

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (★) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}'' \times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than 1 November 2001. They may also be sent by email to crux-editors@cms.math.ca. (It would be appreciated if email proposals and solutions were written in \LaTeX). Graphics files should be in \LaTeX format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.

2620. Correction. *Proposed by Bill Sands, University of Calgary, Calgary, Alberta, dedicated to Murray S. Klamkin, on his 80th birthday.*

Three cards are handed to you. On each card are three non-negative real numbers, written one below the other, so that the sum of the numbers on each card is 1. You are permitted to put the three cards in any order you like, then write down the first number from the first card, the second number from the second card, and the third number from the third card. You add these three numbers together.

Prove that you can always arrange the three cards so that your sum lies in the interval $[\frac{1}{3}, \frac{3}{2}]$.

2633. *Proposed by Mihály Bencze, Brasov, Romania.*

$$\text{Prove that } \frac{n(n+1)}{2e} < \sum_{k=1}^n (k!)^{\frac{1}{k}} < \frac{31}{20} + \frac{n(n+1)}{4}.$$

2626 ★. *Proposed by Achilleas Sinefakopoulos, student, University of Athens, Greece.*

Let $\alpha_n = 2n + \lfloor n\sqrt{2} \rfloor$ for $n = 1, 2, \dots$. Suppose that k and m are positive integers such that α_m is a multiple of 10 and $\alpha_k = \alpha_m + 10j$ for some positive integer j . Prove or disprove that if $j \leq 4$, then $k = m + 3j$.

2627 . Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let x_1, \dots, x_n be positive real numbers and let $s_n = x_1 + \dots + x_n$ ($n \geq 2$). Let a_1, \dots, a_n be non-negative real numbers. Determine the optimum constant $C(n)$ such that

$$\sum_{j=1}^n \frac{a_j (s_n - x_j)}{x_j} \geq C(n) \left(\prod_{j=1}^n a_j \right)^{\frac{1}{n}}.$$

2628 . Proposed by Victor Oxman, University of Haifa, Haifa, Israel.

Four points, X, Y, Z and W are taken inside or on triangle ABC . Prove that there exists a set of three of these points such that the area of the triangle formed by them is less than $\frac{3}{8}$ of the area of the given triangle.

2629 . Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.

In triangle ABC , the symmedian point is denoted by S . Prove that

$$\frac{1}{3} (AS^2 + BS^2 + CS^2) \geq \frac{BC^2 AS^2 + CA^2 BS^2 + AB^2 CS^2}{BC^2 + CA^2 + AB^2}.$$

2630 . Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.

Prove that
$$\sum_{n=1}^{\infty} \frac{1}{n2^n} \sum_{j=1}^n \frac{1}{j} = \frac{\pi^2}{12}.$$

2631 . Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Find the exact value of
$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} \binom{n-k}{k} 2^{n-2k}.$$

2632 . Proposed by Mihály Bencze, Brasov, Romania.

Let $S_m = \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{1 \leq j \leq k \leq n} \cos \left(\frac{(j + (-1)^m k)x}{n} \right)$, where $x \in \mathbb{R}$.

Find exact expressions for S_0 and S_1 .

2634 . Proposed by Mihály Bencze, Brasov, Romania.

Let $P(x) = 1 + \sum_{k=1}^n a_k x^k$, where $a_k \in [0, 2]$ ($k = 1, 2, \dots, n$).

Prove that $P(x)$ is never zero in $(1 - \sqrt{2}, 0]$.

2635 . Proposed by Toshio Seimiya, Kawasaki, Japan.

Consider triangle ABC , and three squares $BCDE$, $CAFG$ and $ABHI$ constructed on its sides, outside the triangle. Let XYZ be the triangle enclosed by the lines EF , DI and GH .

Prove that $[XYZ] \leq (4 - 2\sqrt{3})[ABC]$, where $[PQR]$ denotes the area of $\triangle PQR$.

2636 . Proposed by Toshio Seimiya, Kawasaki, Japan.

Suppose that $A_1A_2 \dots A_n$ is a convex n -gon with $n \geq 5$, and that the angle at each vertex is divided into $(n - 2)$ equal angles by the $(n - 3)$ diagonals through that vertex. Prove that $A_1A_2 \dots A_n$ is a regular n -gon.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

There was a typographical error in the statement of Peter Woo's theorem in the second solution to 2516 — the theorem should read:

Theorem. Let CD , BE be cevians of a triangle ABC where $BD \geq CD$. Let CD intersect BE at P . Then $AD + DP > AE + EP$.

2513. [2000 : 114] Proposed by Waldemar Pompe, University of Darmstadt, Darmstadt, Germany; dedicated to Prof. Toshio Seimiya on his 90th birthday.

A circle is tangent to the sides BC , AD of convex quadrilateral $ABCD$ in points C , D , respectively. The same circle intersects the side AB in points K and L . The lines AC and BD meet in P . Let M be the mid-point of CD . Prove that if $CL = DL$, then the points K , P , M are collinear.

I. Solution by Michel Bataille, Rouen, France.

We suppose first that AD is parallel to BC . In this case, M is the centre of the circle, say Γ , defined in the statement of the problem, L is the mid-point of AB , and $PC/PA = PB/PD = CB/AD$. We also assume $K \neq L$ (otherwise $ABCD$ is a rectangle and the conclusion is obvious), and we call I the intersection of lines MP and BC .

From Menelaus' Theorem applied to the transversal MP of $\triangle BCD$, and since $MC/DM = 1$ and $PD/BP = -AD/CB$, we get

$$\frac{MC}{DM} \cdot \frac{PD}{BP} \cdot \frac{IB}{CI} = -1,$$