{KD \cdot DC}{KC \cdot BC} = \frac{AK \cdot KD}{BK \cdot KC} \quad \text{or} \quad \frac{AD \cdot KD}{KC \cdot BC} = \frac{AK}{BK},$$

which is the same as (3). Thus the points K, P, M are collinear.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; TOSHIO SEIMIYA, Kawasaki, Japan; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

Seimiya's solution treated the cases $AD \nparallel BC$ and $AD \parallel BC$ separately, as did the solution of Benito and Fernández. Two other solvers seemed to consider only the case $AD \nparallel BC$.

Woo sent in two solutions, one similar to Bataille's solution above, and one using more projective geometry which, Woo points out, is valid even if the circle is a general conic. On being sent this latter solution for expert analysis, Chris Fisher interpreted it to arrive at the following generalization.

> *A conic is tangent to the sides BC and AD of convex quadrilateral $ABCD$ at C and D. The same conic intersects the side AB in points K and L. The lines AC and BD meet in P. If R is the point where the given tangents intersect, and M is where RL intersects CD, then the points K, P, M are collinear.*

(The proposer has recently and independently sent in the same generalization.)

Chris's proof is a supercharged version of Solution I, using a transformation that immediately reduces the problem to the case when $ABCD$ is a rectangle. In his words: "Just define Q to be where KL meets CD and note that, if QR misses the conic (which happens if $ABCD$ is convex), there is a projective transformation that simultaneously takes the given conic to a circle and the line QR to infinity. The picture becomes a circle whose diameter is CD with LK parallel to it, and the figure is so symmetric that the statement 'P is on MK' is clear from the picture."

For "projective transformation" one may consult the Yaglom reference again. Readers may like to attempt to prove this general statement (with "conic" specialized to a circle again) by Euclidean means only. It amounts to the proposer's original problem but without the condition that $CL = DL$ and with M defined as the intersection of CD and RL, where R is the intersection of the tangents.

2520. [2000 : 115] *Proposed by Paul Bracken, CRM, Université de Montréal, Montréal, Québec.*

Let a, b be real numbers such that $b \leq 0$ and $1 + ax + bx^2 \geq 0$ for every $x \in [0,1]$. Define

$$F_n(a,b) \;=\; \int_0^1 \left(1 + ax + bx^2\right)^n \, dx \,.$$

Show that the following asymptotic expressions are valid for $F_n(a,b)$ as $n \to \infty$:

1. If $a < 0$ and $b \leq 0$, then

$$F_n(a,b) \;=\; -\frac{1}{an} + \frac{1}{n^2 a}\left(1 - \frac{2b}{a^2}\right) + \mathrm{O}\left(n^{-3}\right) \,.$$

2. If $a \geq 0$ and $b < 0$, then

$$F_n(a,b) \;\sim\; \sqrt{\frac{\pi}{n|b|}}\left(1 - \frac{a^2}{4b}\right)^{n+\frac{1}{2}} \,.$$

Editor's comment: there were a couple of typographical errors in the original statement. These were spotted by all solvers.

Solution to part 1 by Manuel Benito and Emilio Fernández, I.B. Praxedes Mateo Sagasta, Logroño, Spain.

If $a < 0$ and $b \leq 0$, then $y(x) = 1 + ax + bx^2$ is, by the hypothesis, a positive decreasing function on the interval $[0,1)$; since $\ln y(0) = 0$, $\ln y(x) < 0$ in $(0,1)$ and also decreases. Let δ be the positive solution of the equation $1 + ax + bx^2 = 1 + a$; that is, $\delta = \frac{1}{2b}\left(-a - \sqrt{a^2 + 4ab}\right)$. Also, we have $\delta < 1$ and

$$\ln(1 + ax + bx^2) \leq \ln(1 + a) \;<\; 0 \qquad \forall\, x \geq \delta > 0 \,,$$

and so,

$$F_n(a,b) = \int_0^1 e^{n\ln(1+ax+bx^2)}\, dx = \int_0^\delta e^{n\ln(1+ax+bx^2)}\, dx + \mathrm{o}\left(\frac{1}{n^3}\right) , \qquad (1)$$

given that

$$\int_\delta^1 e^{n\ln(1+ax+bx^2)}\, dx \leq (1-\delta)e^{n\ln(1+a)} \;=\; \mathrm{O}\big((1+a)^n\big) \;=\; \mathrm{o}\left(\frac{1}{n^3}\right)$$

as $n \to \infty$.

Let us propose, for the integral on the right side of (1), the change of variable

$$\ln(1 + ax + bx^2) \;=\; au \,;$$

this equation implicitly defines, near 0, a differentiable $x(u)$. It is possible to obtain the first terms of the powers of u series development of $x(u)$ by undetermined coefficients. Let $x = Bu + Cu^2 + Du^3 + \cdots$, and make the identification

$$\begin{aligned} 1 + ax + bx^2 &= 1 + aBu + (aC + bB^2)u^2 + (aD + 2bBC)u^3 + \ldots \\ &\equiv e^{au} = 1 + au + \frac{1}{2}a^2u^2 + \frac{1}{6}a^3u^3 + \ldots . \end{aligned}$$

It follows that

$$B = 1\,, \quad C = \frac{1}{a}\left(\frac{a^2}{2} - b\right)\,, \quad D = \frac{1}{a}\left(\frac{a^3}{6} - ab + \frac{2b^2}{a}\right)\,, \ldots .$$

And thus, near $u = 0$, we have

$$\frac{dx}{du} \;=\; B + 2Cu + 3Du^2 + \ldots \;=\; 1 + \left(a - \frac{2b}{a}\right)u + 3Du^2 + \ldots\,,$$

and by following the calculus on (1), by developing the change on the integral and integrating by parts,

$$\begin{aligned} F_n(a,b) &= \int_0^{\frac{\ln(1+a)}{a}} e^{nau}\left(1 + \left(a - \frac{2b}{a}\right)u + 3Du^2 + \ldots\right)du + \mathrm{o}\left(\frac{1}{n^3}\right) \\ &= \mathrm{o}\big((1+a)^n\big) - \frac{1}{an} + \frac{1}{n^2a^2}\left(a - \frac{2b}{a}\right) - \frac{6D}{n^3a^3} + \ldots \\ &= -\frac{1}{an} + \frac{1}{n^2a}\left(1 - \frac{2b}{a^2}\right) + \mathrm{O}(n^{-3})\,, \end{aligned}$$

as desired.

Editor's comment. There was disagreement amongst solvers about part 2. We shall leave it for the nonce.

Also solved by MICHEL BATAILLE, Rouen, France; and the proposer.

2528. [2000 : 177] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Prove that every rectifiable centrosymmetric curve on a unit sphere in $\mathbb{E}^3$ has length greater than or equal to 2π.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

Let A and A' be two centrosymmetric points on the curve. The result follows since the shortest distance on the sphere between A and A' is π.

Also solved by AUSTRIAN IMO-TEAM 2000 (whose solution was essentially the same as Klamkin's); MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Klamkin commented that a related known result is that any closed curve of length less than 2π lies in an open hemisphere. His open related problem is:

> *if one has a closed curve on the ellipsoid of semi-axes a, b, c, with $a \le b \le c$, and whose length is less than that of an ellipse of semi-axes a, b, then it lies in an open semi-ellipsoid.*

2529. [2000 : 178] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $G = \{A_1, A_2, \ldots, A_n\}$ be a set of points on a unit hemisphere. Let $\widehat{A_iA_j}$ be the spherical distance between the points A_i and A_j. Suppose that $\widehat{A_iA_j} \ge d$. Find $\max d$.

Solution by the proposer.

Let g be the spherical convex cover of G. It is obvious that for the perimeter L of g we have: $L(g) \ge nd$.

The condition on g, to be in the same hemisphere, is $L(g) \le 2\pi$. Therefore, g will be in the same hemisphere when $nd \le 2\pi$. So we must have

$$\max d \le \frac{2\pi}{n}.$$

There were no other solutions submitted for this problem.

2530. [2000 : 178] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let F be a compact convex set in $\mathbb{E}^3$, let T be the translation along a vector $\vec{a}$, and let $F' = T(F)$.

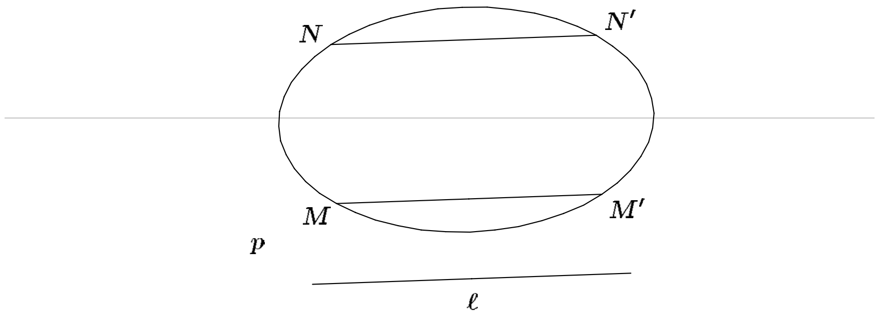
Prove that the intersection of the boundary of F and the boundary of F' is connected.

Solution by the proposer.

Let M, $M' \in F$ and $\overrightarrow{MM'} = \vec{a}$. Our problem is equivalent to proving that the locus of M' is connected.

We consider the line $l \parallel \vec{a}$ outside of F, and the support planes p_1, p_2 through ℓ to F. The dihedral angle of p_1, p_2 is denoted by ϑ; that is $\angle(p_1, p_2) = \vartheta$.

We also consider the plane p through ℓ moving into the dihedral angle $\angle(p_1, p_2)$. There are two positions of p so that the maximum diameters AB, $A'B'$ of c are parallel to ℓ and equal to $|\vec{a}|$. Between these two positions, the plane p intersects F along c', and there are two chords of c', MM' and NN' so that $\overrightarrow{MM'} = \overrightarrow{NN'} = \vec{a}$.



Suppose MM' is between NN' and ℓ. Denote the boundary of F by $\partial(F)$. Then the function $f : \vartheta \to \partial(F)$, so that for $0 \le \vartheta \le \vartheta_0$, $\vartheta \to M$ and $p = p(\vartheta)$ is continuous, so the point M describes a curve k_1 with endpoints A, A' on $\partial(F)$. Similarly, the point N describes a curve k_2 so that $k_1 \cap k_2 = \{A,\ A'\}$. We also easily see that M', N' describe curves k_1', k_2' so that $k_1' = T(k_1)$, $k_2' = T(k_2)$, $k_1' \cup k_2' = T(k_1 \cup k_2)$ and $k_1' \cup k_2' = \partial(F) \cap \partial(F')$.

The interval $[0, \vartheta]$ is connected (and compact), and f is a continuous function. Hence, according to a well-known theorem of topology,

$$k_1' \cup k_2' = \partial(F) \cap \partial(F') \text{ is connected.}$$

Comment: Using the same technique we can prove the following problem:

Suppose that η is the support line of F that is parallel to $\vec{a}$ and that $G = \{M | M \in F \cap \eta\}$. Then the point set G is connected.

There were no other solutions submitted.

2532. [2000 : 178] *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Suppose that a, b and c are positive real numbers satisfying $a^2 + b^2 + c^2 = 1$. Prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \ \ge\ 3 + \frac{2(a^3 + b^3 + c^3)}{abc}\,.$$

I. *Solution by Richard B. Eden, Ateneo de Manila University, Manila, The Philippines.*

$$\begin{aligned}
&\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-3-\frac{2\left(a^3+b^3+c^3\right)}{abc} \\
&\quad= \frac{a^2+b^2+c^2}{a^2}+\frac{a^2+b^2+c^2}{b^2}+\frac{a^2+b^2+c^2}{c^2} \\
&\qquad\quad -3-2\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right) \\
&\quad= a^2\left(\frac{1}{b^2}+\frac{1}{c^2}\right)+b^2\left(\frac{1}{c^2}+\frac{1}{a^2}\right)+c^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right) \\
&\qquad\quad -2\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right) \\
&\quad= a^2\left(\frac{1}{b}-\frac{1}{c}\right)^2+b^2\left(\frac{1}{c}-\frac{1}{a}\right)^2+c^2\left(\frac{1}{a}-\frac{1}{b}\right)^2 \;\geq\; 0\,,
\end{aligned}$$

with equality if and only if $a=b=c$.

II. *Solution by Goran Conar, student, University of Zagreb, Croatia.*

The given inequality is equivalent in sequence to

$$\frac{1-a^2}{a^2}+\frac{1-b^2}{b^2}+\frac{1-c^2}{c^2} \;\geq\; \frac{2\left(a^3+b^3+c^3\right)}{abc}$$

or

$$bc\left(\frac{b^2+c^2}{a}\right)+ca\left(\frac{c^2+a^2}{b}\right)+ab\left(\frac{a^2+b^2}{c}\right) \;\geq\; 2\left(a^3+b^3+c^3\right)$$

or

$$a^3\left(\frac{b}{c}+\frac{c}{b}-2\right)+b^3\left(\frac{c}{a}+\frac{a}{c}-2\right)+c^3\left(\frac{a}{b}+\frac{b}{a}-2\right) \;\geq\; 0\,,$$

which is true since $\frac{x}{y}+\frac{y}{x}\geq 2$ for all positive numbers x and y.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; the AUSTRIAN IMO Team 2000 (2 solutions); MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; JONATHAN CAMPBELL, Chapel Hill High School, Chapel Hill, NC, USA; MIGUEL CARRIÓN ÁLVAREZ, Universidad Complutense de Madrid, Spain; MAUREEN P. COX and ALBERT WHITE, Bonaventure University, St.Bonaventure, NY, USA; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES DIMINNIE and TREY SMITH, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; OLEG IVRII, student (grade 8), Cummer Valley Middle School, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; PHIL McCARTNEY, Northen Kentucky University, Highland Heights, KY, USA; VEDULA N. MURTY, Visakhapatnam, India; JUAN-BOSCO ROMERO MÁRQUEZ, *Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer (2 solutions)*

About half of the submitted solutions use the AM-GM Inequality in one way or another. The two solutions featured above are deviations from this. Solution I shows that, in fact, the inequality holds for all non-zero a, b, and c. This was explicitly pointed out only by the Austrian IMO Team 2000, Leversha, and Murty.

2536. [2000 : 179] *Proposed by Cristinel Mortici, Ovidius University of Constanta, Romania.*

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and periodic function such that for all positive integers n the following inequality holds:

$$\frac{|f(1)|}{1} + \frac{|f(2)|}{2} + \dots + \frac{|f(n)|}{n} \;\leq\; 1\,.$$

Prove that there exists $c \in \mathbb{R}$ such that $f(c) = 0$ and $f(c+1) = 0$.

I. *Solution by Michel Bataille, Rouen, France.*

Let $T > 0$ be a period of f. Since the function $|f|$ is continuous on $[0, T]$, there exists M such that $|f(x)| \leq M$ for all $x \in [0, T]$. By periodicity, we even have $|f(x)| \leq M$ for all $x \in \mathbb{R}$. Consider now the continuous, T-periodic function g defined by

$$g(x) \;=\; |f(x)| + |f(x+1)|\,.$$

We are required to show that there exists c such that $g(c) = 0$. Assume, for the purpose of contradiction, that $g(x) \neq 0$ for all x. Then, since g is a positive continuous function, it achieves a minimum $m > 0$ on $[0, T]$ and, by periodicity, we get $g(x) \geq m$ for all $x \in \mathbb{R}$. For $n = 1, 2, \dots$, let us define

$$S_n \;=\; \sum_{k=1}^{n} \frac{g(k)}{k}\,.$$

For any positive integer n we have, on the one hand,

$$S_n \;\geq\; m \sum_{k=1}^{n} \frac{1}{k}\,,$$

and, on the other hand,

$$\begin{aligned} S_n \;&=\; \sum_{k=1}^{n} \frac{|f(k)|}{k} + \sum_{k=1}^{n} \frac{|f(k+1)|}{k} \\ &=\; \sum_{k=1}^{n} \frac{|f(k)|}{k} + \sum_{k=1}^{n} \frac{|f(k+1)|}{k+1} + \sum_{k=1}^{n} \frac{|f(k+1)|}{k(k+1)} \\ &\leq\; 1 + 1 + M \sum_{k=1}^{n} \frac{1}{k(k+1)} \;\leq\; M + 2\,. \end{aligned}$$

(We have used the hypothesis and $\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1} \leq 1$.) Thus, we obtain $\sum_{k=1}^{n} \frac{1}{k} \leq \frac{M+2}{m}$ for all positive integers n, a clear contradiction to the well-known divergence of the harmonic series. The desired result follows.

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Case (i): Let $p \in \mathbb{Q}^+$ be the period of f; that is, $p = \alpha/\beta$ with $\alpha, \beta \in \mathbb{N}$, and $f(j) = f(j+\beta p) = f(j+\alpha)$ for all $j \in \mathbb{N}$ (even for all $j \in \mathbb{R}$). Assume there is a number $j \in \mathbb{N}$ such that $|f(j)| = \lambda > 0$. Then, because of the boundary condition of f, we have

$$\sum_{k=0}^{n} \frac{\lambda}{j+k\alpha} \;\leq\; \sum_{\ell=1}^{k\alpha+j} \frac{|f(\ell)|}{\ell} \;\leq\; 1\,,$$

a contradiction to $\sum\limits_{k=1}^{\infty} \frac{1}{j+k\alpha} = \infty$. Therefore, $f(x) = 0$ for all $x \in \mathbb{N}$ and we are done.

Case (ii): Let $p \in \mathbb{R}^+ \backslash \mathbb{Q}^+$ be the period of f. Then, the fractional part of the sequence $(np)_{n \geq 1}$ (that is, the sequence $\{a_n\}_{n \geq 1}$ where $a_n = np - \lfloor np \rfloor$) lies dense and equally distributed in the interval $[0, 1)$. Therefore, for any interval of length ε (and contained in $[0, 1)$), roughly speaking, every $1/\varepsilon^{\text{th}}$ member of the sequence a_n defined above lies in this interval. Assume that $f(c) \neq 0$ for some $c \in \mathbb{N}$. Then, $|f(c)| = \alpha > 0$. By continuity, there is an $\varepsilon > 0$ such that $|f(x)| > \alpha/2$ whenever $x \in \left(c - \frac{\varepsilon}{2}, c + \frac{\varepsilon}{2}\right)$. But then, the sums defined by

$$\sum_{j=0}^{n} \frac{|f(c_j)|}{c_j}\,, \qquad n \to \infty\,,$$

with $c_0 = c$, $c_j = c + jp$ (rounded), have to diverge. Thus, $f(c) = 0$ for all $c \in \mathbb{N}$. This in turn implies that $f(x) \equiv 0$ for $x \in \mathbb{R}$.

Also solved by the AUSTRIAN IMO TEAM 2000; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KEE-WAI LAU, Hong Kong; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.