On Improper Integrals

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In this note, we evaluate some improper integrals. For the first proposition, two proofs are given. The first one is a nice application of **integration by parts** and **change of variable** techniques. The second method is more advanced. We make use of the **Mean Value Theorem** and the **Zero Derivative Theorem** and **Chain Rule**. It is shown that $I(a)$ is a differentiable function, and to find its derivative one simply changes the order of integration and differentiation. An application of the **First Fundamental Theorem of Calculus** leads to more interesting results.

Proposition 1 Let a be a positive real number. Then we have that

$$I(a) \;=\; \int_0^\infty \frac{\arctan\left(\frac{x}{a}\right) + \arctan(ax)}{1+x^2}\,dx \;=\; \frac{\pi^2}{4}\,.$$

First proof. Let us consider $\displaystyle\int_0^\infty \frac{\arctan\left(\frac{x}{a}\right)}{1+x^2}\,dx$ and do integration by parts. Take $u = \arctan\left(\frac{x}{a}\right)$ and $dv = \dfrac{dx}{1+x^2}$. Thus, $du = \dfrac{a\,dx}{a^2+x^2}$ and $v = \arctan(x)$. Hence,

$$\begin{aligned}\int_0^\infty \arctan\left(\frac{x}{a}\right)\frac{dx}{1+x^2} \;&=\; \arctan\left(\frac{x}{a}\right)\arctan(x)\Big|_{x=0}^{x\to\infty} \\ &\quad -\int_0^\infty \arctan(x)\,\frac{a\,dx}{a^2+x^2} \\ &=\; \frac{\pi^2}{4} - \int_0^\infty \arctan(x)\,\frac{a\,dx}{a^2+x^2}\,.\end{aligned}$$

In the last integral change x to ax. Since a is positive, the upper and lower limits (0 and ∞) do not change. Hence,

$$\begin{aligned}\int_0^\infty \frac{\arctan(\frac{x}{a})}{1+x^2}\,dx \;&=\; \frac{\pi^2}{4} - \int_0^\infty \arctan(ax)\,\frac{a^2\,dx}{a^2+a^2\,x^2} \\ &=\; \frac{\pi^2}{4} - \int_0^\infty \arctan(ax)\,\frac{dx}{1+x^2}\,.\end{aligned}$$

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As the first proof shows, $I(a)$ is a numerical constant and does not depend on a. Therefore, we should also be able to do this problem by showing that $I' \equiv 0$.

Second proof. Let $J(a) = \displaystyle\int_0^\infty \frac{\arctan(ax)}{1+x^2}\,dx$, $0 < a < \infty$. Since $I(a) = J(a)+J\left(\frac{1}{a}\right)$, we will show that J is differentiable, find $J'(a)$, and use this to show that $I' \equiv 0$. To find J', we can change the order of integration and differentiation. Let $0 < |\Delta a| < \frac{a}{2}$. Eventually, we let $\Delta a \to 0$. By the Mean Value Theorem, there exists $0 \leqslant \theta \leqslant 1$ such that

$$\arctan\big((a+\Delta a)\,x\big) \;-\; \arctan(a\,x) \;=\; \frac{\Delta a\, x}{1+(a+\theta\,\Delta a)^2\,x^2}\,.$$

Of course, θ depends on a and Δa. Thus,

$$\begin{aligned}
&\left|\frac{J(a+\Delta a)-J(a)}{\Delta a} - \int_0^\infty \frac{x}{(1+a^2\,x^2)\,(1+x^2)}\,dx\right| \\
&= \left|\int_0^\infty \left(\frac{\arctan\big((a+\Delta a)\,x\big) \;-\; \arctan(a\,x)}{\Delta a} - \frac{x}{(1+a^2\,x^2)}\right)\frac{dx}{(1+x^2)}\right| \\
&\leqslant \int_0^\infty \left|\frac{x}{1+(a+\theta\,\Delta a)^2\,x^2} - \frac{x}{1+a^2\,x^2}\right|\frac{dx}{1+x^2} \\
&= \int_0^\infty \frac{|2\,a\,\theta\,\Delta a + \theta^2\,(\Delta a)^2|\,x^3}{(1+(a+\theta\,\Delta a)^2\,x^2)\;(1+a^2\,x^2)\,(1+x^2)}\,dx \\
&\leqslant |\Delta a| \int_0^\infty \frac{(2a+\frac{a}{4})\,x^3}{(1+\frac{a^2}{4}\,x^2)\,(1+a^2\,x^2)\,(1+x^2)}\,dx\,.
\end{aligned}$$

The last integral is convergent. Let $\Delta a \to 0$. Hence, for each $a > 0$,

$$J'(a) \;=\; \int_0^\infty \frac{x}{(1+a^2\,x^2)\,(1+x^2)}\,dx\,.$$

We are able to evaluate this integral. For $a \neq 1$

$$\begin{aligned}
J'(a) \;&=\; \int_0^\infty \frac{x}{(1+a^2\,x^2)(1+x^2)}\,dx \;=\; \int_0^\infty \left(\frac{\frac{-a^2}{1-a^2}\,x}{1+a^2\,x^2} + \frac{\frac{1}{1-a^2}\,x}{1+x^2}\right)dx \\
&= \frac{1}{2(1-a^2)}\,\ln\left(\frac{1+x^2}{1+a^2\,x^2}\right)\Bigg|_{x=0}^{x\to\infty} \;=\; \frac{\ln a}{a^2-1}\,.
\end{aligned}$$

It is even simpler to show that $J'(1) = \frac{1}{2} = \lim\limits_{a\to 1}\dfrac{\ln a}{a^2-1}$. Therefore, by the Chain Rule, for each $a > 0$,

$$I'(a) \;=\; J'(a) - \frac{1}{a^2}\,J'\left(\frac{1}{a}\right) \;=\; \frac{\ln a}{a^2-1} \;-\; \frac{1}{a^2}\,\frac{\ln\left(\frac{1}{a}\right)}{\frac{1}{a^2}-1} \;=\; 0\,.$$

Thus, by the Zero Derivative Theorem, I is a constant function. For $a = 1$,

$$I(1) \;=\; \int_0^\infty \frac{2\arctan(x)}{1+x^2}\,dx \;=\; \arctan^2(x)\,\Big|_{x=0}^{x\to\infty} \;=\; \left(\frac{\pi}{2}\right)^2 - 0 \;=\; \frac{\pi^2}{4}\,.$$

Therefore, $I \equiv \frac{\pi^2}{4}$.

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Since we have shown that J is differentiable on $(0,\infty)$, it is also continuous on this interval. One can show that J' is not differentiable at zero, but, nevertheless, a similar reasoning, as in the second proof, shows that J is actually continuous on $(-\infty,\infty)$. Continuity at zero is implicitly used in the following corollary.

Corollary 1

$$\int_0^1 \frac{\ln x}{x^2-1}\,dx \;=\; \frac{\pi^2}{8} \qquad \text{and} \qquad \int_0^\infty \frac{\ln x}{x^2-1}\,dx \;=\; \frac{\pi^2}{4}\,.$$

Proof. It is easy to show $J(0) = 0$, $J(1) = \dfrac{\pi^2}{8}$ and $\lim\limits_{a\to\infty} J(a) = \dfrac{\pi^2}{4}$. Thus, by the First Fundamental Theorem of Calculus,

$$\int_0^a \frac{\ln x}{x^2-1}\,dx \;=\; \int_0^a J'(x)\,dx \;=\; J(a) - J(0) \;=\; J(a)\,.$$

Put $a = 1$, and also let $a \to \infty$, to get

$$\int_0^1 \frac{\ln x}{x^2-1}\,dx \;=\; \frac{\pi^2}{8} \qquad \text{and} \qquad \int_0^\infty \frac{\ln x}{x^2-1}\,dx \;=\; \frac{\pi^2}{4}\,.$$

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