THE SKOLIAD CORNER

No. 53

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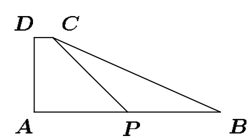
Last number we gave the problems of the Final Round of the Senior Mathematics Contest of the British Columbia Colleges. This issue we give the "official solutions". Our thanks go to Jim Totten, The University College of the Cariboo, and one of the contest organizers, for arranging these for our use.

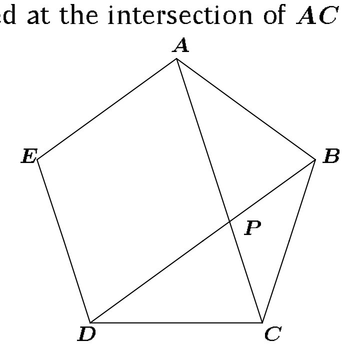
BRITISH COLUMBIA COLLEGES
Senior High School Mathematics Contest
Part A — Final Round — May 5, 2000

1. In the diagram, DC is parallel to AB, and DA is perpendicular to AB. If $DC = 1$, $DA = 4$, $AB = 10$, and the area of quadrilateral $APCD$ equals the area of triangle CPB, then PB equals:



Solution. The answer is (e). The area of the trapezoid $ABCD$ is $\frac{1}{2} \cdot 4 \cdot (1 + 10) = 22$ square units. We want to choose the point P such that the area of $\triangle CPB$ is half this area; that is, 11 square units. Let the base PB have length x. Then the area of $\triangle CPB = \frac{1}{2} \cdot 4 \cdot x = 2x$. Since this must be 11, we have $x = 5\frac{1}{2}$.

2. Label the vertices of a **regular** pentagon with A, B, C, D, and E, so that edges of the pentagon are line segments AB, BC, CD, DE, and EA. One of the angles formed at the intersection of AC and BD has measure:



Solution. The answer is (a). Let P be the intersection of AC and BD as in the diagram above. The sum of the interior angles of a (regular) pentagon is $(5 - 2)180° = 540°$. Thus, each interior angle in a regular pentagon has measure $540^\circ/5 = 108^\circ$. Since $\triangle ABC$ is isosceles with vertex angle equal to 108°, the base angles, namely $\angle BAC$ and $\angle BCA$, both have measure 36°. Similarly, $\angle CBD = \angle CDB = 36^\circ$. Since $\angle ABC = 108^\circ$ and $\angle CBD = 36^\circ$, we see that $\angle ABP = 108^\circ - 36^\circ = 72^\circ$. This means that the third angle in triangle ABP, namely $\angle APB$, has measure $180^\circ - 36^\circ - 72^\circ = 72^\circ$. Thus, the angles at P have measure 72° and $180^\circ - 72^\circ = 108^\circ$.

3. Three children are all under the age of 15. If I tell you that the product of their ages is 90, you do not have enough information to determine their ages. If I also tell you the sum of their ages, you still do not have enough information to determine their ages. Which of the following is *not* a possible age for one of the children?

Solution. The answer is **(d)**. Let a, b, c be the ages (in integers) of the three children. We may assume that $a \le b \le c < 15$. Since the product $abc = 90$, we also know that $a > 0$. The values we seek for a, b, and c are integers which divide evenly into 90 and lie between the values 1 and 14 (inclusive). The only such integers are 1, 2, 3, 5, 6, 9, and 10. We will now look at all possible products which satisfy the above conditions, and for each one we will compute the sum of the ages:

a	b	c	sum
1	9	10	20
2	5	9	16
3	3	10	16
3	5	6	14

We are further told that even knowing the sum of the ages would NOT allow us to determine the three ages. Thus, we are forced to conclude that the sum must be 16, since there are two distinct sets of ages which sum to 16 in the above table. The only ages found in these two sets are 2, 3, 5, 9, and 10. We notice that 1 and 6 do not appear.

4. I know I can fill my bathtub in 10 minutes if I put the hot water tap on full, and that it takes 8 minutes if I put the cold water on full. I was in a hurry so I put both on full. Unfortunately, I forgot to put in the plug. A full tub empties in 5 minutes. How long, in minutes, will it take for the tub to fill?

Solution. The answer is **(c)**. Let V be the volume of a full tub (in litres, say). Then the rate at which the hot water can fill the tub is $V/10$ litres per minute. Similarly the rate at which the cold water can fill the tub is $V/8$ litres per minute. On the other hand a full tub empties at the rate of $V/5$ litres per minute. If all three are happening at the same time, then the rate at which the tub fills is:

$$\frac{V}{10} + \frac{V}{8} - \frac{V}{5} = \frac{4V + 5V - 8V}{40} = \frac{V}{40}$$

litres per minute, which means it takes 40 minutes to fill the tub.

5. The smallest positive integer k such that

$$(k+1)+(k+2)+\cdots+(k+19)$$

is a perfect square is:

Solution. The answer is **(b)**. We first need to recall that the sum of the first n integers is given by $n(n+1)/2$. The sum we are presented with is the difference between the sums of the first $k+19$ integers and the first k integers. Using the above formula we have:

$$\begin{aligned}(k+1)+(k+2)+\cdots(k+19) &= \frac{(k+19)(k+20)}{2}-\frac{k(k+1)}{2}\\ &= \frac{k^2+39k+380-k^2-k}{2}\\ &= \frac{38k+380}{2} = 19(k+10)\,.\end{aligned}$$

Since 19 is prime, in order for $19(k+10)$ to be a perfect square, $k+10$ must contain 19 as a factor. The smallest such value occurs when $k+10=19$; that is, when $k=9$, and we indeed get a perfect square in this case, namely 19^2.

6. A six-digit number begins with 1. If this digit is moved from the extreme left to the extreme right without changing the order of the other digits, the new number is three times the original. The sum of the digits in either number is:

Solution. The answer is **(d)**. Let n be the number in question. Then n can be written as 10^5+a where a is a number with at most 5 digits. Moving the left-most digit (the digit 1) to the extreme right produces a number $10a+1$. The information in the problem now tells us that $10a+1=3(10^5+a)=300000+3a$, or $7a=299999$. This yields $a=42857$. Thus, $n=142857$ (and the other number we created is 428571), the sum of whose digits is $1+4+2+8+5+7=27$.

7. A cube of edge 5 cm is cut into smaller cubes, not all the same size, in such a way that the smallest possible number of cubes is formed. If the edge of each of the smaller cubes is a whole number of centimetres, how many cubes with edge 2 cm are formed?

Solution. The answer is **(d)**. Let us organize this solution by considering the size of the largest cube in the subdivision of the original cube. The largest could have a side of size 4 cm, 3 cm, 2 cm, or 1 cm. In each of these 4 cases we will determine the minimum number of cubes possible. In the first case, when there is a cube of side 4 cm present, we can only include cubes of side 1 cm to complete the subdivision, and we would need 61 of them, since the cube of side 4 cm uses up 64 cm^3 of the 125 cm^3 in the original cube. Thus, in this case we have 62 cubes in the subdivision. If we look at the other extreme case, namely when the largest cube in the subdivision has side length of 1 cm, we clearly need 125 cubes for the subdivision. We also note here that we will certainly decrease the number of cubes in a subdivision if we try to replace sets of 1×1 cubes by larger cubes whenever possible. Now consider the case when there is a cube of side length 3 cm present. If we place it anywhere but in a corner, the subdivision can only be completed by cubes of side length 1 cm, which gives us $1 + 98 = 99$ cubes in total. If we place it in a corner, we can then place 4 cubes of side length 2 cm on one side of the larger cube, 2 more such cubes on a second side and a third such cube on the third side; this gives us 1 large cube and 7 medium cubes for a total volume of $27 + 7(8) = 83$ cm^3, which means we still have 42 small cubes, for a grand total of 50 cubes. It is easy to see that if the largest cube has side length 2 cm we can place at most 8 of them in the original cube and the remainder of the volume must be made up of cubes of side length 1 cm; this gives a total of $8 + 61 = 69$ cubes. Thus, the smallest number of cubes possible is 50 and in this case there are 7 cubes of side length 2 cm.

Note that the restriction in the problem statement "not all the same size" can be dropped without changing the solution, since the above solution completely ignored that restriction.

8. The nine councillors on the student council are not all on speaking terms. The table below shows the current relationship between each pair of councillors, where '1' means 'is on speaking terms', '0' means 'is not on speaking terms', and the letters stand for the councillors' names.

	A	B	C	D	E	F	G	H	I
A	–	0	0	1	0	0	1	0	0
B	0	–	1	1	1	1	1	1	1
C	0	1	–	0	0	0	1	1	0
D	1	1	0	–	1	0	1	0	1
E	0	1	0	1	–	0	1	0	0
F	0	1	0	0	0	–	0	0	1
G	1	1	1	1	1	0	–	0	0
H	0	1	1	0	0	0	0	–	0
I	0	1	0	1	0	1	0	0	–

Councillor A recently started a rumour. It was heard by each councillor once and only once. Each councillor heard it from and passed it to a councillor with whom he or she was on speaking terms. Counting councillor A as zero, councillor E was the eighth and last to hear it. Who was the fourth councillor to hear the rumour?

Solution. The answer is **(a)**. We need to find a sequence of all nine councillors beginning with A and ending with E such that each pair of consecutive councillors are 'on speaking terms' with each other. When one first looks at the table provided, it looks a little daunting. However, a first observation is that among the councillors other than A and E (who need to appear at the ends of the sequence) councillors F and H are only 'on speaking terms' with two others, one of which is councillor B. Thus, councillor F must receive the rumour from B and pass it to I, or vice versa. Similarly, councillor H must hear the rumour from B and pass it to C, or vice versa. Thus, we must have either I–F–B–H–C or C–H–B–F–I as consecutive councillors in the sequence. Since A and E lie on the ends, and neither of them are 'on speaking terms with' either councillors C or I, we see that councillors D and G must be placed one on either end of the above subsequence of five councillors. This leaves us with either D–I–F–B–H–C–G or G–C–H–B–F–I–D. Councillors A and E can be placed on the front and rear of either of these sequences to gives the final sequence as either A–D–I–F–B–H–C–G–E or A–G–C–H–B–F–I–D–E. In either case the fourth person after councillor A (who started the rumour) to hear the rumour was councillor B.

9. A, B, and C are thermometers with different scales. When A reads 10° and 34°, B reads 15° and 31°, respectively. When B reads 30° and 42°, C reads 5° and 77°, respectively. If the temperature drops 18° using A's scale, how many degrees does it drop using C's scale?

Solution. The answer is **(e)**. Let us examine the temperature differences on the respective pairs of thermometers. A difference of 24° on A corresponds to a difference of 16° on B; thus, they are in the ratio of $3 : 2$. A difference of 12° on B corresponds to a difference of 72° on C; thus, they are in the ratio of $1 : 6$. Now a temperature drop of 18° on A means a drop of 12° on B (using the ratio $3 : 2$). This results in a temperature drop of 72° on C (using the ratio $1 : 6$).

10. The number of positive integers between 200 and 2000 that are multiples of 6 or 7 but not both is:

Solution. The answer is **(b)**. The number of positive integers less than or equal to n which are multiples of k is the integer part of n/k (that is, perform the division and discard the decimal fraction, if any). This integer is commonly denoted $\lfloor n/k \rfloor$. Thus, the number of positive integers between 200 and 2000 which are multiples of 6 is

$$\left\lfloor \frac{2000}{6} \right\rfloor - \left\lfloor \frac{200}{6} \right\rfloor = 333 - 33 = 300 .$$

Similarly, the number of positive integers between 200 and 2000 which are multiples of 7 is

$$\left\lfloor \frac{2000}{7} \right\rfloor - \left\lfloor \frac{200}{7} \right\rfloor = 285 - 28 = 257 .$$

In order to count the number of positive integers between 200 and 2000 which are multiples of 6 or 7 we could add the above numbers. This, however, would count the multiples of both 6 and 7 twice; that is, the multiples of 42 would be counted twice. Thus, we need to subtract from this sum the number of positive integers between 200 and 2000 which are multiples of 42. That number is

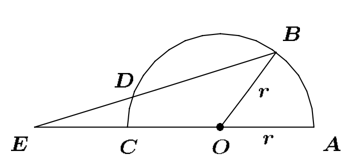
$$\left\lfloor \frac{2000}{42} \right\rfloor - \left\lfloor \frac{200}{42} \right\rfloor = 47 - 4 = 43 .$$

Therefore, the number of positive integers between 200 and 2000 which are multiples of 6 or 7 is $300 + 257 - 43 = 514$. But we are asked for the number of positive integers which are multiples of 6 or 7, but NOT BOTH. Thus, we need to again subtract the number of multiples of 42 in this range, namely 43. The final answer is $514 - 43 = 471$.

Part B — Final Round — May 5, 2000

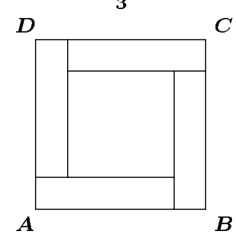
1. In the diagram O is the centre of a circle with radius r, and $ED = r$.

The angle $\angle DEC = k\angle BOA$. Find k.



Solution. First draw the radius OD. Let $\angle AEB = \alpha$. Since $DE = r = OD$, $\triangle DOE$ is isosceles. Therefore, $\angle DOE = \alpha$. Since $\angle BDO$ is an exterior angle to $\triangle DOE$, it is equal in measure to the sum of the opposite interior angles of the triangle; that is, $\angle BDO = 2\alpha$. Now, $\triangle BOD$ is isosceles, since two of its sides are radii of the circle. Thus, $\angle DBO = \angle BDO = 2\alpha$. Since $\angle BOA$ is an exterior angle to $\triangle BOE$, it is equal in measure to the sum of $\angle AEB = \alpha$ and $\angle EBO = 2\alpha$. Thus, $\angle BOA = 3\alpha$, which means that the value k in the problem is $\frac{1}{3}$.

2. The square $ABCD$, whose area is 180 square units, is divided into five rectangular regions of equal area, four of which are congruent as shown. What are the dimensions of one of the rectangular regions which is not a square?



Solution. Since there are five regions of equal area which sum to 180 square units, each region has area 36 square units. The dimensions of the inner square are clearly 6 units on a side, and of the outer square are $\sqrt{180} = 6\sqrt{5}$ units on a side. Let x and y be the dimensions of one of the four congruent regions, where $x < y$. Then $x + y = 6\sqrt{5}$ and $y - x = 6$. On adding these and dividing by 2 we get $y = 3(\sqrt{5} + 1)$, and then it easily follows that $x = 3(\sqrt{5} - 1)$.

3. An integer i *evenly divides* an integer j if there exists an integer k such that $j = ik$; that is, if j is an integer multiple of i.

(a) Recall $n! = (n)(n-1)(n-2)\cdots(2)(1)$. Find the largest value of n such that 25 evenly divides $n! + 1$.

(b) Show that if 3 evenly divides $x + 2y$, then 3 evenly divides $y + 2x$.

Solution. (a) Note that $5! = 5 \cdot 4 \cdot 3 \cdot 2 = 120$, which ends in the digit 0. Thus, $n!$, where $n > 5$, must also end in the digit 0, since $5!$ evenly divides $n!$, for $n > 5$. Thus, $n! + 1$ ends in the digit 1 whenever $n \geq 5$, which means that 25 can never evenly divide $n! + 1$ when $n \geq 5$. We are left to examine the cases $n = 4$, 3, 2, and 1. Since $4! + 1 = 24 + 1 = 25$, we see that 25 divides evenly $4! + 1$. Thus, $n = 4$ is the largest value of n such that 25 evenly divides $n! + 1$.

(b) Note first that $(x + 2y) + (y + 2x) = 3x + 3y = 3(x + y)$. Thus, three evenly divides the sum of the two numbers. This can be rewritten as $y + 2x = 3(x + y) - (x + 2y)$. Suppose that 3 evenly divides $x + 2y$. This means that $x + 2y = 3k$ for some integer k. Thus, $y + 2x = 3(x + y) - 3k = 3(x + y - k)$, which means that 3 evenly divides $y + 2x$.

4. A circular coin is placed on a table. Then identical coins are placed around it so that each coin touches the first coin and its other two neighbours. It is known that exactly six coins can be so placed.

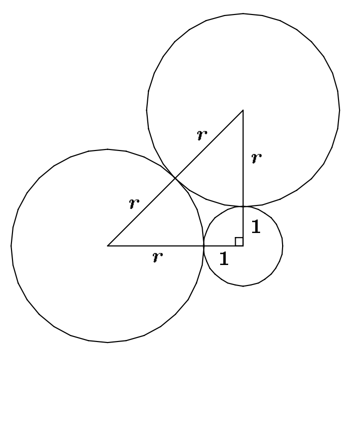
(a) If the radius of all seven coins is 1, find the total area of the spaces between the inner coin and the six outer coins.

(b) If the inner coin has radius 1, find the radius of a larger coin, so that exactly four such larger coins fit around the outside of the coin of radius 1.

Solution. (a) Same as Problem #5(b) on the Junior Paper (Part B) given on [2001: 33].

(b) Let r be the radius of each of the four larger coins which surround the coin of radius 1. Then by considering two such neighbouring coins and the coin of radius 1 (as in the diagram below) we have a right-angled triangle when we connect the three centres. The Theorem of Pythagoras then implies that

$$\begin{aligned} (2r)^2 &= (r+1)^2 + (r+1)^2\,, \\ 4r^2 &= 2r^2 + 4r + 2\,, \\ 2r^2 - 4r - 2 &= 0\,, \\ r^2 - 2r - 1 &= 0\,. \end{aligned}$$

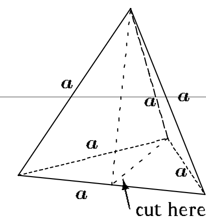


This quadratic has solutions

$$r = \frac{2 \pm \sqrt{4 - 4(1)(-1)}}{2} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}\,.$$

When we reject the negative root, we are left with $r = 1 + \sqrt{2}$.

5. The 6 edges of a regular tetrahedron are of length a. The tetrahedron is sliced along one of its edges to form two identical solids.



(a) Find the perimeter of the slice.

(b) Find the area of the slice.

Solution. (a) The slice is in the shape of an isosceles triangle with two sides equal to the altitude of the equilateral triangular faces of side a and the third side of length a. By using the Theorem of Pythagoras on half of one equilateral triangular face we see that its altitude is given by $a\sqrt{3}/2$. Thus, the perimeter of the triangular slice is $a + 2(a\sqrt{3}/2) = a(1 + \sqrt{3})$.

(b) We must now find the area of the triangular slice whose sides we found in part (a) above. Consider the altitude h which splits the isosceles slice into two congruent halves. Each half is a right-angled triangle with hypotenuse $a\sqrt{3}/2$ and one side of length $a/2$. The third side is h, which can be found by the Theorem of Pythagoras:

$$\begin{aligned} h^2 &= \left(\frac{a\sqrt{3}}{2}\right)^2 - \left(\frac{a}{2}\right)^2 = \frac{3a^2}{4} - \frac{a^2}{4} = \frac{a^2}{2} \\ h &= \frac{a}{\sqrt{2}} = \frac{a\sqrt{2}}{2}. \end{aligned}$$

Thus, the area of the slice is

$$\frac{1}{2} \cdot a \cdot \frac{a\sqrt{2}}{2} = \frac{a^2\sqrt{2}}{4}.$$

That completes the *Skoliad Corner* for this issue.

Who wrote this?

The mathematician is fascinated with the marvellous beauty of the forms he constructs, and in their beauty he finds everlasting truth.