THE OLYMPIAD CORNER

No. 213

R.E. Woodrow

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We start this number with a set of five Klamkin Quickies. Many thanks to Murray Klamkin for sending them to us.

FIVE KLAMKIN QUICKIES

1. Prove that

$$a+b+c \;\geq\; \sqrt{b^2+c^2-a^2}+\sqrt{c^2+a^2-b^2}+\sqrt{a^2+b^2-c^2}\,,$$

where a, b, c are sides of a non-obtuse triangle.

2. Determine the extreme values of the area of a triangle ABC given the lengths of the two altitudes h_a, h_b and the side $BC = a$.

3. Determine the maximum area of a triangle ABC given the perimeter p and the angle A.

4. Determine the minimum value of

$$\sum\left[\frac{a_2+a_3+a_4+a_5}{a_1}\right]^{1/2}$$

where the sum is cyclic over the positive numbers a_1, a_2, a_3, a_4, a_5.

5. $ABCD$ and $AB'C'D'$ are any two parallelograms in a plane with A opposite to C and C'. Prove that BB', CC' and DD' are possible sides of a triangle.

Next we give the problems of the two days of the Vietnamese Mathematical Olympiad 1997. My thanks go to Richard Nowakowski, Canadian Team Leader at the IMO in Argentina for collecting them.

VIETNAMESE MATHEMATICAL COMPETITION 1997

First Day — March 14, 1997

Time: 3 hours

1. In a plane, let there be given a circle with centre O, with radius R and a point P inside the circle, $OP = d < R$. Among all convex quadrilaterals $ABCD$, inscribed in the circle such that their diagonals AC and BD cut each other orthogonally at P, determine the ones which have the greatest perimeter and the ones which have the least perimeter. Calculate these perimeters in terms of R and d.

2. Let there be given a whole number $n > 1$, not divisible by 1997. Consider two sequences of numbers $\{a_i\}$ and $\{b_j\}$ defined by:

$$a_i = i + \frac{ni}{1997} \quad (i = 1, 2, 3, \ldots, 1996),$$
$$b_j = j + \frac{1997j}{n} \quad (j = 1, 2, 3, \ldots, n-1).$$

By arranging the numbers of these two sequences in increasing order, we get the sequence $c_1 \leq c_2 \leq \cdots \leq c_{1995+n}$.

Prove that $c_{k+1} - c_k < 2$ for every $k = 1, 2, \ldots, 1994 + n$.

3. How many functions $f : \mathbb{N}^* \to \mathbb{N}^*$ are there that simultaneously satisfy the two following conditions:

(i) $f(1) = 1$,

(ii) $f(n) \cdot f(n+2) = (f(n+1))^2 + 1997$ for all $n \in \mathbb{N}^*$?

($\mathbb{N}^*$ denotes the set of all positive integers.)

Second Day — March 15, 1997

Time: 3 hours

4. (a) Find all polynomials of least degree, with rational coefficients such that

$$f(\sqrt[3]{3} + \sqrt[3]{9}) = 3 + \sqrt[3]{3}.$$

(b) Does there exist a polynomial with integer coefficients such that

$$f(\sqrt[3]{3} + \sqrt[3]{9}) = 3 + \sqrt[3]{3}?$$

5. Prove that for every positive integer n, there exists a positive integer k such that $19^k - 97$ is divisible by 2^n.

6. Let there be given 75 points, where no three of them are collinear, inside a cube, of which the length of an edge is 1. Prove that there exists a triangle whose vertices are among these 75 points and such that its area does not exceed $\frac{7}{72}$.

The next problem set gives the problems of the Team Selection Examination for Turkey for the 38^{th} IMO. My thanks go to Richard Nowakowski, Canadian Team Leader to the IMO in Argentina for collecting them.

TURKEY TEAM SELECTION EXAMINATION FOR THE 38^{th} IMO

First Day — April 12, 1997

Time: 4.5 hours

1. In a triangle ABC which has a right angle at A, let H denote the foot of the altitude belonging to the hypotenuse. Show that the sum of the radii of the incircles of the triangles ABC, ABH and AHC is equal to $|AH|$.

2. The sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ are defined through $a_1 = \alpha$, $b_1 = \beta$, and $a_{n+1} = \alpha a_n - \beta b_n$, $b_{n+1} = \beta a_n + \alpha b_n$ for all $n \geq 1$. How many pairs (α, β) of real numbers are there such that

$$a_{1997} = b_1 \quad \text{and} \quad b_{1997} = a_1\,?$$

3. In a soccer league, when a player is transferred from a team X with x players to a team Y with y players, the federation is paid $y - x$ billion liras by Y if $y \geq x$, while the federation pays $x - y$ billion liras to X if $x > y$. A player is allowed to change as many teams as he wishes during a season. In a league consisting of 18 teams, each team starts the season with 20 players. At the end of the season, 12 of these turn out again to have 20 players, while the remaining 6 teams end up having 16, 16, 21, 22, 22 and 23 players, respectively. What is the maximal amount the federation may have won during this season?

Second Day — April 13, 1997

Time: 4.5 hours

4. The edge AE of a convex pentagon $ABCDE$ whose vertices lie on the unit circle passes through the centre of this circle. If $|AB| = a$, $|BC| = b$, $|CD| = c$, $|DE| = d$ and $ab = cd = \frac{1}{4}$, compute $|AC| + |CE|$ in terms of a, b, c, d.

5. Prove that, for each prime number $p \geq 7$, there exists a positive integer n and integers $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ which are not divisible by p, such that

$$\begin{aligned}
x_1^2 + y_1^2 &\equiv x_2^2 \pmod{p}\,, \\
x_2^2 + y_2^2 &\equiv x_3^2 \pmod{p}\,, \\
&\vdots \\
x_{n-1}^2 + y_{n-1}^2 &\equiv x_n^2 \pmod{p}\,, \\
x_n^2 + y_n^2 &\equiv x_1^2 \pmod{p}\,.
\end{aligned}$$

6. Given an integer $n \geq 2$, find the minimal value of

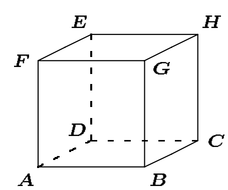
$$\frac{x_1^5}{x_2+x_3+\cdots+x_n}+\frac{x_2^5}{x_1+x_3+\cdots+x_n}+\cdots+\frac{x_n^5}{x_1+x_2+\cdots+x_{n-1}}$$

subject to $x_1^2+x_2^2+\cdots+x_n^2=1$, where x_1, x_2, ..., x_n are positive real numbers.

As a final collection of problems this number we give those of the Chilean Mathematical Olympiads 1994–1995. Thanks go to Raul A. Simon Lamb, Santiago, Chile, for forwarding the set to us.

CHILEAN MATHEMATICAL OLYMPIADS 1994–95

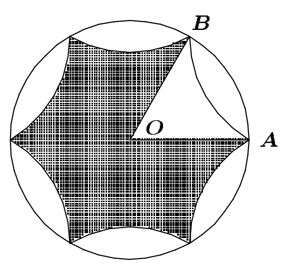
1. Given three straight lines in a plane, that concur at point O, consider the three consecutive angles between them (which, naturally, add up to 180°). Let P be a point in the plane not on any of these lines and let A, B, C be the feet of the perpendiculars drawn from P to the three lines. Show that the internal angles of $\triangle ABC$ are equal to those between the given lines.

2. 

$ABCDEFGH$ is a cube of edge 2. Let M be the mid-point of $\overline{BC}$ and N the mid-point of $\overline{EF}$. Compute the area of the quadrilateral $AMHN$.

3. Given a trapezoid $ABCD$, where $\overline{AB}$ and $\overline{DC}$ are parallel, and $\overline{AD}=\overline{DC}=\overline{AB}/2$, determine $\angle ACB$.

4. In a circle of radius 1 are drawn six equal arcs of circles, radius 1, cutting the original circle as in the figure. Calculate the shaded area.



5. In right triangle ABC the altitude $h_c = \overline{CD}$ is drawn to the hypotenuse $\overline{AB}$. Let P, P_1, P_2 be the radii of the circles inscribed in the triangles ABC, ADC, BCD respectively. Show that $P+P_1+P_2=h_c$.

6. Consider the product of all the positive multiples of 6 that are less than 1000. Find the number of zeros with which this product ends.

7. Let x be an integer of the form

$$x = \underbrace{1\,1\,1\,\ldots\,1}_{n}.$$

Show that, if x is a prime, then n is a prime.

8. Let x be a number such that

$$x + \frac{1}{x} = -1.$$

Compute

$$x^{1994} + \frac{-1}{x^{1994}}.$$

9. Let $ABCD$ be an $m \times n$ rectangle, with $m, n \in \mathbb{N}$. Consider a ray of light that starts from A, is reflected at an angle of $45°$ on another side of the rectangle, and goes on reflecting in this way.

(a) Show that the ray will finally hit a vertex.

(b) Suppose m and n have no common factor greater than 1. Determine the number of reflections undergone by the ray before it hits a vertex.

10. Let a be a natural number. Show that the equation

$$x^2 - y^2 = a^3$$

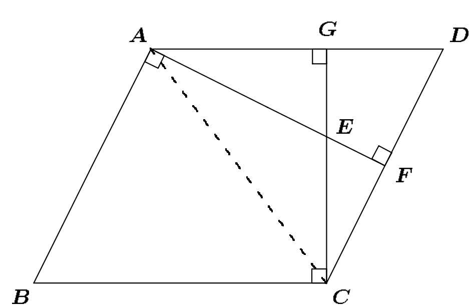
always has integer solutions for x and y.

Next we turn to readers' solutions to the problems of the Third Macedonian Mathematical Olympiad given [1999 : 198].

1. Let $ABCD$ be a parallelogram which is not a rectangle and E be a point in its plane, such that $AE \perp AB$ and $BC \perp EC$. Prove that $\angle DAE = \angle CEB$. [Ed. We know this is incorrect — can any reader supply the correct version?]

Correction and solution by Luyun-Zhong-Qiao, Columbia International College, Hamilton, Ontario. [Ed. The solver considers only the case when E is interior to the parallelogram.]

Try to prove that $\angle DAE = \angle DCE$.



Proof. Extend AE to meet CD at F; CE to meet AD at G. Therefore, $ABCD$ is a parallelogram. Therefore, $AB \parallel DC$, $AD \parallel BC$, and

$$\left.\begin{array}{l} EC \perp BC \implies EC \perp AD \\ AE \perp AB \implies AE \perp CD \end{array}\right\} \implies \left\{\begin{array}{l} CG \perp AD\,, \\ AF \perp CD\,. \end{array}\right.$$

With base AC, and since $\angle AGC = \angle AFC = 90°$, we have that $ACFG$ is a cyclic quadrilateral. Therefore, $\angle DAE = \angle DCE$.

Next we give an analysis of what the question, as presented, entails, and the resulting correction, provided by D.J. Smeenk, Zaltbommel, the Netherlands.

Let M be the intersection point of AC and BD. Let S be the projection point of D into AB. Then $\triangle ABS \simeq \triangle ABD$ (See figure on page 172).

Let F be the projection of B onto AD, and G the reflection of S onto AD (or its production).

$$\text{Quadrilateral } ABCE \quad \text{is inscribable.} \tag{1}$$

$$\angle BAD \;=\; \alpha \implies \angle DAE \;=\; 90° - \alpha\,. \tag{2}$$

Next, we see what must hold in order that

$$\angle DAE \;=\; \angle CEB \;=\; 90° - \alpha\,. \tag{3}$$

(1), (2) and (3) $\implies \angle CAB = 90° - \alpha$. Consider $\triangle ABC$.

$$\left.\begin{array}{rcl} \angle ABC &=& 180° - \alpha \\ \angle CAB &=& 90° - \alpha \end{array}\right\} \tag{4}$$

(4) $\implies \angle ACB = 2\alpha - 90°$ (so that $\alpha > 45°$). Apply the Sine Law to $\triangle ABC$; $AB = a$, $BC = b$. $a : b = \sin(2\alpha - 90°) : \sin(90° - \alpha)$, or

$$a\cos\alpha + b\cos 2\alpha \;=\; 0 \tag{5}$$

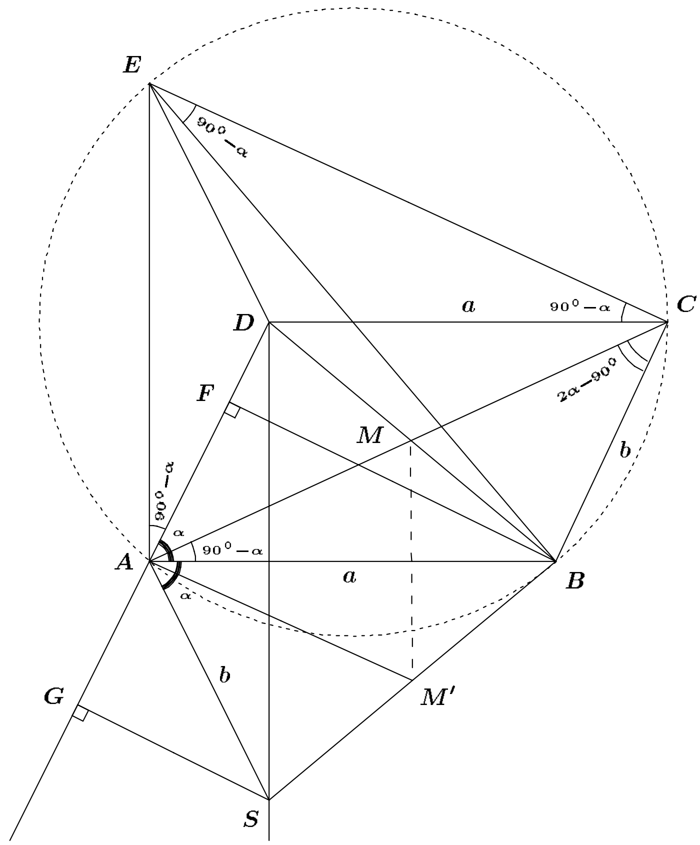
Thus, if $\angle DAE = \angle CEB = 90° - \alpha$, then (5) holds, and the reverse holds as well.

The geometrical meaning of (5) is the following: $a\cos\alpha = \overline{AF}$ signifies the projection onto AD of $\overline{AB}$; $b\cos 2\alpha = \overline{AG}$ signifies the projection onto AD of $\overline{AS}$. F and G lie on different sides of A, or A is the mid-point of segment $\overline{FG}$.

M' is the mid-point of $BS \implies AM' \perp AD \implies AC \perp AS$.

Thus, $AC \perp AS$ is a geometrical translation of (5), and that is the condition to be added to the hypotheses to correct the problem.

A trivial case is: $ABCD$ is a rhombus, and $\angle DAB = 60°$.


E
90° − α
a
90° − α
C
D
2α − 90°
F
M
b
90° − α
α
90° − α
A
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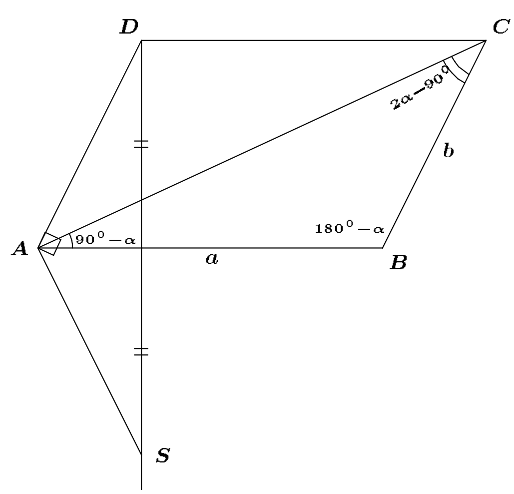
M′
S


D
C
2α − 90°
b
A
90° − α
180° − α
a
B
S

2. Let $\mathcal{P}$ be the set of all polygons in the plane and let $M : \mathcal{P} \to \mathbb{R}$ be a mapping which satisfies:

(i) $M(P) \geq 0$ for each polygon P;

(ii) $M(P) = x^2$ if P is an equilateral triangle of side x;

(iii) If P is a polygon separated into two polygons S and T, then

$$M(P) \;=\; M(S) + M(T)\,; \quad \text{and}$$

(iv) If P and T are congruent polygons, then $M(P) = M(T)$.

Find $M(P)$ if P is a rectangle with edges x and y.

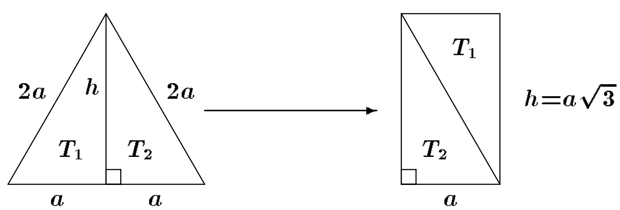
Solution by Pierre Bornsztein, Courdimanche, France.

We will prove that, if P is a rectangle with edges x and y, then

$$M(P) \;=\; \frac{4xy}{\sqrt{3}}\,.$$

Lemma. Let a be a positive real number. Denote by R_a the rectangle with edges a and $a\sqrt{3}$. Then $M(R_a) = 4a^2$.

Proof of the Lemma. Let T be an equilateral triangle with edges $2a$. From (ii), we have $M(T) = 4a^2$. Using a median, we separate T into two congruent right-angled triangles T_1 and T_2.



From (iii) we have

$$M(T) \;=\; M(T_1) + M(T_2)\,.$$

With these two right-angled triangles, we can form a rectangle R_a with edges a and $h = \sqrt{3}a$.

From (iii) and (iv) we have $M(R_a) = M(T_1) + M(T_2) = 4a^2$. ■

Now, let x, y be two positive real numbers. Denote by P a rectangle with edges x and y.

Let $n \in \mathbb{N}^*$ such that

$$0 < \frac{1}{n} < x \quad \text{and} \quad 0 < \frac{1}{n} < \frac{y}{\sqrt{3}}\,. \qquad (1)$$

Let p, q be the largest positive integers such that

$$\frac{p}{n} \le x \quad \text{and} \quad \frac{q}{n} \le \frac{y}{\sqrt{3}}\,. \tag{1'}$$

It follows that

$$\frac{pq}{n^2} \le \frac{xy}{\sqrt{3}} \tag{2}$$

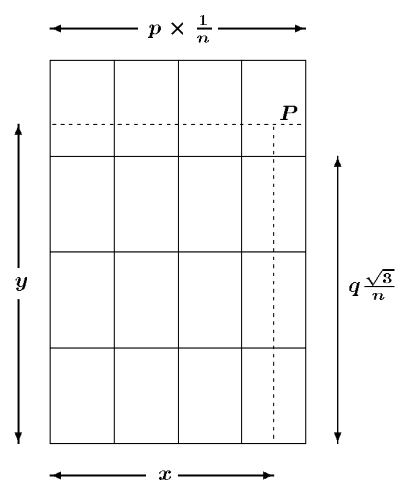
and

$$x < \frac{p+1}{n}\,, \quad \frac{y}{\sqrt{3}} < \frac{q+1}{n}\,. \tag{3}$$

Thus,

$$\left(x - \frac{1}{n}\right)\left(\frac{y}{\sqrt{3}} - \frac{1}{n}\right) < \frac{pq}{n^2}\,. \tag{4}$$

From (1'), note that pq rectangles $R_{1/n}$ can be placed, without overlapping, into P. Note that the part of P which is not covered by these rectangles is a polygon P_1.



From (iii) and (i), we have

$$\begin{aligned} M(P) &= pqM(R_{1/n}) + M(P_1) \\ &\ge pqM(R_{1/n}) \quad \text{from (i)}, \\ &= \frac{4pq}{n^2} \quad \text{from the lemma}, \\ &\ge 4\left(x - \frac{1}{n}\right)\left(\frac{y}{\sqrt{3}} - \frac{1}{n}\right) \quad \text{from (4)}\,. \end{aligned}$$

From (3), P is covered by $(p+1)(q+1)$ rectangles $R_{1/n}$.

As above, we have

$$\begin{aligned} M(P) &\le (p+1)(q+1)M(R_{1/n}) \;=\; \frac{4(p+1)(q+1)}{n^2} \\ &= \frac{4pq}{n^2}+\frac{4p}{n^2}+\frac{4q}{n^2}+\frac{4}{n^2} \\ &\le \frac{4xy}{\sqrt{3}}+\frac{4x}{n}+\frac{4y}{n\sqrt{3}}+\frac{4}{n^2} \end{aligned}$$

(from (1′) and (2)).

Finally, we have

$$4\left(x-\frac{1}{n}\right)\left(\frac{y}{\sqrt{3}}-\frac{1}{n}\right) \;\le\; M(P) \;\le\; \frac{4xy}{\sqrt{3}}+\frac{4x}{n}+\frac{4y}{n\sqrt{3}}+\frac{4}{n^2}\,.$$

As n tends to infinity, we get

$$M(P) \;=\; \frac{4xy}{\sqrt{3}}\,,$$

as claimed.

3. Prove that if α, β and γ are angles of a triangle, then

$$\frac{1}{\sin\alpha}+\frac{1}{\sin\beta} \;\ge\; \frac{8}{3+2\cos\gamma}\,.$$

Solutions by Pierre Bornsztein, Courdimanche, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by Luyun-Zhong-Qiao, Columbia International College, Hamilton, Ontario; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give Klamkin's solution.

Since $\dfrac{1}{\sin x}$ is convex for $0 \le x \le \pi$, we have

$$\frac{1}{\sin\alpha}+\frac{1}{\sin\beta} \;\ge\; \frac{2}{\sin\frac{(\alpha+\beta)}{2}} \;=\; \frac{2}{\cos\frac{\gamma}{2}}\,.$$

The problem will be done once we establish

$$\frac{2}{\cos\frac{\gamma}{2}} \;\ge\; \frac{8}{(3+2\cos\gamma)}\,.$$

Replacing $\cos\gamma$ by $2\cos^2\frac{\gamma}{2}-1$ and cross multiplying, we get

$$\left(2\cos\tfrac{\gamma}{2}-1\right)^2 \;\ge\; 0\,.$$

There is equality if and only if $\gamma=\frac{2\pi}{3}$, $\alpha=\beta=\frac{\pi}{3}$.

4. A polygon is called "good" if the following conditions are satisfied:

(i) all angles belong to $(0, \pi) \cup (\pi, 2\pi)$;

(ii) two non-neighbouring sides do not have any common point; and

(iii) for any three sides, at least two are parallel and equal.

Find all non-negative integers n such that there exists a "good" polygon with n sides.

Solution by Pierre Bornsztein, Courdimanche, France.

We will prove that there exists a "good" polygon with n sides if and only if $n = 4k$ where $k \in \mathbb{N}^*$, $k \neq 2$.

Let n be a non-negative integer such that there exists a "good" polygon $\mathcal{P}_n$ with n sides (such n will be called "good" too). Obviously we have $n \geq 4$.

Denote by M_1, M_2, ..., M_n the vertices of $\mathcal{P}_n$ (subscripts will be read modulo n).

From (i), any two consecutive sides are never equal. And, from (iii), for any three consecutive sides at least two are parallel and equal. It follows that, for each $i \geq 1$, we have

$$\overrightarrow{M_{2i-1}M_{2i}} \;=\; \varepsilon_i \overrightarrow{M_1M_2}$$

and

$$\overrightarrow{M_{2i}M_{2i+1}} \;=\; \varepsilon_i' \overrightarrow{M_2M_3}\,, \quad \text{where} \quad \varepsilon_i, \varepsilon_i' \in \{-1,\, 1\}\,.$$

Consider the coordinate system with origin M_1 and unit vectors $\overrightarrow{M_1M_2}$, and $\overrightarrow{M_2M_3}$.

Then, for each i, M_i belongs to the "integer lattice".

We will say that we have moved "to the right" when $\overrightarrow{M_iM_{i+1}} = \varepsilon \overrightarrow{M_1M_2}$ with $\varepsilon \in \{-1,\, 1\}$. Movements to the left, up, and down are defined in the same way. Then a move to the right (and horizontally) can be described by $(x, y) \mapsto (x+1, y)$ (the others are $(x, y) \mapsto (x-1, y)$ (left), $(x, y) \mapsto (x, y+1)$ (up), $(x, y) \mapsto (x, y-1)$ (down)).

Denote by h, v, r the numbers of moves made horizontally, vertically and to the right, respectively.

From the above, as we alternate vertical and horizontal moves, from M_2 to M_1, we have $h = v$. The total number of moves is $n = h + v = 2h$. By the same reasoning, we have $h = 2r$ because the number of right moves equals the number of left moves. Thus, $n = 4r$.

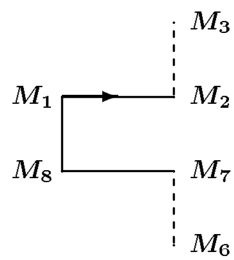
Conversely:

Case 1. when $n = 4$, choose a rectangle;

Case 2. when $n = 8$, suppose for a contradiction that P_8 exists.

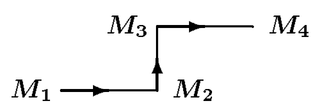
Using reflections, (and maybe renumbering the vertices…) we can suppose that we are in one of the two following cases:

First Case.



Then, the positions of M_3 and M_6 are fixed. Thus, there are at least three vertical moves: $r \geq 3$, then $n \geq 12$, a contradiction.

Second Case. [Ed. The two cases are disjoint, and cover all possibilities.]



We use two right moves. Then we do not move to the right anymore.

If M_5 is "under" M_4, then $M_6 = M_2$, which contradicts (ii).

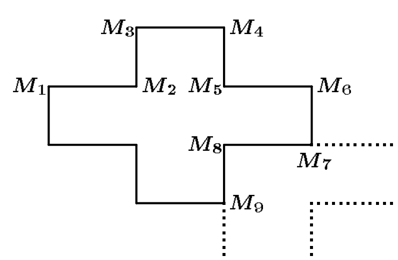
If M_4 is "under" M_5, we have used the two up-moves.

Then M_3 is necessarily "under" M_6.

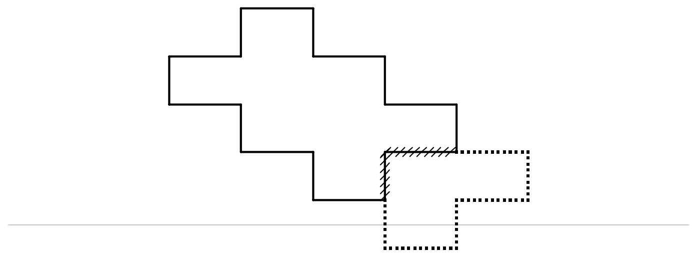
Thus, $M_7 = M_3$, which contradicts (ii).

Then, in each case, we obtain a contradiction. Thus, 8 is not a "good" integer.

If $n = 4k$, $k \geq 3$, starting from $\mathcal{P}_{12}$ (in solid lines),



we add the dotted part, deleting $[M_8M_9]$ and $[M_7M_8]$:



$\mathcal{P}_{4(k+1)}$ is obtained from $\mathcal{P}_{4k}$ by the same construction.

Then, every integer of the form $4k$ with $k \geq 3$ is good.

We are done.

5. Find the biggest number n such that there exist n straight lines in space, $\mathbb{R}^3$, which pass through one point, and the angle between each two lines is the same. (The angle between two intersecting straight lines is defined to be the smaller one of the two angles between these two lines.)

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

More generally, the result is known for $\mathbb{R}^n$. (It is given in one of my *Olympiad Corners* way back).

Let A_n, $n = 0, 1, \ldots, n$ denote unit vectors from the centroid of a regular n-dimensional simplex to the vertices. Then these $n+1$ vectors make equal angles with each other and there cannot be more than $n+1$. Furthermore, the common angle θ between the vectors is obtained from

$$(A_0 + A_1 + \cdots + A_n)^2 \;=\; 0 \;=\; n + 1 + [n(n+1)]\cos\theta\,.$$

Thus, $\cos\theta = -1/n$.

We continue this number of the *Corner* with readers' solutions to problems of the Ninth Irish Mathematical Olympiad [1999 : 199-200].

1. For each positive integer n, let $f(n)$ denote the greatest common divisor of $n!+1$ and $(n+1)!$ (where ! denotes "factorial"). Find, with proof, a formula for $f(n)$ for each n.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by George Evagelopoulos, Athens, Greece; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

We show that

$$f(n) \;=\; \begin{cases} n+1 & \text{if } n+1 \text{ is a prime,} \\ 1 & \text{otherwise .} \end{cases}$$

For convenience of notation, denote $f(n)$ by d. Since $d \mid n! + 1$ and $d \mid (n+1)!$ we have $d \mid (n+1)(n!+1) - (n+1)!$; that is, $d \mid n+1$. If $n+1$ is a prime, then $n+1 \mid n!+1$ by Wilson's Theorem. Since clearly $n+1 \mid (n+1)!$ we have $n+1 \mid d$. Hence, $d = n+1$. If $n+1$ is a composite, then $n+1 = ab$ for some integers a and b such that $1 < a \leq b < n$. If $d = n+1$ then $ab = d$ and so, $a \mid d$. Since $d \mid n!+1$ we have $a \mid n!+1$. On the other hand, since $a < n$ we also have $a \mid n!$. Hence, $a \mid 1$ which implies that $a = 1$, a contradiction. Thus, $d \leq n$. Then, $d \mid n!$ together with $d \mid n!+1$ imply that $d = 1$, and the proof is complete.

2. For each positive integer n, let $S(n)$ denote the sum of the digits of n (when n is written in base 10). Prove that for every positive integer n

$$S(2n) \;\leq\; 2S(n) \;\leq\; 10S(2n)\,.$$

Prove also that there exists a positive integer n with

$$S(n) \;=\; 1996S(3n)\,.$$

Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Courdimanche, France. We give Bornsztein's solution.

Let $n \in \mathbb{N}^*$, $n = d_0 + 10d_1 + \cdots + 10^k d_k$ (decimal expansion). Thus,

$$S(n) = \sum_{i=0}^{n} d_i\,.$$

It is well known that $S(2n) = 2\sum_{i=0}^{k} d_i - 9\lambda_n$ where λ_n denotes the number of d_i such that $d_i \geq 5$. It follows that $S(2n) \leq 2S(n)$.

Moreover,

$$\begin{aligned} S(n) \;&=\; \sum_{d_i \leq 4} d_i + \sum_{d_i \geq 5} d_i \;\geq\; \sum_{d_i \geq 5} d_i \\ &\geq\; 5\sum_{d_i \geq 5} 1 \;=\; 5\lambda_n\,. \end{aligned}$$

Then

$$S(2n) - \lambda_n \;=\; 2S(n) - 10\lambda_n \;\geq\; 0$$

and

$$S(2n) \;\geq\; \lambda_n\,.$$

Thus,

$$2S(n) \;=\; S(2n) + 9\lambda_n \;\leq\; 10S(2n)\,.$$

Finally,

$$S(2n) \;\leq\; 2S(n) \;\leq\; 10S(2n)\,.$$

For $n = \underbrace{3\,3\,\ldots\ldots\ldots\,3\,3}_{5986 \text{ digits ``3''}}6$, we have

$$S(n) \;=\; 3 \times 5986 + 6 \;=\; 17964 \;=\; 9 \times 1996$$

and then,

$$S(n) \;=\; 1996S(3n)\,.$$

3. Let K be the set of all real numbers x with $0 \le x \le 1$. Let f be a function from K to the set of all real numbers $\mathbb{R}$ with the following properties:

(i) $f(1) = 1$.

(ii) $f(x) \ge 0$ for all $x \in K$.

(iii) if x, y and $x + y$ are all in K, then

$$f(x+y) \;\ge\; f(x) + f(y)\,.$$

Prove that $f(x) \le 2x$ for all $x \in K$.

Solutions by Michel Bataille, Rouen, France; and by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea. We give Lee's solution.

We first prove the following lemma.

Lemma. If $0 \le x \le \frac{1}{n}$ for $n \in \mathbb{N}$, then $f(x) \le \frac{1}{n}$.

Proof of Lemma. Let $0 \le x \le \frac{1}{n}$ for $n \in \mathbb{N}$. Then, we have

$$1 \;=\; f(1) \;=\; f(1-nx+nx) \;\ge\; f(1-nx)+f(nx) \;\ge\; f(nx)\,,$$

and we have

$$f(nx) \;=\; f(\overbrace{x+\cdots+x}^{n \text{ times}}) \;\ge\; \overbrace{f(x)+\cdots+f(x)}^{n \text{ times}}$$

from (iii).

Hence, we get

$$1 \;\ge\; f(nx) \;\ge\; nf(x) \quad \text{or} \quad \frac{1}{n} \;\ge\; f(x)\,,$$

as desired.

We shall prove that $f(x) \le 2x$ for $0 < x \le 1$.

Let $0 < x \le 1$. Then, there exists a natural number x such that $\frac{1}{n+1} < x \le \frac{1}{n}$. Then we have $f(x) \le \frac{1}{n}$ from the above lemma.

So, we have $f(x) \le \frac{1}{n} \le \frac{2}{n+1} < 2x$ or $f(x) < 2x$, as desired.

Now, we prove $f(0) \le 0$.

Since $0 \le \frac{1}{n}$ for all $n \in \mathbb{N}$, we have $f(0) \le \frac{1}{n}$ for all $n \in \mathbb{N}$ from the above lemma. This implies that $f(0) \le 0$ since $\lim_{n\to\infty} \frac{1}{n} = 0$.

Comments by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France.

Aassila points out that the problem was proposed at the 8^{th} All-Union Mathematical Olympiad, 1974 held in Erevan. A solution can be found, for example, in N.B. Vassil'ev and A.A. Egorov, *The Problems of the All-Union Mathematical Competitions*, Moscow, Nauka., 1988 (in Russian), ISBN 5-02-013730-8.

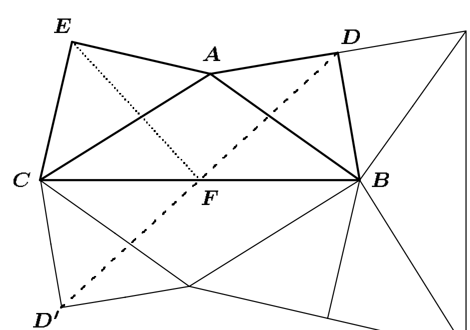
Bornsztein reminds us that this is the same problem as problem No. 5 – Grade X of the Georgian Mathematical Olympiad [1998 : 388]. Moreover, in that problem it was also proved that the number 2 cannot be replaced by any number $k < 2$.

4. Let F be the mid-point of the side BC of the triangle ABC. Isosceles right-angled triangles ABD and ACE are constructed externally on the sides AB and AC with the right angles at D and E, respectively.

Prove that DEF is a right-angled isosceles triangle.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by Luyun-Zhong-Qiao, Columbia International College, Hamilton, Ontario; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give Bataille's solution.

Let R_1 be the rotation with centre D which transforms B into A and R_2 be the rotation with centre E which transforms A into C. Then $S = R_2 \circ R_1$ is a rotation by angle $180°$; that is, a symmetry about a point, and, since $S(B) = C$, this point is F.



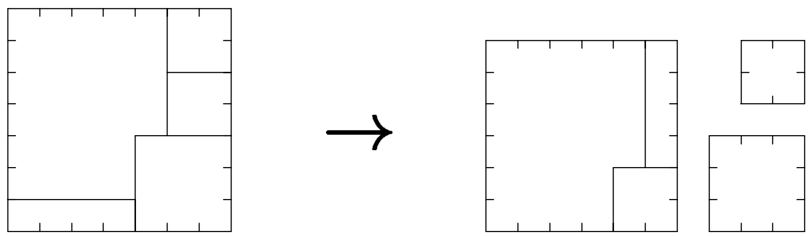
Now, $S(D) = R_2 \circ R_1(D) = R_2(D) = D'$ (say).

From $S(D) = D'$, we see that F is the mid-point of DD', and from $R_2(D) = D'$, we deduce that $\triangle DED'$ is isosceles and right-angled at E.

Therefore, $FD = FD' = FE$ and $EF \perp DD'$ and the result follows.

5. Show, with proof, how to dissect a square into at most five pieces in such a way that the pieces can be reassembled to form three squares no two of which have the same area.

Solution by Mohammed Aassila, Strasbourg, France.
They say a picture is worth a thousand words.



6. The Fibonacci sequence $F_0, F_1, F_2, \ldots$ is defined as follows: $F_0 = 0$, $F_1 = 1$ and for all $n \geq 0$

$$F_{n+2} = F_n + F_{n+1} .$$

(Thus, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $\ldots$) Prove that

(i) The statement "$F_{n+k} - F_n$ is divisible by 10 for all positive integers n" is true if $k = 60$ but it is not true for any positive integer $k < 60$.

(ii) The statement "$F_{n+t} - F_n$ is divisible by 100 for all positive integers n" is true if $t = 300$ but it is not true for any positive integer $t < 300$.

Solutions by Pierre Bornsztein, Courdimanche, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Klamkin's solution.

(i) The smallest m such that $F_m \equiv 0 \pmod 2$ is $m = 3$. Then, $F_{n+3} - F_n = 2F_{n+1} \equiv 0 \pmod 2$.

The smallest k such that $F_{n+k} - F_n$ is divisible by 10 for $n = 0, 1, 2, \ldots, 20$ is $k = 20$. Then $F_{n+20} = F_{n+19} + F_{n+18} = \cdots = F_{20}F_{n+1} + F_{19}F_n$ so that $F_{n+20} - F_n = F_{20}F_{n+1} + (F_{19} - 1)F_n \equiv 0 \pmod 5$ since $F_{20} = 6765$ and $F_{19} - 1 = 4180$.

Hence, the smallest k such that $F_{n+k} - F_n$ is divisible by 10 for all positive integers n is $3 \times 20 = 60$.

(ii) The smallest m such that $F_m \equiv 0 \pmod 4$ is $m = 6$. Then $F_{n+6} - F_n = 8F_{n+1} + 4F_n \equiv 0 \pmod 4$.

The smallest t such that $F_{n+t} - F_n$ is divisible by 25 for $n = 0, 1, \ldots, 100$ is $t = 100$ (by examining a table of the F_n's). Then,

$$F_{n+100} - F_n = F_{100}F_{n+1} + (F_{99} - 1)F_n \equiv 0 \pmod{25}$$

since $F_{100} = 354224848179261915075$ and $F_{99} = 218922995834555169026$.

Finally, the smallest t is the lowest common multiple of 6 and 100, or 300.

7. Prove that the inequality

$$2^{1/2}\cdot 4^{1/4}\cdot 8^{1/8}\cdots(2^n)^{1/2^n} \;<\; 4$$

holds for all positive integers n.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Luyun-Zhong-Qiao, Columbia International College, Hamilton, Ontario; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Luyun's solution.
Proof.

$$\begin{aligned} 2^{\frac12}\cdot 4^{\frac14}\cdot 8^{\frac18}\cdots(2^n)^{\frac{1}{2^n}} &= 2^{\frac12}\cdot 2^{\frac{2}{2^2}}\cdot 2^{\frac{3}{2^3}}\cdots(2)^{\frac{n}{2^n}} \\ &= 2^{\frac12+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{n}{2^n}} \qquad (\text{see } \textbf{Aside}) \\ &< 2^2 \;=\; 4\,, \end{aligned}$$

since $y = 2^x$ is increasing.

We have $2^{\frac12}\cdot 4^{\frac14}\cdot 8^{\frac18}\cdots(2^n)^{\frac{1}{2^n}} < 4$ for all $n \in \mathbb{N}$.

Aside. Now

$$\frac12+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{n}{2^n}\,.$$

is an arithmetic-geometric series. Let

$$\begin{aligned} S &= \frac12+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{n}{2^n} \\ \frac12 S &= \qquad\frac{1}{2^2}+\frac{2}{2^3}+\cdots+\frac{n-1}{2^n}+\frac{n}{2^{n+1}}\,. \end{aligned}$$

Subtracting,

$$\frac12 S \;=\; \frac12+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}-\frac{n}{2^{n+1}}\,.$$

The first part is geometric with $r = \frac12$, $a = \frac12$, so

$$\frac12 S \;=\; \frac{\frac12[1-(\frac12)^n]}{1-\frac12}-\frac{n}{2^{n+1}}\,,$$

or

$$S \;=\; 2-\frac{1}{2^{n-1}}-\frac{n}{2^n}\,,$$

since $\dfrac{1}{2^{n-1}} \;>\; 0$ for any n, and $\dfrac{n}{2^n} \;>\; 0$, we get $S \;<\; 2$.

When $n \to +\infty$,

$$\lim_{n\to\infty}\left(\frac12\right)^n \;=\; 0\,; \qquad \lim_{n\to\infty}\frac{n}{2^n} \;=\; 0\,; \qquad \text{therefore,} \quad S \to 2^-\,.$$

8. Let p be a prime number and a and n positive integers. Prove that if $2^p + 3^p = a^n$, then $n = 1$.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Heinz-Jürgen Seiffert, Berlin, Germany. We give Aassila's solution.

If $p = 2$, we have $2^2 + 3^2 = 13$ and $n = 1$. If $p > 2$, then p is odd, and hence, 5 divides $2^p + 3^p$ and then 5 divides a. Now, if $n > 1$, then 25 divides a^n and 5 divides

$$\frac{2^p + 3^p}{2+3} = 2^{p-1} - 2^{p-2} \cdot 3 + \cdots 3^{p-1} \pmod{5},$$

a contradiction if $p \neq 5$. Finally, if $p = 5$, then $2^5 + 3^5 = 753$ is not a perfect power, so that $n = 1$.

9. Let ABC be an acute-angled triangle and let D, E, F be the feet of the perpendiculars from A, B, C onto the sides BC, CA, AB, respectively. Let P, Q, R be the feet of the perpendiculars from A, B, C onto the lines EF, FD, DE respectively. Prove that the lines AP, BQ, CR (extended) are concurrent.

Solution by Michel Bataille, Rouen, France and a comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

Let H_B and H_C be the points symmetrical to the orthocentre H about the lines AC and AB, respectively. Then E, F are the mid-points of HH_B, HH_C, respectively, so that $H_BH_C \| EF$. Hence, $AP \perp H_BH_C$, and, since $AH_B = AH_C$ $(= AH)$, AP is the perpendicular bisector of the segment H_BH_C. Since, as is well known, H_B and H_C lie on the circumcircle of $\triangle ABC$, we can conclude: AP passes through the circumcentre O of $\triangle ABC$.

Similarly, BQ and CR pass through O. Thus, the lines AP, BQ, CR are concurrent (at O).

Comment. This is a known result due to Steiner, (*Werke*, I, p. 157) and is given as follows: If lines drawn from three points A, B, C respectively, perpendicular to the joins, $B'C'$, $C'A'$, $A'B'$, of three other points, meet in a point, then the lines drawn from A', B', C', respectively perpendicular to BC, CA, AB, also meet in a point.

More generally, if lines drawn from A, B, C, respectively conjugate to $B'C'$, $C'A'$, $A'B'$, in regard to any conic, meet in a point, then the lines drawn from A', B', C', respectively conjugate to BC, CA, AB, in regard to this conic, also meet in a point.

That completes the *Olympiad Corner* for this issue. Now is the time of year to collect Olympiad problem sets and forward them to me. We always appreciate your nice solutions and generalizations.