THE ACADEMY CORNER

No. 40

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We start with some solutions sent in by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario to problems of the Memorial University Undergraduate Mathematics Competition, written in March 2000 [2000: 257].

1. Find all roots of $(b-c)x^2+(c-a)x+a-b=0$ if a, b, c are in arithmetic progression (in the order listed).

Let d denote the common difference of the arithmetic progression, so that $b=a+d$ and $c=b+d=a+2d$.

Then, $a-b=b-c=-d$ and $c-a=2d$. Hence, the given equation becomes $-dx^2+2dx-d=0$.

If $d=0$, then the polynomial is the zero polynomial, and thus any number is a root.

If $d\neq 0$, then, from $x^2-2x+1=0$, we conclude that the only root is $x=1$, with multiplicity two.

2. Evaluate x^3+y^3 where $x+y=1$ and $x^2+y^2=2$.

$$\begin{aligned} x^3+y^3 &= (x+y)(x^2-xy+y^2) \;=\; 2-xy \\ &= 2-\tfrac{1}{2}\left((x+y)^2-(x^2+y^2)\right) \\ &= 2-\tfrac{1}{2}(1-2) \;=\; \tfrac{5}{2}\,. \end{aligned}$$

3. In triangle ABC, we have $\angle ABC=\angle ACB=80^\circ$. P is chosen on line segment AB such that $\angle BPC=30^\circ$. Prove that $AP=BC$.

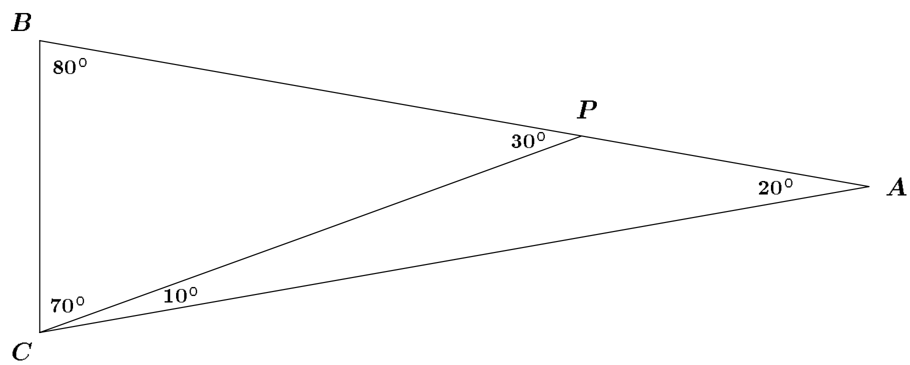
From the given assumptions, we have

$$\angle PAC=20^\circ \qquad \angle APC=150^\circ \quad \text{and} \quad \angle ACP=10^\circ\,.$$

(See figure on page 162.)

By applying the Law of Sines to $\triangle APC$ and $\triangle ABC$, respectively, we have

$$\frac{AP}{\sin 10^\circ}=\frac{AC}{\sin 150^\circ} \quad\text{and}\quad \frac{BC}{\sin 20^\circ}=\frac{AC}{\sin 80^\circ}\,.$$



Hence, $AP = BC$ if and only if $\dfrac{\sin 10^\circ}{\sin 150^\circ} = \dfrac{\sin 20^\circ}{\sin 80^\circ}$, which is true since

$$\frac{\sin 20^\circ}{\sin 80^\circ} = \frac{2\sin 10^\circ \cos 10^\circ}{\sin 80^\circ} = 2\sin 10^\circ = \frac{\sin 10^\circ}{\sin 30^\circ} = \frac{\sin 10^\circ}{\sin 150^\circ}.$$

4. Show that $\binom{2000}{3} = (1)(1998) + (2)(1997) + \ldots + k(1999 - k) + \ldots + (1997)(2) + (1998)(1)$.

This is (virtually) the same problem as **H 241**, which appeared in this journal [1998 : 289], and is the special case, when $n = 1998$, of the more general identity $\sum_{k=1}^{n} k(n-k+1) = \binom{n+2}{3}$. This identity can be verified easily by direct computation. However, there is a short and elegant combinatorial proof which appeared on [1999 : 350].

5. Let a_1, a_2, $\ldots$, a_6 be 6 consecutive integers. Show that the set $\{a_1, a_2, \ldots, a_6\}$ cannot be divided into two disjoint subsets so that the product of the members of one set is equal to the product of the members of the other. (Hint: First consider the case where one of the integers is divisible by 7.)

If any one of the given integers is divisible by 7, then it is the only one with this property, since the six integers are consecutive. Hence, the product of the members of the set containing this integer is divisible by 7, while the product of the members of the other set is not.

Hence, we may assume that $a_i \not\equiv 0 \pmod 7$ for $i = 1, 2, \ldots, 6$.

Then, $a_i \equiv 1, 2, 3, 4, 5, 6$, in some order.

Suppose that $\{a_1, a_2, \ldots, a_6\} = S \cup T$ is a partition such that $\prod_{x\in S} x = \prod_{y\in T} y = c$. Then $c^2 = \prod_{i=1}^{6} a_i \equiv 6! \equiv 6 \pmod 7$.

However, it is easy to check that, for any integer k, we have $k^2 \equiv 0, 1, 2$, or $4 \pmod 7$, a contradiction.

6. Let $f(x) = x(x-1)(x-2)\dots(x-n)$.

   (a) Show that $f'(0) = (-1)^n n!$

   (b) More generally, show that if $0 \le k \le n$, then $f'(k) = (-1)^{n-k} k!(n-k)!$

   It suffices to prove (b). Differentiating, we have

$$f'(x) \;=\; \sum_{j=0}^{n} x(x-1)\cdots\overline{(x-j)}\cdots(x-n)\,,$$

where $\overline{(x-j)}$ indicates that the factor $x-j$ is missing.

For each fixed $j = 0, 1, \dots, n$, consider the corresponding summand $P_j(x) = x(x-1)\cdots\overline{(x-j)}\cdots(x-n)$.

If $j \neq k$, then $x-k$ is a factor of $P_j(x)$, and thus, $P_j(k) = 0$. Hence, $f'(k) = P_k(k)$, where $P_k(x) = x(x-1)\cdots(x-[k-1])(x-[k+1])\cdots(x-n)$, with the conventions that the first factor is $(x-1)$ if $k=0$, and the last factor is $(x-n+1)$ if $k=n$. Therefore,

$$f'(k) \;=\; k(k-1)\cdots 2\cdot 1\cdot(-1)(-2)\cdots(-(k-n)) \;=\; (-1)^{n-k}k!(n-k)!$$

7. For each integer $n \ge 1$, let $\alpha_n = \sum_{j=1}^{n} 10^{-(j!)}$.

   (a) Show that $\lim_{n\to\infty} \alpha_n$ exists. (b) Show that $\lim_{n\to\infty} \alpha_n$ is irrational.

(a) Note first that $\lim_{n\to\infty} \alpha_n = \sum_{j=1}^{\infty} 10^{-(j!)}$.

Since $\lim_{j\to\infty} \dfrac{10^{-((j+1)!)}}{10^{-(j!)}} = \lim_{j\to\infty} 10^{-j(j!)} = 0 < 1$, the series is convergent by the ratio test; that is, $\lim_{n\to\infty} \alpha_n$ exists.

(b) Suppose that

$$\sum_{j=1}^{\infty} 10^{-(j!)} \;=\; \frac{a}{b} \quad \text{for some } a,\, b \in \mathbb{N}. \tag{1}$$

Choose an n sufficiently large so that $10^{n!} - 1 > b$. Multiplying both sides of (1) by $10^{n!}b$, we get

$$b\left(10^{n!-1!} + 10^{n!-2!} + \cdots + 10^{n!-(n-1)!} + 1\right) + b\sum_{j=1}^{\infty} 10^{-((n+j)!-n!)} \;=\; a\,,$$

which implies that $b\sum_{j=1}^{\infty} 10^{-((n+j)!-n!)}$ is an integer.

By induction on n, it can be shown easily that $(n+j)! \geq (j+1)n!$ and, thus, that $(n+j)! - n! \geq jn!$. Hence,

$$\begin{aligned} \sum_{j=1}^{\infty} 10^{-((n+j)!-n!)} &\leq \sum_{j=1}^{\infty} 10^{-j\,n!} = \sum_{j=1}^{\infty} \left(10^{-n!}\right)^j \\ &= \frac{1}{10^{n!}-1} < \frac{1}{b}, \end{aligned}$$

which implies that $b\sum_{j=1}^{\infty} 10^{-((n+j)!-n!)} < 1$, a contradiction.

Also, in this issue, we have some readers' solutions to problems of the Memorial University Undergraduate Mathematics Competition, written in September 2000. [2000 : 449]

1. (a) Prove that the sum of a positive real number and its reciprocal is greater than or equal to two.

   (b) Let a be a positive real number and let x, y and z be real numbers such that $x+y+z=0$.
   Prove that $\log_2(1+a^x)+\log_2(1+a^y)+\log_2(1+a^z) \geq 3$.

Solution by José Díaz Iriberri and Luis Díaz Iriberri, high school students, Barcelona, Spain.

(a) Since $(x-1)^2 \geq 0$, we have $x^2-2x+1 \geq 0$ and so, $x^2+1 \geq 2x$. Since $x > 0$, we can divide both sides of the last inequality by x, to get $x + \frac{1}{x} \geq 2$, and we are done.

(b) The left hand side of the given inequality can be written as

$$\log_2(1+a^x)+\log_2(1+a^y)+\log_2(1+a^z) = \log_2\left((1+a^x)(1+a^y)(1+a^z)\right)$$

$$= \log_2\left(1+a^x+a^y+a^z+a^{x+y}+a^{y+z}+a^{z+x}+a^{x+y+z}\right). \quad (2)$$

From $x+y+z=0$, we have $x+y=-z$, $y+z=-x$ and $z+x=-y$. Hence, the right hand side of (2) can be re-written as

$$\log_2\left(1+\left(a^x+a^{-x}\right)+\left(a^y+a^{-y}\right)+\left(a^z+a^{-z}\right)+1\right).$$

This is greater than or equal to

$$\log_2(1+2+2+2+1) = \log_2(8) = \log_2\left(2^3\right) = 3.$$

Note that, in the preceding inequality, we have taken into account that $\log_2(x)$ is an increasing function for positive values of the variable x and part (a). This completes the proof of part (b).

Also solved by JEAN-CLAUDE ANDRIEUX, Beaune, France; and MARCUS EMMANUEL BARNES, student, York University, Toronto, Ontario.

2. Given any five points inside an equilateral triangle with side length 2, prove that you can always find two points whose distance apart is less than 1 unit.

Solution by Marcus Emmanuel Barnes, student, York University, Toronto, Ontario.

Draw the medial equilateral triangle Δ with side 2. Thus, Δ is composed of four smaller equilateral triangles with side length 1.

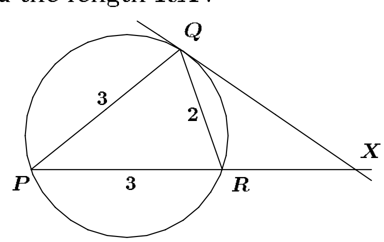
Since we have five points and four equilateral triangles with side length 1 which compose Δ, by the Pigeonhole Principle, at least one of these smaller triangles will contain at least two points.

Note that in an equilateral triangle of side length 1, the points which as farthest apart are the vertices of that equilateral triangle. Thus, they are one unit apart.

But, for all of the four equilateral triangles of side length 1, their vertices lie on the sides of Δ. The given points are required to be contained within Δ.

Hence, the two points that belong to one of the four smaller equilateral triangles cannot be vertices. Thus, their distance apart is less than one unit.

4. Triangle PQR is isosceles, with $PQ = PR = 3$ and $QR = 2$ as shown below. The tangent to the circumcircle at Q meets (the extension of) PR at X as shown. Find the length RX.



Solution by Jean-Claude Andrieux, Beaune, France.

On a $\angle RQX = \angle QPX$, donc $\triangle QXR \sim \triangle PXQ$. On en déduit

$$\frac{XR}{XQ} = \frac{QX}{PX} = \frac{QR}{PQ},$$

soit

$$QX^2 = XR \times XP, \qquad QR \times PX = QX \times PQ$$

De la seconde égalité on tire $QX = \frac{2}{3}PX$.

La première égalité donne alors

$$\tfrac{4}{9}PX^2 = XR \times XP \iff \tfrac{4}{9}PX = XR \iff \tfrac{4}{9}PX = PX - 3\,.$$

Finalement, $PX = \frac{27}{5}$, donc $RX = PX - 3 = \frac{12}{5}$.

Also solved by MARCUS EMMANUEL BARNES, student, York University, Toronto, Ontario.