THE ACADEMY CORNER

No. 40

Bruce Shawyer

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We start with some solutions sent in by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario to problems of the Memorial University Undergraduate Mathematics Competition, written in March 2000 [2000: 257].

1. Find all roots of $(b - c)x^2 + (c - a)x + a - b = 0$ if a, b, c are in arithmetic progression (in the order listed).

Let d denote the common difference of the arithmetic progression, so that $b = a + d$ and $c = b + d = a + 2d$.

Then, $a - b = b - c = -d$ and $c - a = 2d$. Hence, the given equation becomes $-dx^2 + 2dx - d = 0$.

If $d = 0$, then the polynomial is the zero polynomial, and thus any number is a root.

If $d \neq 0$, then, from $x^2 - 2x + 1 = 0$, we conclude that the only root is $x = 1$, with multiplicity two.

2. Evaluate $x^3 + y^3$ where $x + y = 1$ and $x^2 + y^2 = 2$.

$$\begin{aligned} x^3 + y^3 &= (x+y)(x^2 - xy + y^2) \;=\; 2 - xy \\ &= 2 - \tfrac{1}{2}\left((x+y)^2 - (x^2+y^2)\right) \\ &= 2 - \tfrac{1}{2}(1-2) \;=\; \tfrac{5}{2}\,. \end{aligned}$$

3. In triangle ABC, we have $\angle ABC = \angle ACB = 80^\circ$. P is chosen on line segment AB such that $\angle BPC = 30^\circ$. Prove that $AP = BC$.

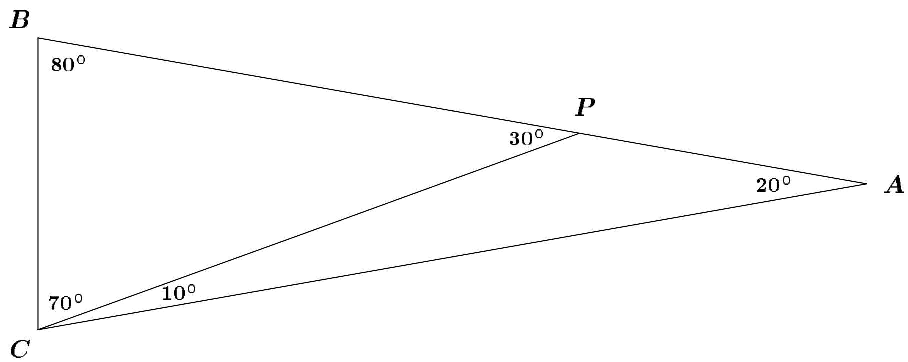
From the given assumptions, we have

$$\angle PAC = 20^\circ \qquad \angle APC = 150^\circ \quad \text{and} \quad \angle ACP = 10^\circ\,.$$

(See figure on page 162.)

By applying the Law of Sines to $\triangle APC$ and $\triangle ABC$, respectively, we have

$$\frac{AP}{\sin 10^\circ} = \frac{AC}{\sin 150^\circ} \quad \text{and} \quad \frac{BC}{\sin 20^\circ} = \frac{AC}{\sin 80^\circ}\,.$$



Hence, $AP = BC$ if and only if $\dfrac{\sin 10^\circ}{\sin 150^\circ} = \dfrac{\sin 20^\circ}{\sin 80^\circ}$, which is true since

$$\frac{\sin 20^\circ}{\sin 80^\circ} = \frac{2\sin 10^\circ \cos 10^\circ}{\sin 80^\circ} = 2\sin 10^\circ = \frac{\sin 10^\circ}{\sin 30^\circ} = \frac{\sin 10^\circ}{\sin 150^\circ}.$$

4. Show that $\binom{2000}{3} = (1)(1998) + (2)(1997) + \ldots + k(1999-k) + \ldots + (1997)(2) + (1998)(1)$.

This is (virtually) the same problem as **H 241**, which appeared in this journal [1998 : 289], and is the special case, when $n = 1998$, of the more general identity $\sum\limits_{k=1}^{n} k(n-k+1) = \dbinom{n+2}{3}$. This identity can be verified easily by direct computation. However, there is a short and elegant combinatorial proof which appeared on [1999 : 350].

5. Let a_1, a_2, $\ldots$, a_6 be 6 consecutive integers. Show that the set $\{a_1, a_2, \ldots, a_6\}$ cannot be divided into two disjoint subsets so that the product of the members of one set is equal to the product of the members of the other. (Hint: First consider the case where one of the integers is divisible by 7.)

If any one of the given integers is divisible by 7, then it is the only one with this property, since the six integers are consecutive. Hence, the product of the members of the set containing this integer is divisible by 7, while the product of the members of the other set is not.

Hence, we may assume that $a_i \not\equiv 0 \pmod 7$ for $i = 1, 2, \ldots, 6$.

Then, $a_i \equiv 1, 2, 3, 4, 5, 6$, in some order.

Suppose that $\{a_1, a_2, \ldots, a_6\} = S \cup T$ is a partition such that $\prod\limits_{x\in S} x = \prod\limits_{y \in T} y = c$. Then $c^2 = \prod\limits_{i=1}^{6} a_i \equiv 6! \equiv 6 \pmod 7$.

However, it is easy to check that, for any integer k, we have $k^2 \equiv 0, 1, 2$, or $4 \pmod 7$, a contradiction.

6. Let $f(x) = x(x-1)(x-2)\dots(x-n)$.

   (a) Show that $f'(0) = (-1)^n n!$

   (b) More generally, show that if $0 \le k \le n$, then $f'(k) = (-1)^{n-k} k!(n-k)!$

   It suffices to prove (b). Differentiating, we have

$$f'(x) \;=\; \sum_{j=0}^{n} x(x-1)\cdots\overline{(x-j)}\cdots(x-n)\,,$$

where $\overline{(x-j)}$ indicates that the factor $x-j$ is missing.

For each fixed $j = 0, 1, \dots, n$, consider the corresponding summand $P_j(x) = x(x-1)\cdots\overline{(x-j)}\cdots(x-n)$.

If $j \neq k$, then $x-k$ is a factor of $P_j(x)$, and thus, $P_j(k) = 0$. Hence, $f'(k) = P_k(k)$, where $P_k(x) = x(x-1)\cdots(x-[k-1])(x-[k+1])\cdots(x-n)$, with the conventions that the first factor is $(x-1)$ if $k=0$, and the last factor is $(x-n+1)$ if $k=n$. Therefore,

$$f'(k) \;=\; k(k-1)\cdots 2\cdot 1\cdot(-1)(-2)\cdots(-(k-n)) \;=\; (-1)^{n-k}k!(n-k)!$$

7. For each integer $n \ge 1$, let $\alpha_n = \sum_{j=1}^{n} 10^{-(j!)}$.

   (a) Show that $\lim_{n\to\infty} \alpha_n$ exists. (b) Show that $\lim_{n\to\infty} \alpha_n$ is irrational.

(a) Note first that $\lim_{n\to\infty} \alpha_n = \sum_{j=1}^{\infty} 10^{-(j!)}$.

Since $\lim_{j\to\infty} \dfrac{10^{-((j+1)!)}}{10^{-(j!)}} \;=\; \lim_{j\to\infty} 10^{-j(j!)} \;=\; 0 \;<\; 1$, the series is convergent by the ratio test; that is, $\lim_{n\to\infty} \alpha_n$ exists.

(b) Suppose that

$$\sum_{j=1}^{\infty} 10^{-(j!)} \;=\; \frac{a}{b} \quad \text{for some } a,\, b \in \mathbb{N}. \tag{1}$$

Choose an n sufficiently large so that $10^{n!} - 1 > b$. Multiplying both sides of (1) by $10^{n!}b$, we get

$$b\left(10^{n!-1!} + 10^{n!-2!} + \cdots + 10^{n!-(n-1)!} + 1\right) + b\sum_{j=1}^{\infty} 10^{-((n+j)!-n!)} \;=\; a\,,$$

which implies that $b\sum_{j=1}^{\infty} 10^{-((n+j)!-n!)}$ is an integer.

By induction on n, it can be shown easily that $(n+j)! \geq (j+1)n!$ and, thus, that $(n+j)! - n! \geq jn!$. Hence,

$$\begin{aligned} \sum_{j=1}^{\infty} 10^{-((n+j)!-n!)} &\leq \sum_{j=1}^{\infty} 10^{-j\,n!} = \sum_{j=1}^{\infty} \left(10^{-n!}\right)^j \\ &= \frac{1}{10^{n!}-1} < \frac{1}{b}, \end{aligned}$$

which implies that $b\sum_{j=1}^{\infty} 10^{-((n+j)!-n!)} < 1$, a contradiction.

Also, in this issue, we have some readers' solutions to problems of the Memorial University Undergraduate Mathematics Competition, written in September 2000. [2000 : 449]

1. (a) Prove that the sum of a positive real number and its reciprocal is greater than or equal to two.
   (b) Let a be a positive real number and let x, y and z be real numbers such that $x+y+z=0$.
   Prove that $\log_2(1+a^x)+\log_2(1+a^y)+\log_2(1+a^z) \geq 3$.

Solution by José Díaz Iriberri and Luis Díaz Iriberri, high school students, Barcelona, Spain.

(a) Since $(x-1)^2 \geq 0$, we have $x^2-2x+1 \geq 0$ and so, $x^2+1 \geq 2x$. Since $x > 0$, we can divide both sides of the last inequality by x, to get $x + \frac{1}{x} \geq 2$, and we are done.

(b) The left hand side of the given inequality can be written as

$$\log_2(1+a^x)+\log_2(1+a^y)+\log_2(1+a^z) = \log_2\left((1+a^x)(1+a^y)(1+a^z)\right)$$

$$= \log_2\left(1+a^x+a^y+a^z+a^{x+y}+a^{y+z}+a^{z+x}+a^{x+y+z}\right). \qquad (2)$$

From $x+y+z=0$, we have $x+y=-z$, $y+z=-x$ and $z+x=-y$. Hence, the right hand side of (2) can be re-written as

$$\log_2\left(1+\left(a^x+a^{-x}\right)+\left(a^y+a^{-y}\right)+\left(a^z+a^{-z}\right)+1\right).$$

This is greater than or equal to

$$\log_2(1+2+2+2+1) = \log_2(8) = \log_2\left(2^3\right) = 3.$$

Note that, in the preceding inequality, we have taken into account that $\log_2(x)$ is an increasing function for positive values of the variable x and part (a). This completes the proof of part (b).

Also solved by JEAN-CLAUDE ANDRIEUX, Beaune, France; and MARCUS EMMANUEL BARNES, student, York University, Toronto, Ontario.

2. Given any five points inside an equilateral triangle with side length 2, prove that you can always find two points whose distance apart is less than 1 unit.

Solution by Marcus Emmanuel Barnes, student, York University, Toronto, Ontario.

Draw the medial equilateral triangle Δ with side 2. Thus, Δ is composed of four smaller equilateral triangles with side length 1.

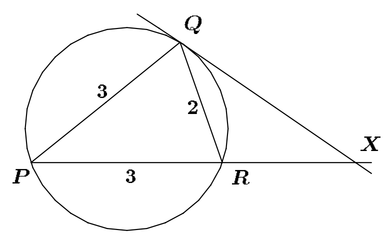
Since we have five points and four equilateral triangles with side length 1 which compose Δ, by the Pigeonhole Principle, at least one of these smaller triangles will contain at least two points.

Note that in an equilateral triangle of side length 1, the points which as farthest apart are the vertices of that equilateral triangle. Thus, they are one unit apart.

But, for all of the four equilateral triangles of side length 1, their vertices lie on the sides of Δ. The given points are required to be contained within Δ.

Hence, the two points that belong to one of the four smaller equilateral triangles cannot be vertices. Thus, their distance apart is less than one unit.

4. Triangle PQR is isosceles, with $PQ = PR = 3$ and $QR = 2$ as shown below. The tangent to the circumcircle at Q meets (the extension of) PR at X as shown. Find the length RX.



Solution by Jean-Claude Andrieux, Beaune, France.

On a $\angle RQX = \angle QPX$, donc $\triangle QXR \sim \triangle PXQ$. On en déduit

$$\frac{XR}{XQ} = \frac{QX}{PX} = \frac{QR}{PQ},$$

soit

$$QX^2 = XR \times XP, \qquad QR \times PX = QX \times PQ$$

De la seconde égalité on tire $QX = \frac{2}{3}PX$.

La première égalité donne alors

$$\tfrac{4}{9}PX^2 = XR \times XP \iff \tfrac{4}{9}PX = XR \iff \tfrac{4}{9}PX = PX - 3\,.$$

Finalement, $PX = \frac{27}{5}$, donc $RX = PX - 3 = \frac{12}{5}$.

Also solved by MARCUS EMMANUEL BARNES, student, York University, Toronto, Ontario.

THE OLYMPIAD CORNER

No. 213

R.E. Woodrow

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We start this number with a set of five Klamkin Quickies. Many thanks to Murray Klamkin for sending them to us.

FIVE KLAMKIN QUICKIES

1. Prove that

$$a+b+c \ \geq \ \sqrt{b^2+c^2-a^2}+\sqrt{c^2+a^2-b^2}+\sqrt{a^2+b^2-c^2}\,,$$

where a, b, c are sides of a non-obtuse triangle.

2. Determine the extreme values of the area of a triangle ABC given the lengths of the two altitudes h_a, h_b and the side $BC = a$.

3. Determine the maximum area of a triangle ABC given the perimeter p and the angle A.

4. Determine the minimum value of

$$\sum\left[\frac{a_2+a_3+a_4+a_5}{a_1}\right]^{1/2}$$

where the sum is cyclic over the positive numbers a_1, a_2, a_3, a_4, a_5.

5. $ABCD$ and $AB'C'D'$ are any two parallelograms in a plane with A opposite to C and C'. Prove that BB', CC' and DD' are possible sides of a triangle.

Next we give the problems of the two days of the Vietnamese Mathematical Olympiad 1997. My thanks go to Richard Nowakowski, Canadian Team Leader at the IMO in Argentina for collecting them.

VIETNAMESE MATHEMATICAL COMPETITION 1997

First Day — March 14, 1997

Time: 3 hours

1. In a plane, let there be given a circle with centre O, with radius R and a point P inside the circle, $OP = d < R$. Among all convex quadrilaterals $ABCD$, inscribed in the circle such that their diagonals AC and BD cut each other orthogonally at P, determine the ones which have the greatest perimeter and the ones which have the least perimeter. Calculate these perimeters in terms of R and d.

2. Let there be given a whole number $n > 1$, not divisible by 1997. Consider two sequences of numbers $\{a_i\}$ and $\{b_j\}$ defined by:

$$a_i \;=\; i + \frac{ni}{1997} \quad (i \;=\; 1, 2, 3, \ldots, 1996)\,,$$

$$b_j \;=\; j + \frac{1997j}{n} \quad (j \;=\; 1, 2, 3, \ldots, n-1)\,.$$

By arranging the numbers of these two sequences in increasing order, we get the sequence $c_1 \le c_2 \le \cdots \le c_{1995+n}$.

Prove that $c_{k+1} - c_k < 2$ for every $k = 1, 2, \ldots, 1994 + n$.

3. How many functions $f : \mathbb{N}^* \to \mathbb{N}^*$ are there that simultaneously satisfy the two following conditions:

(i) $f(1) = 1$,

(ii) $f(n) \cdot f(n+2) \;=\; (f(n+1))^2 + 1997$ for all $n \in \mathbb{N}^*$?

($\mathbb{N}^*$ denotes the set of all positive integers.)

Second Day — March 15, 1997

Time: 3 hours

4. (a) Find all polynomials of least degree, with rational coefficients such that

$$f(\sqrt[3]{3} + \sqrt[3]{9}) \;=\; 3 + \sqrt[3]{3}\,.$$

(b) Does there exist a polynomial with integer coefficients such that

$$f(\sqrt[3]{3} + \sqrt[3]{9}) \;=\; 3 + \sqrt[3]{3}\,?$$

5. Prove that for every positive integer n, there exists a positive integer k such that $19^k - 97$ is divisible by 2^n.

6. Let there be given 75 points, where no three of them are collinear, inside a cube, of which the length of an edge is 1. Prove that there exists a triangle whose vertices are among these 75 points and such that its area does not exceed $\frac{7}{72}$.

The next problem set gives the problems of the Team Selection Examination for Turkey for the 38^{th} IMO. My thanks go to Richard Nowakowski, Canadian Team Leader to the IMO in Argentina for collecting them.

TURKEY TEAM SELECTION EXAMINATION FOR THE 38^{th} IMO

First Day — April 12, 1997

Time: 4.5 hours

1. In a triangle ABC which has a right angle at A, let H denote the foot of the altitude belonging to the hypotenuse. Show that the sum of the radii of the incircles of the triangles ABC, ABH and AHC is equal to $|AH|$.

2. The sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ are defined through $a_1 = \alpha$, $b_1 = \beta$, and $a_{n+1} = \alpha a_n - \beta b_n$, $b_{n+1} = \beta a_n + \alpha b_n$ for all $n \geq 1$. How many pairs (α, β) of real numbers are there such that

$$a_{1997} = b_1 \quad \text{and} \quad b_{1997} = a_1\,?$$

3. In a soccer league, when a player is transferred from a team X with x players to a team Y with y players, the federation is paid $y - x$ billion liras by Y if $y \geq x$, while the federation pays $x - y$ billion liras to X if $x > y$. A player is allowed to change as many teams as he wishes during a season. In a league consisting of 18 teams, each team starts the season with 20 players. At the end of the season, 12 of these turn out again to have 20 players, while the remaining 6 teams end up having 16, 16, 21, 22, 22 and 23 players, respectively. What is the maximal amount the federation may have won during this season?

Second Day — April 13, 1997

Time: 4.5 hours

4. The edge AE of a convex pentagon $ABCDE$ whose vertices lie on the unit circle passes through the centre of this circle. If $|AB| = a$, $|BC| = b$, $|CD| = c$, $|DE| = d$ and $ab = cd = \frac{1}{4}$, compute $|AC| + |CE|$ in terms of a, b, c, d.

5. Prove that, for each prime number $p \geq 7$, there exists a positive integer n and integers $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ which are not divisible by p, such that

$$\begin{aligned} x_1^2 + y_1^2 &\equiv x_2^2 \pmod{p}, \\ x_2^2 + y_2^2 &\equiv x_3^2 \pmod{p}, \\ &\vdots \\ x_{n-1}^2 + y_{n-1}^2 &\equiv x_n^2 \pmod{p}, \\ x_n^2 + y_n^2 &\equiv x_1^2 \pmod{p}. \end{aligned}$$

6. Given an integer $n \geq 2$, find the minimal value of

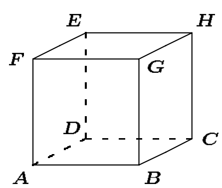
$$\frac{x_1^5}{x_2 + x_3 + \cdots + x_n} + \frac{x_2^5}{x_1 + x_3 + \cdots + x_n} + \cdots + \frac{x_n^5}{x_1 + x_2 + \cdots + x_{n-1}}$$

subject to $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, where $x_1, x_2, \ldots, x_n$ are positive real numbers.

As a final collection of problems this number we give those of the Chilean Mathematical Olympiads 1994–1995. Thanks go to Raul A. Simon Lamb, Santiago, Chile, for forwarding the set to us.

CHILEAN MATHEMATICAL OLYMPIADS 1994–95

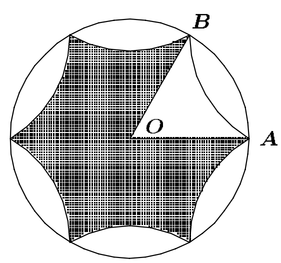
1. Given three straight lines in a plane, that concur at point O, consider the three consecutive angles between them (which, naturally, add up to $180°$). Let P be a point in the plane not on any of these lines and let A, B, C be the feet of the perpendiculars drawn from P to the three lines. Show that the internal angles of $\triangle ABC$ are equal to those between the given lines.

2. 

$ABCDEFGH$ is a cube of edge 2. Let M be the mid-point of $\overline{BC}$ and N the mid-point of $\overline{EF}$. Compute the area of the quadrilateral $AMHN$.

3. Given a trapezoid $ABCD$, where $\overline{AB}$ and $\overline{DC}$ are parallel, and $\overline{AD} = \overline{DC} = \overline{AB}/2$, determine $\angle ACB$.

4. In a circle of radius 1 are drawn six equal arcs of circles, radius 1, cutting the original circle as in the figure. Calculate the shaded area.



5. In right triangle ABC the altitude $h_c = \overline{CD}$ is drawn to the hypotenuse $\overline{AB}$. Let P, P_1, P_2 be the radii of the circles inscribed in the triangles ABC, ADC, BCD respectively. Show that $P + P_1 + P_2 = h_c$.

6. Consider the product of all the positive multiples of 6 that are less than 1000. Find the number of zeros with which this product ends.

7. Let x be an integer of the form

$$x = \underbrace{1\,1\,1\,\ldots\,1}_{n} .$$

Show that, if x is a prime, then n is a prime.

8. Let x be a number such that

$$x + \frac{1}{x} = -1 .$$

Compute

$$x^{1994} + \frac{-1}{x^{1994}} .$$

9. Let $ABCD$ be an $m \times n$ rectangle, with $m, n \in \mathbb{N}$. Consider a ray of light that starts from A, is reflected at an angle of $45°$ on another side of the rectangle, and goes on reflecting in this way.

(a) Show that the ray will finally hit a vertex.

(b) Suppose m and n have no common factor greater than 1. Determine the number of reflections undergone by the ray before it hits a vertex.

10. Let a be a natural number. Show that the equation

$$x^2 - y^2 = a^3$$

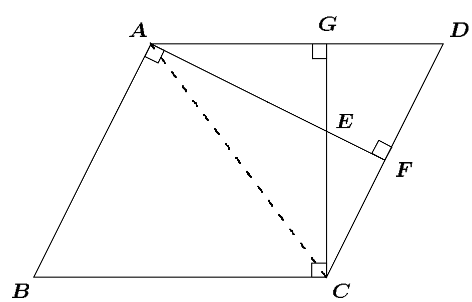
always has integer solutions for x and y.

Next we turn to readers' solutions to the problems of the Third Macedonian Mathematical Olympiad given [1999 : 198].

1. Let $ABCD$ be a parallelogram which is not a rectangle and E be a point in its plane, such that $AE \perp AB$ and $BC \perp EC$. Prove that $\angle DAE = \angle CEB$. [Ed. We know this is incorrect — can any reader supply the correct version?]

Correction and solution by Luyun-Zhong-Qiao, Columbia International College, Hamilton, Ontario. [Ed. The solver considers only the case when E is interior to the parallelogram.]

Try to prove that $\angle DAE = \angle DCE$.



Proof. Extend AE to meet CD at F; CE to meet AD at G. Therefore, $ABCD$ is a parallelogram. Therefore, $AB \parallel DC$, $AD \parallel BC$, and

$$\left.\begin{array}{l} EC \perp BC \implies EC \perp AD \\ AE \perp AB \implies AE \perp CD \end{array}\right\} \implies \left\{\begin{array}{l} CG \perp AD\,, \\ AF \perp CD\,. \end{array}\right.$$

With base AC, and since $\angle AGC = \angle AFC = 90^\circ$, we have that $ACFG$ is a cyclic quadrilateral. Therefore, $\angle DAE = \angle DCE$.

Next we give an analysis of what the question, as presented, entails, and the resulting correction, provided by D.J. Smeenk, Zaltbommel, the Netherlands.

Let M be the intersection point of AC and BD. Let S be the projection point of D into AB. Then $\triangle ABS \simeq \triangle ABD$ (See figure on page 172).

Let F be the projection of B onto AD, and G the reflection of S onto AD (or its production).

$$\text{Quadrilateral } ABCE \quad \text{is inscribable.} \tag{1}$$

$$\angle BAD = \alpha \implies \angle DAE = 90^\circ - \alpha\,. \tag{2}$$

Next, we see what must hold in order that

$$\angle DAE = \angle CEB = 90^\circ - \alpha\,. \tag{3}$$

(1), (2) and (3) $\implies \angle CAB = 90^\circ - \alpha$. Consider $\triangle ABC$.

$$\left.\begin{array}{rcl} \angle ABC &=& 180^\circ - \alpha \\ \angle CAB &=& 90^\circ - \alpha \end{array}\right\} \tag{4}$$

(4) $\implies \angle ACB = 2\alpha - 90^\circ$ (so that $\alpha > 45^\circ$). Apply the Sine Law to $\triangle ABC$; $AB = a$, $BC = b$. $a : b = \sin(2\alpha - 90^\circ) : \sin(90^\circ - \alpha)$, or

$$a\cos\alpha + b\cos 2\alpha = 0 \tag{5}$$

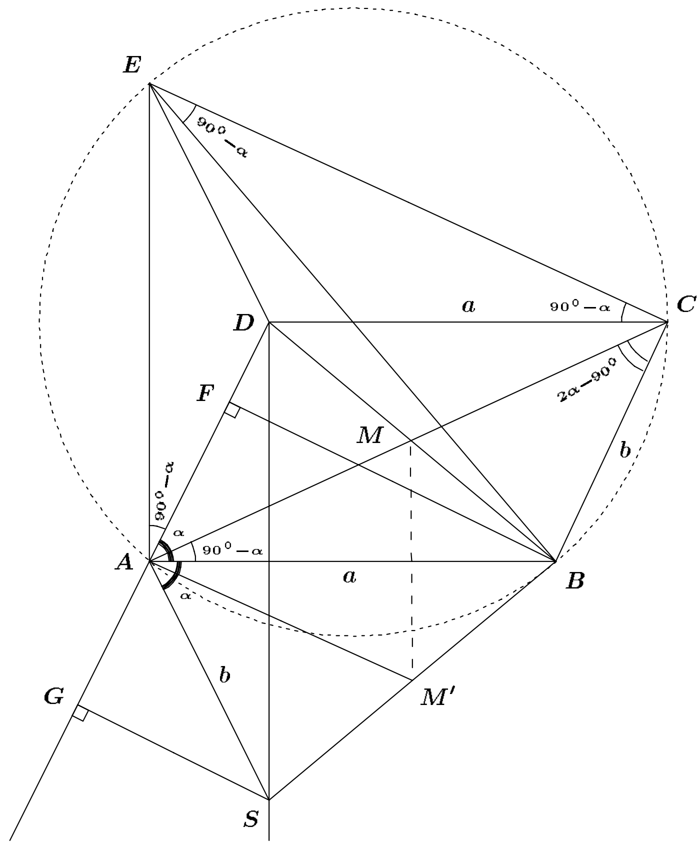
Thus, if $\angle DAE = \angle CEB = 90^\circ - \alpha$, then (5) holds, and the reverse holds as well.

The geometrical meaning of (5) is the following: $a\cos\alpha = \overline{AF}$ signifies the projection onto AD of $\overline{AB}$; $b\cos 2\alpha = \overline{AG}$ signifies the projection onto AD of $\overline{AS}$. F and G lie on different sides of A, or A is the mid-point of segment $\overline{FG}$.

M' is the mid-point of $BS \implies AM' \perp AD \implies AC \perp AS$.

Thus, $AC \perp AS$ is a geometrical translation of (5), and that is the condition to be added to the hypotheses to correct the problem.

A trivial case is: $ABCD$ is a rhombus, and $\angle DAB = 60^\circ$.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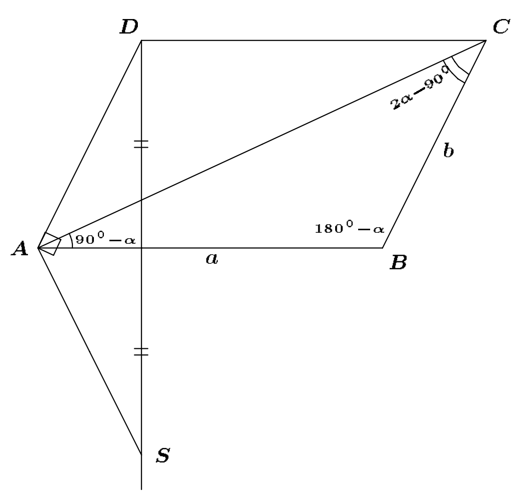
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2. Let $\mathcal{P}$ be the set of all polygons in the plane and let $M : \mathcal{P} \to \mathbb{R}$ be a mapping which satisfies:

(i) $M(P) \geq 0$ for each polygon P;

(ii) $M(P) = x^2$ if P is an equilateral triangle of side x;

(iii) If P is a polygon separated into two polygons S and T, then

$$M(P) \;=\; M(S) + M(T)\,; \quad \text{and}$$

(iv) If P and T are congruent polygons, then $M(P) = M(T)$.

Find $M(P)$ if P is a rectangle with edges x and y.

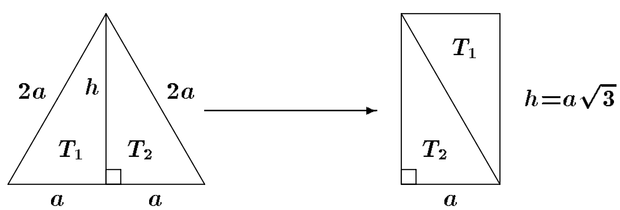
Solution by Pierre Bornsztein, Courdimanche, France.

We will prove that, if P is a rectangle with edges x and y, then

$$M(P) \;=\; \frac{4xy}{\sqrt{3}}\,.$$

Lemma. Let a be a positive real number. Denote by R_a the rectangle with edges a and $a\sqrt{3}$. Then $M(R_a) = 4a^2$.

Proof of the Lemma. Let T be an equilateral triangle with edges $2a$. From (ii), we have $M(T) = 4a^2$. Using a median, we separate T into two congruent right-angled triangles T_1 and T_2.



From (iii) we have

$$M(T) \;=\; M(T_1) + M(T_2)\,.$$

With these two right-angled triangles, we can form a rectangle R_a with edges a and $h = \sqrt{3}a$.

From (iii) and (iv) we have $M(R_a) = M(T_1) + M(T_2) = 4a^2$. ∎

Now, let x, y be two positive real numbers. Denote by P a rectangle with edges x and y.

Let $n \in \mathbb{N}^*$ such that

$$0 < \frac{1}{n} < x \quad \text{and} \quad 0 < \frac{1}{n} < \frac{y}{\sqrt{3}}\,. \tag{1}$$

Let p, q be the largest positive integers such that

$$\frac{p}{n} \leq x \quad \text{and} \quad \frac{q}{n} \leq \frac{y}{\sqrt{3}}\,. \tag{1'}$$

It follows that

$$\frac{pq}{n^2} \leq \frac{xy}{\sqrt{3}} \tag{2}$$

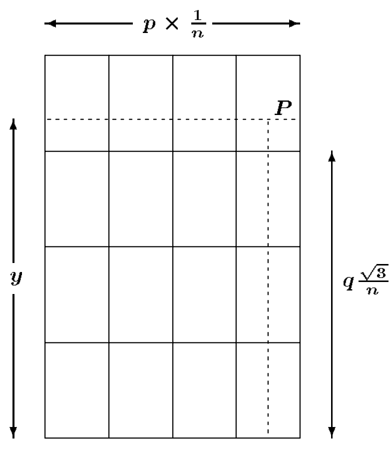
and

$$x < \frac{p+1}{n}\,, \quad \frac{y}{\sqrt{3}} < \frac{q+1}{n}\,. \tag{3}$$

Thus,

$$\left(x - \frac{1}{n}\right)\left(\frac{y}{\sqrt{3}} - \frac{1}{n}\right) < \frac{pq}{n^2}\,. \tag{4}$$

From (1'), note that pq rectangles $R_{1/n}$ can be placed, without overlapping, into P. Note that the part of P which is not covered by these rectangles is a polygon P_1.



From (iii) and (i), we have

$$\begin{aligned} M(P) &= pqM(R_{1/n}) + M(P_1) \\ &\geq pqM(R_{1/n}) \quad \text{from (i)}, \\ &= \frac{4pq}{n^2} \quad \text{from the lemma}, \\ &\geq 4\left(x - \frac{1}{n}\right)\left(\frac{y}{\sqrt{3}} - \frac{1}{n}\right) \quad \text{from (4)}\,. \end{aligned}$$

From (3), P is covered by $(p+1)(q+1)$ rectangles $R_{1/n}$.

As above, we have

$$\begin{aligned} M(P) &\le (p+1)(q+1)M(R_{1/n}) = \frac{4(p+1)(q+1)}{n^2} \\ &= \frac{4pq}{n^2} + \frac{4p}{n^2} + \frac{4q}{n^2} + \frac{4}{n^2} \\ &\le \frac{4xy}{\sqrt{3}} + \frac{4x}{n} + \frac{4y}{n\sqrt{3}} + \frac{4}{n^2} \end{aligned}$$

(from (1′) and (2)).

Finally, we have

$$4\left(x - \frac{1}{n}\right)\left(\frac{y}{\sqrt{3}} - \frac{1}{n}\right) \le M(P) \le \frac{4xy}{\sqrt{3}} + \frac{4x}{n} + \frac{4y}{n\sqrt{3}} + \frac{4}{n^2}.$$

As n tends to infinity, we get

$$M(P) = \frac{4xy}{\sqrt{3}},$$

as claimed.

3. Prove that if α, β and γ are angles of a triangle, then

$$\frac{1}{\sin\alpha} + \frac{1}{\sin\beta} \ge \frac{8}{3 + 2\cos\gamma}.$$

Solutions by Pierre Bornsztein, Courdimanche, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by Luyun-Zhong-Qiao, Columbia International College, Hamilton, Ontario; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give Klamkin's solution.

Since $\frac{1}{\sin x}$ is convex for $0 \le x \le \pi$, we have

$$\frac{1}{\sin\alpha} + \frac{1}{\sin\beta} \ge \frac{2}{\sin\frac{(\alpha+\beta)}{2}} = \frac{2}{\cos\frac{\gamma}{2}}.$$

The problem will be done once we establish

$$\frac{2}{\cos\frac{\gamma}{2}} \ge \frac{8}{(3 + 2\cos\gamma)}.$$

Replacing $\cos\gamma$ by $2\cos^2\frac{\gamma}{2} - 1$ and cross multiplying, we get

$$\left(2\cos\tfrac{\gamma}{2} - 1\right)^2 \ge 0.$$

There is equality if and only if $\gamma = \frac{2\pi}{3}$, $\alpha = \beta = \frac{\pi}{3}$.

4. A polygon is called "good" if the following conditions are satisfied:

(i) all angles belong to $(0, \pi) \cup (\pi, 2\pi)$;

(ii) two non-neighbouring sides do not have any common point; and

(iii) for any three sides, at least two are parallel and equal.

Find all non-negative integers n such that there exists a "good" polygon with n sides.

Solution by Pierre Bornsztein, Courdimanche, France.

We will prove that there exists a "good" polygon with n sides if and only if $n = 4k$ where $k \in \mathbb{N}^*$, $k \neq 2$.

Let n be a non-negative integer such that there exists a "good" polygon $\mathcal{P}_n$ with n sides (such n will be called "good" too). Obviously we have $n \geq 4$.

Denote by M_1, M_2, ..., M_n the vertices of $\mathcal{P}_n$ (subscripts will be read modulo n).

From (i), any two consecutive sides are never equal. And, from (iii), for any three consecutive sides at least two are parallel and equal. It follows that, for each $i \geq 1$, we have

$$\overrightarrow{M_{2i-1}M_{2i}} = \varepsilon_i \overrightarrow{M_1M_2}$$

and

$$\overrightarrow{M_{2i}M_{2i+1}} = \varepsilon_i' \overrightarrow{M_2M_3}, \quad \text{where} \quad \varepsilon_i, \varepsilon_i' \in \{-1, 1\}.$$

Consider the coordinate system with origin M_1 and unit vectors $\overrightarrow{M_1M_2}$, and $\overrightarrow{M_2M_3}$.

Then, for each i, M_i belongs to the "integer lattice".

We will say that we have moved "to the right" when $\overrightarrow{M_iM_{i+1}} = \varepsilon \overrightarrow{M_1M_2}$ with $\varepsilon \in \{-1, 1\}$. Movements to the left, up, and down are defined in the same way. Then a move to the right (and horizontally) can be described by $(x, y) \mapsto (x+1, y)$ (the others are $(x, y) \mapsto (x-1, y)$ (left), $(x, y) \mapsto (x, y+1)$ (up), $(x, y) \mapsto (x, y-1)$ (down)).

Denote by h, v, r the numbers of moves made horizontally, vertically and to the right, respectively.

From the above, as we alternate vertical and horizontal moves, from M_2 to M_1, we have $h = v$. The total number of moves is $n = h + v = 2h$. By the same reasoning, we have $h = 2r$ because the number of right moves equals the number of left moves. Thus, $n = 4r$.

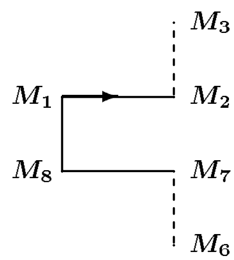
Conversely:

Case 1. when $n = 4$, choose a rectangle;

Case 2. when $n = 8$, suppose for a contradiction that P_8 exists.

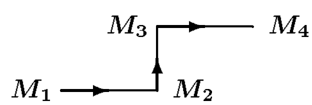
Using reflections, (and maybe renumbering the vertices...) we can suppose that we are in one of the two following cases:

First Case.



Then, the positions of M_3 and M_6 are fixed. Thus, there are at least three vertical moves: $r \geq 3$, then $n \geq 12$, a contradiction.

Second Case. [Ed. The two cases are disjoint, and cover all possibilities.]



We use two right moves. Then we do not move to the right anymore.

If M_5 is "under" M_4, then $M_6 = M_2$, which contradicts (ii).

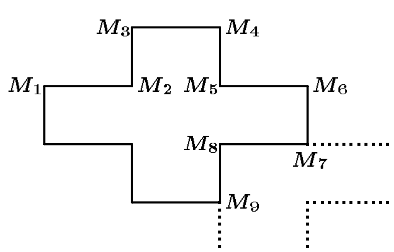
If M_4 is "under" M_5, we have used the two up-moves.

Then M_3 is necessarily "under" M_6.

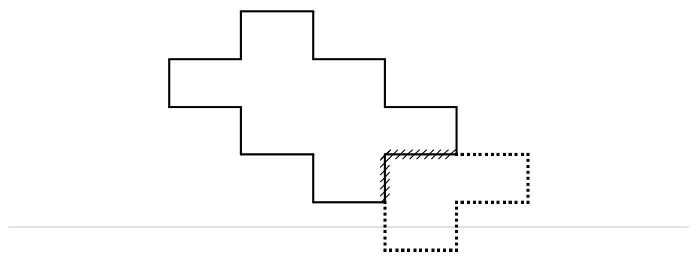
Thus, $M_7 = M_3$, which contradicts (ii).

Then, in each case, we obtain a contradiction. Thus, 8 is not a "good" integer.

If $n = 4k$, $k \geq 3$, starting from $\mathcal{P}_{12}$ (in solid lines),



we add the dotted part, deleting $[M_8M_9]$ and $[M_7M_8]$:



$\mathcal{P}_{4(k+1)}$ is obtained from $\mathcal{P}_{4k}$ by the same construction.

Then, every integer of the form $4k$ with $k \geq 3$ is good.

We are done.

5. Find the biggest number n such that there exist n straight lines in space, $\mathbb{R}^3$, which pass through one point, and the angle between each two lines is the same. (The angle between two intersecting straight lines is defined to be the smaller one of the two angles between these two lines.)

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

More generally, the result is known for $\mathbb{R}^n$. (It is given in one of my *Olympiad Corners* way back).

Let A_n, $n = 0, 1, \ldots, n$ denote unit vectors from the centroid of a regular n-dimensional simplex to the vertices. Then these $n+1$ vectors make equal angles with each other and there cannot be more than $n+1$. Furthermore, the common angle θ between the vectors is obtained from

$$(A_0 + A_1 + \cdots + A_n)^2 \;=\; 0 \;=\; n + 1 + [n(n+1)]\cos\theta\,.$$

Thus, $\cos\theta = -1/n$.

We continue this number of the *Corner* with readers' solutions to problems of the Ninth Irish Mathematical Olympiad [1999 : 199-200].

1. For each positive integer n, let $f(n)$ denote the greatest common divisor of $n! + 1$ and $(n+1)!$ (where ! denotes "factorial"). Find, with proof, a formula for $f(n)$ for each n.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by George Evagelopoulos, Athens, Greece; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

We show that

$$f(n) \;=\; \begin{cases} n+1 & \text{if } n+1 \text{ is a prime,} \\ 1 & \text{otherwise .} \end{cases}$$

For convenience of notation, denote $f(n)$ by d. Since $d \mid n! + 1$ and $d \mid (n+1)!$ we have $d \mid (n+1)(n!+1) - (n+1)!$; that is, $d \mid n+1$. If $n+1$ is a prime, then $n+1 \mid n!+1$ by Wilson's Theorem. Since clearly $n+1 \mid (n+1)!$ we have $n+1 \mid d$. Hence, $d = n+1$. If $n+1$ is a composite, then $n+1 = ab$ for some integers a and b such that $1 < a \leq b < n$. If $d = n+1$ then $ab = d$ and so, $a \mid d$. Since $d \mid n!+1$ we have $a \mid n!+1$. On the other hand, since $a < n$ we also have $a \mid n!$. Hence, $a \mid 1$ which implies that $a = 1$, a contradiction. Thus, $d \leq n$. Then, $d \mid n!$ together with $d \mid n!+1$ imply that $d = 1$, and the proof is complete.

2. For each positive integer n, let $S(n)$ denote the sum of the digits of n (when n is written in base 10). Prove that for every positive integer n

$$S(2n) \leq 2S(n) \leq 10S(2n).$$

Prove also that there exists a positive integer n with

$$S(n) = 1996S(3n).$$

Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Courdimanche, France. We give Bornsztein's solution.

Let $n \in \mathbb{N}^*$, $n = d_0 + 10d_1 + \cdots + 10^k d_k$ (decimal expansion). Thus,

$$S(n) = \sum_{i=0}^{n} d_i .$$

It is well known that $S(2n) = 2\sum_{i=0}^{k} d_i - 9\lambda_n$ where λ_n denotes the number of d_i such that $d_i \geq 5$. It follows that $S(2n) \leq 2S(n)$.

Moreover,

$$\begin{aligned} S(n) &= \sum_{d_i \leq 4} d_i + \sum_{d_i \geq 5} d_i \geq \sum_{d_i \geq 5} d_i \\ &\geq 5 \sum_{d_i \geq 5} 1 = 5\lambda_n . \end{aligned}$$

Then

$$S(2n) - \lambda_n = 2S(n) - 10\lambda_n \geq 0$$

and

$$S(2n) \geq \lambda_n .$$

Thus,

$$2S(n) = S(2n) + 9\lambda_n \leq 10S(2n).$$

Finally,

$$S(2n) \leq 2S(n) \leq 10S(2n).$$

For $n = \underbrace{3\,3\,\ldots\ldots\ldots\,3\,3}_{5986 \text{ digits "3"}}6$, we have

$$S(n) = 3 \times 5986 + 6 = 17964 = 9 \times 1996$$

and then,

$$S(n) \;=\; 1996S(3n)\,.$$

3. Let K be the set of all real numbers x with $0 \le x \le 1$. Let f be a function from K to the set of all real numbers $\mathbb{R}$ with the following properties:

(i) $f(1) = 1$.

(ii) $f(x) \ge 0$ for all $x \in K$.

(iii) if x, y and $x + y$ are all in K, then

$$f(x+y) \;\ge\; f(x) + f(y)\,.$$

Prove that $f(x) \le 2x$ for all $x \in K$.

Solutions by Michel Bataille, Rouen, France; and by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea. We give Lee's solution.

We first prove the following lemma.

Lemma. If $0 \le x \le \frac{1}{n}$ for $n \in \mathbb{N}$, then $f(x) \le \frac{1}{n}$.

Proof of Lemma. Let $0 \le x \le \frac{1}{n}$ for $n \in \mathbb{N}$. Then, we have

$$1 \;=\; f(1) \;=\; f(1 - nx + nx) \;\ge\; f(1-nx) + f(nx) \;\ge\; f(nx)\,,$$

and we have

$$f(nx) \;=\; f(\overbrace{x + \cdots + x}^{n \text{ times}}) \;\ge\; \overbrace{f(x) + \cdots + f(x)}^{n \text{ times}}$$

from (iii).

Hence, we get

$$1 \;\ge\; f(nx) \;\ge\; nf(x) \quad \text{or} \quad \frac{1}{n} \;\ge\; f(x)\,,$$

as desired.

We shall prove that $f(x) \le 2x$ for $0 < x \le 1$.

Let $0 < x \le 1$. Then, there exists a natural number x such that $\frac{1}{n+1} < x \le \frac{1}{n}$. Then we have $f(x) \le \frac{1}{n}$ from the above lemma.

So, we have $f(x) \le \frac{1}{n} \le \frac{2}{n+1} < 2x$ or $f(x) < 2x$, as desired.

Now, we prove $f(0) \le 0$.

Since $0 \le \frac{1}{n}$ for all $n \in \mathbb{N}$, we have $f(0) \le \frac{1}{n}$ for all $n \in \mathbb{N}$ from the above lemma. This implies that $f(0) \le 0$ since $\lim_{n\to\infty} \frac{1}{n} = 0$.

Comments by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France.

Aassila points out that the problem was proposed at the 8^{th} All-Union Mathematical Olympiad, 1974 held in Erevan. A solution can be found, for example, in N.B. Vassil'ev and A.A. Egorov, *The Problems of the All-Union Mathematical Competitions*, Moscow, Nauka., 1988 (in Russian), ISBN 5-02-013730-8.

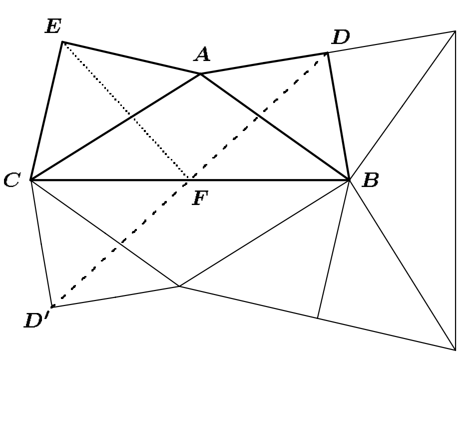
Bornsztein reminds us that this is the same problem as problem No. 5 – Grade X of the Georgian Mathematical Olympiad [1998 : 388]. Moreover, in that problem it was also proved that the number 2 cannot be replaced by any number $k < 2$.

4. Let F be the mid-point of the side BC of the triangle ABC. Isosceles right-angled triangles ABD and ACE are constructed externally on the sides AB and AC with the right angles at D and E, respectively.

Prove that DEF is a right-angled isosceles triangle.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by Luyun-Zhong-Qiao, Columbia International College, Hamilton, Ontario; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give Bataille's solution.

Let R_1 be the rotation with centre D which transforms B into A and R_2 be the rotation with centre E which transforms A into C. Then $S = R_2 \circ R_1$ is a rotation by angle $180°$; that is, a symmetry about a point, and, since $S(B) = C$, this point is F.



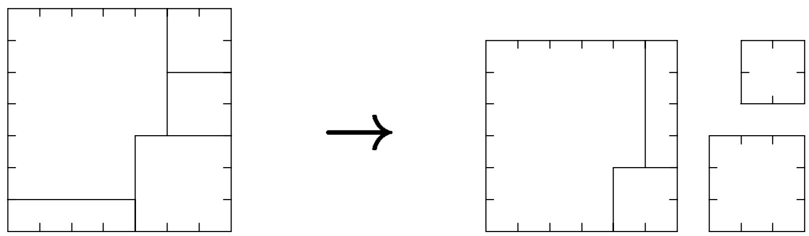
Now, $S(D) = R_2 \circ R_1(D) = R_2(D) = D'$ (say).

From $S(D) = D'$, we see that F is the mid-point of DD', and from $R_2(D) = D'$, we deduce that $\triangle DED'$ is isosceles and right-angled at E.

Therefore, $FD = FD' = FE$ and $EF \perp DD'$ and the result follows.

5. Show, with proof, how to dissect a square into at most five pieces in such a way that the pieces can be reassembled to form three squares no two of which have the same area.

Solution by Mohammed Aassila, Strasbourg, France.
They say a picture is worth a thousand words.



6. The Fibonacci sequence $F_0, F_1, F_2, \ldots$ is defined as follows: $F_0 = 0$, $F_1 = 1$ and for all $n \geq 0$

$$F_{n+2} \;=\; F_n + F_{n+1}\,.$$

(Thus, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8, \ldots$) Prove that

(i) The statement "$F_{n+k} - F_n$ is divisible by 10 for all positive integers n" is true if $k = 60$ but it is not true for any positive integer $k < 60$.

(ii) The statement "$F_{n+t} - F_n$ is divisible by 100 for all positive integers n" is true if $t = 300$ but it is not true for any positive integer $t < 300$.

Solutions by Pierre Bornsztein, Courdimanche, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Klamkin's solution.

(i) The smallest m such that $F_m \equiv 0 \pmod 2$ is $m = 3$. Then, $F_{n+3} - F_n = 2F_{n+1} \equiv 0 \pmod 2$.

The smallest k such that $F_{n+k} - F_n$ is divisible by 10 for $n = 0, 1, 2, \ldots, 20$ is $k = 20$. Then $F_{n+20} = F_{n+19} + F_{n+18} = \cdots = F_{20}F_{n+1} + F_{19}F_n$ so that $F_{n+20} - F_n = F_{20}F_{n+1} + (F_{19} - 1)F_n \equiv 0 \pmod 5$ since $F_{20} = 6765$ and $F_{19} - 1 = 4180$.

Hence, the smallest k such that $F_{n+k} - F_n$ is divisible by 10 for all positive integers n is $3 \times 20 = 60$.

(ii) The smallest m such that $F_m \equiv 0 \pmod 4$ is $m = 6$. Then $F_{n+6} - F_n = 8F_{n+1} + 4F_n \equiv 0 \pmod 4$.

The smallest t such that $F_{n+t} - F_n$ is divisible by 25 for $n = 0, 1, \ldots, 100$ is $t = 100$ (by examining a table of the F_n's). Then,

$$F_{n+100} - F_n \;=\; F_{100}F_{n+1} + (F_{99} - 1)F_n \;\equiv\; 0 \pmod{25}$$

since $F_{100} = 354224848179261915075$ and $F_{99} = 218922995834555169026$.

Finally, the smallest t is the lowest common multiple of 6 and 100, or 300.

7. Prove that the inequality

$$2^{1/2}\cdot 4^{1/4}\cdot 8^{1/8}\cdots(2^n)^{1/2^n} \;<\; 4$$

holds for all positive integers n.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Luyun-Zhong-Qiao, Columbia International College, Hamilton, Ontario; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Luyun's solution.

Proof.

$$\begin{aligned} 2^{\frac{1}{2}}\cdot 4^{\frac{1}{4}}\cdot 8^{\frac{1}{8}}\cdots(2^n)^{\frac{1}{2^n}} &= 2^{\frac{1}{2}}\cdot 2^{\frac{2}{2^2}}\cdot 2^{\frac{3}{2^3}}\cdots(2)^{\frac{n}{2^n}} \\ &= 2^{\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{n}{2^n}} \qquad \text{(see **Aside**)} \\ &< 2^2 \;=\; 4\,, \end{aligned}$$

since $y = 2^x$ is increasing.

We have $2^{\frac{1}{2}}\cdot 4^{\frac{1}{4}}\cdot 8^{\frac{1}{8}}\cdots(2^n)^{\frac{1}{2^n}} < 4$ for all $n \in \mathbb{N}$.

Aside. Now

$$\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{n}{2^n}\,.$$

is an arithmetic-geometric series. Let

$$\begin{aligned} S &= \frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{n}{2^n} \\ \frac{1}{2}S &= \qquad \frac{1}{2^2}+\frac{2}{2^3}+\cdots+\frac{n-1}{2^n}+\frac{n}{2^{n+1}}\,. \end{aligned}$$

Subtracting,

$$\frac{1}{2}S \;=\; \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}-\frac{n}{2^{n+1}}\,.$$

The first part is geometric with $r = \frac{1}{2}$, $a = \frac{1}{2}$, so

$$\frac{1}{2}S \;=\; \frac{\frac{1}{2}[1-(\frac{1}{2})^n]}{1-\frac{1}{2}} - \frac{n}{2^{n+1}}\,,$$

or

$$S \;=\; 2-\frac{1}{2^{n-1}}-\frac{n}{2^n}\,,$$

since $\dfrac{1}{2^{n-1}} \;>\; 0$ for any n, and $\dfrac{n}{2^n} \;>\; 0$, we get $S \;<\; 2$.

When $n \to +\infty$,

$$\lim_{n\to\infty}\left(\frac{1}{2}\right)^n \;=\; 0\,; \qquad \lim_{n\to\infty}\frac{n}{2^n} \;=\; 0\,; \qquad \text{therefore,} \quad S \to 2^-\,.$$

8. Let p be a prime number and a and n positive integers. Prove that if $2^p + 3^p = a^n$, then $n = 1$.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Heinz-Jürgen Seiffert, Berlin, Germany. We give Aassila's solution.

If $p = 2$, we have $2^2 + 3^2 = 13$ and $n = 1$. If $p > 2$, then p is odd, and hence, 5 divides $2^p + 3^p$ and then 5 divides a. Now, if $n > 1$, then 25 divides a^n and 5 divides

$$\frac{2^p + 3^p}{2+3} = 2^{p-1} - 2^{p-2} \cdot 3 + \cdots 3^{p-1} \pmod{5},$$

a contradiction if $p \neq 5$. Finally, if $p = 5$, then $2^5 + 3^5 = 753$ is not a perfect power, so that $n = 1$.

9. Let ABC be an acute-angled triangle and let D, E, F be the feet of the perpendiculars from A, B, C onto the sides BC, CA, AB, respectively. Let P, Q, R be the feet of the perpendiculars from A, B, C onto the lines EF, FD, DE respectively. Prove that the lines AP, BQ, CR (extended) are concurrent.

Solution by Michel Bataille, Rouen, France and a comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

Let H_B and H_C be the points symmetrical to the orthocentre H about the lines AC and AB, respectively. Then E, F are the mid-points of HH_B, HH_C, respectively, so that $H_BH_C \| EF$. Hence, $AP \perp H_BH_C$, and, since $AH_B = AH_C$ $(= AH)$, AP is the perpendicular bisector of the segment H_BH_C. Since, as is well known, H_B and H_C lie on the circumcircle of $\triangle ABC$, we can conclude: AP passes through the circumcentre O of $\triangle ABC$.

Similarly, BQ and CR pass through O. Thus, the lines AP, BQ, CR are concurrent (at O).

Comment. This is a known result due to Steiner, (*Werke*, I, p. 157) and is given as follows: If lines drawn from three points A, B, C respectively, perpendicular to the joins, $B'C'$, $C'A'$, $A'B'$, of three other points, meet in a point, then the lines drawn from A', B', C', respectively perpendicular to BC, CA, AB, also meet in a point.

More generally, if lines drawn from A, B, C, respectively conjugate to $B'C'$, $C'A'$, $A'B'$, in regard to any conic, meet in a point, then the lines drawn from A', B', C', respectively conjugate to BC, CA, AB, in regard to this conic, also meet in a point.

That completes the *Olympiad Corner* for this issue. Now is the time of year to collect Olympiad problem sets and forward them to me. We always appreciate your nice solutions and generalizations.

BOOK REVIEWS

ALAN LAW

Revolutions in Differential Equations:
Exploring ODEs with Modern Technology,
edited by Michael J. Kallaher,
published by the Mathematical Association of America, 1999,
ISBN 0-88385-160-1 softcover, 100 pages, $18.75 (US).
Reviewed by **John Whitehead**, *Memorial University of Newfoundland, St. John's, Newfoundland.*

It is generally acknowledged that numerical techniques should comprise an essential part of the undergraduate education for all students in mathematics and the physical sciences. The importance of the skill sets required to perform scientific computation has grown over the last decade, with the ubiquitous presence of computers in both industry and academia. The raw computer power that a modest desktop, or even laptop, provides would have been hard to credit a generation ago.

Prior to the revolution in microcomputer technology, the vast majority of numerical problems in the mathematical and physical sciences were exclusively the domain of large mainframe computers. To apply the techniques of numerical analysis required familiarity with programming languages such as Fortran or C, combined with a knowledge and understanding of a wide variety of sophisticated numerical techniques. Different indeed from the current situation, where programs such as Mathematica, Maple and Matlab, provide robust root finding algorithms, numerical integration packages, ode and pde solvers, numerical evaluation of special functions and a host of matrix manipulation routines, all tightly integrated with sophisticated graphing and visualisation procedures and, in the case of Mathematica and Maple, an advanced computer algebra system. All of this running on machines that can be purchased for $1,000 and up!

While it is certainly the case that these packages are not without their pitfalls for the unwary, they nevertheless redefine the resources that a typical undergraduate student can bring to bear on problems in applied mathematics and the physical sciences. While this revolution in technology holds out the promise to transform the way mathematics is taught and utilised within the undergraduate science curriculum, there exists a reluctance on the part of many educators to fully exploit the potential that it offers.

The reasons behind this reluctance are many and complex. In part they arise from a well justified uneasiness with providing students, often with limited mathematical skills, a black box to solve complex mathematical problems. This is compounded by the fact there does not exist a clear consensus as to how these techniques are best integrated into the undergraduate curriculum at a fundamental level.

The text "Revolutions in Differential Equations: Exploring ODEs with Modern Technology" is a short collection of papers in the MAA Notes series. The series is designed to address issues in undergraduate mathematics. The text consists of eight articles by a number of authors describing their experience in the development and application of numerical techniques and visualisation in the teaching of ODEs in the undergraduate mathematics curriculum. The authors are all active researchers in the area of ODEs and have had considerable experience in integrating modern analytical, numerical and graphical methods into the teaching of ODEs. The papers are replete with examples as well as informed discussion of the approaches that this technology allows. As such, the text serves as an excellent introduction to instructors wishing to incorporate computer based methods into their teaching.

Four of the eight papers (Borrelli and Coleman, Boyce, Branton and Hale, Manoranjan) describe in considerable detail how ODE solvers and graphical tools can be used to analyse and visualise the solutions of a wide range of very diverse ODEs, in the context of the undergraduate mathematics curriculum. A common element in all these papers is how this technology allows students to study a range of complex topics in ODEs such as bifurcation, stability, oscillations, fixed points, etc., in a relatively informal manner. The approaches described by the different authors encourage, in many instances require, students to explore the solutions to select ODEs for different initial conditions and parameters, and in this way develop insight into the nature of the solutions to ODEs. Much of this material would be inaccessible to undergraduates within the framework of a conventional ODE course, given the limited range of analytic techniques that can realistically be covered. In addition, the techniques described in these articles provide students with a powerful set of tools with which to tackle a wide variety of problems in mathematical modeling and scientific computation.

The article by Lomen examines another aspect in the use of computers in the study of ODEs at the undergraduate level. In this article Lomen describes how computers facilitate the use of data in constructing mathematical models that can be expressed in terms of ODEs. The data serves not only to motivate the model but as a check on the applicability of the model. This approach is echoed in the article by Cooper and LoFaro, who provide an example of how the Web can serve as a useful source of data and problems in mathematical modeling. As with the four papers referred to previously, the examples chosen demonstrate how the use of computer based ODE solvers and visualistion software allow students to explore and analyse "real world" problems mathematically.

A reminder of the many complexities in numerical analysis is provided by the article by Shampine and Gladwell, who discuss general aspects regarding the teaching of numerical methods. The article contains a brief overview of the fundamentals of numerical analysis and touches on some of the difficulties inherent in the various approaches to the numerical solution of ODEs. The article also includes a description of some quality software packages that are available to solve ODEs and how to locate these packages. The authors also touch on the ODE solvers used in Mathematica and Maple and how such programs can utilise the more sophisticated routines included in the IMSL and NAG libraries.

All in all, I found this an extremely interesting and informative book. The articles are, on the whole, well written and provide explicit examples, as well as many useful and practical ideas from several innovative and capable practitioners in this rapidly evolving area of undergraduate education.

My only disappointment with the book is that it does not discuss the effect of this technology on student outcomes. While the article by West, "Technology in Differential Equation Courses: My experiences, student reactions" includes some discussion of the student experience, it is largely anecdotal in nature. While I do not doubt the effectiveness of the approaches described in the various articles, I would like to see some conclusive data that clearly demonstrates the effectiveness of these techniques. Is it a demonstratively better way of teaching ODEs than the traditional analytical approach? Do these methods improve students' attitudes towards mathematics? Does the use of this technology provide students with a useful skill set that they can apply to problems in mathematical modeling and scientific computation? I would like to think that the answer to these questions is a definite yes. Possibly it is.

Who wrote this?

The biologist can push it back to the original protist, and the chemist can push it back to the crystal, but none of them touch the real question of why or how the thing began at all. The astronomer goes back untold million of years and ends in gas and emptiness, and then the mathematician sweeps the whole cosmos into unreality and leaves one with mind as the only thing of which we have any immediate apprehension. *Cogito ergo sum, ergo omnia esse videntur*. All this bother, and we are no further than Descartes. Have you noticed that the astronomers and mathematicians are much the most cheerful people of the lot? I suppose that perpetually contemplating things on so vast a scale makes them feel either that it doesn't matter a hoot anyway, or that anything so large and elaborate must have some sense in it somewhere.

On Improper Integrals

Javad Mashreghi

In this note, we evaluate some improper integrals. For the first proposition, two proofs are given. The first one is a nice application of **integration by parts** and **change of variable** techniques. The second method is more advanced. We make use of the **Mean Value Theorem** and the **Zero Derivative Theorem** and **Chain Rule**. It is shown that $I(a)$ is a differentiable function, and to find its derivative one simply changes the order of integration and differentiation. An application of the **First Fundamental Theorem of Calculus** leads to more interesting results.

Proposition 1 Let a be a positive real number. Then we have that

$$I(a) \;=\; \int_0^\infty \frac{\arctan\left(\frac{x}{a}\right) + \arctan(ax)}{1+x^2}\,dx \;=\; \frac{\pi^2}{4}\,.$$

First proof. Let us consider $\displaystyle\int_0^\infty \frac{\arctan\left(\frac{x}{a}\right)}{1+x^2}\,dx$ and do integration by parts. Take $u = \arctan\left(\frac{x}{a}\right)$ and $dv = \dfrac{dx}{1+x^2}$. Thus, $du = \dfrac{a\,dx}{a^2+x^2}$ and $v = \arctan(x)$. Hence,

$$\begin{aligned}\int_0^\infty \arctan\left(\frac{x}{a}\right)\frac{dx}{1+x^2} \;&=\; \arctan\left(\frac{x}{a}\right)\arctan(x)\Big|_{x=0}^{x\to\infty} \\ &\quad -\int_0^\infty \arctan(x)\,\frac{a\,dx}{a^2+x^2} \\ &=\; \frac{\pi^2}{4} - \int_0^\infty \arctan(x)\,\frac{a\,dx}{a^2+x^2}\,.\end{aligned}$$

In the last integral change x to ax. Since a is positive, the upper and lower limits (0 and ∞) do not change. Hence,

$$\begin{aligned}\int_0^\infty \frac{\arctan\left(\frac{x}{a}\right)}{1+x^2}\,dx \;&=\; \frac{\pi^2}{4} - \int_0^\infty \arctan(ax)\,\frac{a^2\,dx}{a^2+a^2x^2} \\ &=\; \frac{\pi^2}{4} - \int_0^\infty \arctan(ax)\,\frac{dx}{1+x^2}\,.\end{aligned}$$

■

As the first proof shows, $I(a)$ is a numerical constant and does not depend on a. Therefore, we should also be able to do this problem by showing that $I' \equiv 0$.

Second proof. Let $J(a) = \displaystyle\int_0^\infty \frac{\arctan(ax)}{1+x^2}\,dx$, $0 < a < \infty$. Since $I(a) = J(a) + J\left(\frac{1}{a}\right)$, we will show that J is differentiable, find $J'(a)$, and use this to show that $I' \equiv 0$. To find J', we can change the order of integration and differentiation. Let $0 < |\Delta a| < \frac{a}{2}$. Eventually, we let $\Delta a \to 0$. By the Mean Value Theorem, there exists $0 \leqslant \theta \leqslant 1$ such that

$$\arctan\left((a+\Delta a)\,x\right) \;-\; \arctan(a\,x) \;=\; \frac{\Delta a\, x}{1+(a+\theta\,\Delta a)^2\,x^2}\,.$$

Of course, θ depends on a and Δa. Thus,

$$\begin{aligned}
&\left|\frac{J(a+\Delta a)-J(a)}{\Delta a} - \int_0^\infty \frac{x}{(1+a^2\,x^2)\,(1+x^2)}\,dx\right| \\
&= \left|\int_0^\infty \left(\frac{\arctan\left((a+\Delta a)\,x\right) - \arctan(a\,x)}{\Delta a} - \frac{x}{(1+a^2\,x^2)}\right)\frac{dx}{(1+x^2)}\right| \\
&\leqslant \int_0^\infty \left|\frac{x}{1+(a+\theta\,\Delta a)^2\,x^2} - \frac{x}{1+a^2\,x^2}\right|\frac{dx}{1+x^2} \\
&= \int_0^\infty \frac{|2\,a\,\theta\,\Delta a + \theta^2\,(\Delta a)^2|\,x^3}{(1+(a+\theta\,\Delta a)^2\,x^2)\,(1+a^2\,x^2)\,(1+x^2)}\,dx \\
&\leqslant |\Delta a| \int_0^\infty \frac{(2a+\frac{a}{4})\,x^3}{(1+\frac{a^2}{4}\,x^2)\,(1+a^2\,x^2)\,(1+x^2)}\,dx\,.
\end{aligned}$$

The last integral is convergent. Let $\Delta a \to 0$. Hence, for each $a > 0$,

$$J'(a) \;=\; \int_0^\infty \frac{x}{(1+a^2\,x^2)\,(1+x^2)}\,dx\,.$$

We are able to evaluate this integral. For $a \neq 1$

$$\begin{aligned}
J'(a) &= \int_0^\infty \frac{x}{(1+a^2\,x^2)(1+x^2)}\,dx \;=\; \int_0^\infty \left(\frac{\frac{-a^2}{1-a^2}\,x}{1+a^2\,x^2} + \frac{\frac{1}{1-a^2}\,x}{1+x^2}\right)dx \\
&= \frac{1}{2(1-a^2)}\ln\left(\frac{1+x^2}{1+a^2\,x^2}\right)\Bigg|_{x=0}^{x\to\infty} \;=\; \frac{\ln a}{a^2-1}\,.
\end{aligned}$$

It is even simpler to show that $J'(1) = \frac{1}{2} = \lim\limits_{a\to 1}\dfrac{\ln a}{a^2-1}$. Therefore, by the Chain Rule, for each $a > 0$,

$$I'(a) \;=\; J'(a) - \frac{1}{a^2}\,J'\left(\frac{1}{a}\right) \;=\; \frac{\ln a}{a^2-1} - \frac{1}{a^2}\,\frac{\ln\left(\frac{1}{a}\right)}{\frac{1}{a^2}-1} \;=\; 0\,.$$

Thus, by the Zero Derivative Theorem, I is a constant function. For $a = 1$,

$$I(1) = \int_0^\infty \frac{2\arctan(x)}{1+x^2}\,dx = \arctan^2(x)\Big|_{x=0}^{x\to\infty} = \left(\frac{\pi}{2}\right)^2 - 0 = \frac{\pi^2}{4}\,.$$

Therefore, $I \equiv \frac{\pi^2}{4}$.

■

Since we have shown that J is differentiable on $(0,\infty)$, it is also continuous on this interval. One can show that J' is not differentiable at zero, but, nevertheless, a similar reasoning, as in the second proof, shows that J is actually continuous on $(-\infty,\infty)$. Continuity at zero is implicitly used in the following corollary.

Corollary 1

$$\int_0^1 \frac{\ln x}{x^2-1}\,dx = \frac{\pi^2}{8} \quad \text{and} \quad \int_0^\infty \frac{\ln x}{x^2-1}\,dx = \frac{\pi^2}{4}\,.$$

Proof. It is easy to show $J(0) = 0$, $J(1) = \dfrac{\pi^2}{8}$ and $\lim\limits_{a\to\infty} J(a) = \dfrac{\pi^2}{4}$. Thus, by the First Fundamental Theorem of Calculus,

$$\int_0^a \frac{\ln x}{x^2-1}\,dx = \int_0^a J'(x)\,dx = J(a) - J(0) = J(a)\,.$$

Put $a = 1$, and also let $a \to \infty$, to get

$$\int_0^1 \frac{\ln x}{x^2-1}\,dx = \frac{\pi^2}{8} \quad \text{and} \quad \int_0^\infty \frac{\ln x}{x^2-1}\,dx = \frac{\pi^2}{4}\,.$$

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THE SKOLIAD CORNER

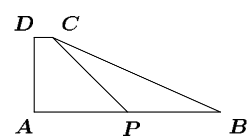
No. 53

R.E. Woodrow

Last number we gave the problems of the Final Round of the Senior Mathematics Contest of the British Columbia Colleges. This issue we give the "official solutions". Our thanks go to Jim Totten, The University College of the Cariboo, and one of the contest organizers, for arranging these for our use.

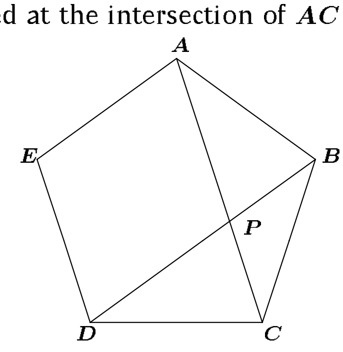
BRITISH COLUMBIA COLLEGES
Senior High School Mathematics Contest
Part A — Final Round — May 5, 2000

1. In the diagram, DC is parallel to AB, and DA is perpendicular to AB. If $DC = 1$, $DA = 4$, $AB = 10$, and the area of quadrilateral $APCD$ equals the area of triangle CPB, then PB equals:



Solution. The answer is **(e)**. The area of the trapezoid $ABCD$ is $\frac{1}{2} \cdot 4 \cdot (1 + 10) = 22$ square units. We want to choose the point P such that the area of $\triangle CPB$ is half this area; that is, 11 square units. Let the base PB have length x. Then the area of $\triangle CPB = \frac{1}{2} \cdot 4 \cdot x = 2x$. Since this must be 11, we have $x = 5\frac{1}{2}$.

2. Label the vertices of a **regular** pentagon with A, B, C, D, and E, so that edges of the pentagon are line segments AB, BC, CD, DE, and EA. One of the angles formed at the intersection of AC and BD has measure:



Solution. The answer is **(a)**. Let P be the intersection of AC and BD as in the diagram above. The sum of the interior angles of a (regular) pentagon is $(5 - 2)180° = 540°$. Thus, each interior angle in a regular pentagon has measure $540°/5 = 108°$. Since $\triangle ABC$ is isosceles with vertex angle equal to $108°$, the base angles, namely $\angle BAC$ and $\angle BCA$, both have measure $36°$. Similarly, $\angle CBD = \angle CDB = 36°$. Since $\angle ABC = 108°$ and $\angle CBD = 36°$, we see that $\angle ABP = 108° - 36° = 72°$. This means that the third angle in triangle ABP, namely $\angle APB$, has measure $180° - 36° - 72° = 72°$. Thus, the angles at P have measure $72°$ and $180° - 72° = 108°$.

3. Three children are all under the age of 15. If I tell you that the product of their ages is 90, you do not have enough information to determine their ages. If I also tell you the sum of their ages, you still do not have enough information to determine their ages. Which of the following is *not* a possible age for one of the children?

Solution. The answer is **(d)**. Let a, b, c be the ages (in integers) of the three children. We may assume that $a \leq b \leq c < 15$. Since the product $abc = 90$, we also know that $a > 0$. The values we seek for a, b, and c are integers which divide evenly into 90 and lie between the values 1 and 14 (inclusive). The only such integers are 1, 2, 3, 5, 6, 9, and 10. We will now look at all possible products which satisfy the above conditions, and for each one we will compute the sum of the ages:

a	b	c	sum
1	9	10	20
2	5	9	16
3	3	10	16
3	5	6	14

We are further told that even knowing the sum of the ages would NOT allow us to determine the three ages. Thus, we are forced to conclude that the sum must be 16, since there are two distinct sets of ages which sum to 16 in the above table. The only ages found in these two sets are 2, 3, 5, 9, and 10. We notice that 1 and 6 do not appear.

4. I know I can fill my bathtub in 10 minutes if I put the hot water tap on full, and that it takes 8 minutes if I put the cold water on full. I was in a hurry so I put both on full. Unfortunately, I forgot to put in the plug. A full tub empties in 5 minutes. How long, in minutes, will it take for the tub to fill?

Solution. The answer is **(c)**. Let V be the volume of a full tub (in litres, say). Then the rate at which the hot water can fill the tub is $V/10$ litres per minute. Similarly the rate at which the cold water can fill the tub is $V/8$ litres per minute. On the other hand a full tub empties at the rate of $V/5$ litres per minute. If all three are happening at the same time, then the rate at which the tub fills is:

$$\frac{V}{10} + \frac{V}{8} - \frac{V}{5} = \frac{4V + 5V - 8V}{40} = \frac{V}{40}$$

litres per minute, which means it takes 40 minutes to fill the tub.

5. The smallest positive integer k such that

$$(k+1)+(k+2)+\cdots+(k+19)$$

is a perfect square is:

Solution. The answer is **(b)**. We first need to recall that the sum of the first n integers is given by $n(n+1)/2$. The sum we are presented with is the difference between the sums of the first $k+19$ integers and the first k integers. Using the above formula we have:

$$\begin{aligned}(k+1)+(k+2)+\cdots(k+19) &= \frac{(k+19)(k+20)}{2}-\frac{k(k+1)}{2}\\ &= \frac{k^2+39k+380-k^2-k}{2}\\ &= \frac{38k+380}{2} = 19(k+10)\,.\end{aligned}$$

Since 19 is prime, in order for $19(k+10)$ to be a perfect square, $k+10$ must contain 19 as a factor. The smallest such value occurs when $k+10=19$; that is, when $k=9$, and we indeed get a perfect square in this case, namely 19^2.

6. A six-digit number begins with 1. If this digit is moved from the extreme left to the extreme right without changing the order of the other digits, the new number is three times the original. The sum of the digits in either number is:

Solution. The answer is **(d)**. Let n be the number in question. Then n can be written as 10^5+a where a is a number with at most 5 digits. Moving the left-most digit (the digit 1) to the extreme right produces a number $10a+1$. The information in the problem now tells us that $10a+1 = 3(10^5+a) = 300000+3a$, or $7a = 299999$. This yields $a = 42857$. Thus, $n = 142857$ (and the other number we created is 428571), the sum of whose digits is $1+4+2+8+5+7=27$.

7. A cube of edge 5 cm is cut into smaller cubes, not all the same size, in such a way that the smallest possible number of cubes is formed. If the edge of each of the smaller cubes is a whole number of centimetres, how many cubes with edge 2 cm are formed?

Solution. The answer is **(d)**. Let us organize this solution by considering the size of the largest cube in the subdivision of the original cube. The largest could have a side of size 4 cm, 3 cm, 2 cm, or 1 cm. In each of these 4 cases we will determine the minimum number of cubes possible. In the first case, when there is a cube of side 4 cm present, we can only include cubes of side 1 cm to complete the subdivision, and we would need 61 of them, since the cube of side 4 cm uses up 64 cm^3 of the 125 cm^3 in the original cube. Thus, in this case we have 62 cubes in the subdivision. If we look at the other extreme case, namely when the largest cube in the subdivision has side length of 1 cm, we clearly need 125 cubes for the subdivision. We also note here that we will certainly decrease the number of cubes in a subdivision if we try to replace sets of 1×1 cubes by larger cubes whenever possible. Now consider the case when there is a cube of side length 3 cm present. If we place it anywhere but in a corner, the subdivision can only be completed by cubes of side length 1 cm, which gives us $1 + 98 = 99$ cubes in total. If we place it in a corner, we can then place 4 cubes of side length 2 cm on one side of the larger cube, 2 more such cubes on a second side and a third such cube on the third side; this gives us 1 large cube and 7 medium cubes for a total volume of $27 + 7(8) = 83$ cm^3, which means we still have 42 small cubes, for a grand total of 50 cubes. It is easy to see that if the largest cube has side length 2 cm we can place at most 8 of them in the original cube and the remainder of the volume must be made up of cubes of side length 1 cm; this gives a total of $8 + 61 = 69$ cubes. Thus, the smallest number of cubes possible is 50 and in this case there are 7 cubes of side length 2 cm.

Note that the restriction in the problem statement "not all the same size" can be dropped without changing the solution, since the above solution completely ignored that restriction.

8. The nine councillors on the student council are not all on speaking terms. The table below shows the current relationship between each pair of councillors, where '1' means 'is on speaking terms', '0' means 'is not on speaking terms', and the letters stand for the councillors' names.

	A	B	C	D	E	F	G	H	I
A	–	0	0	1	0	0	1	0	0
B	0	–	1	1	1	1	1	1	1
C	0	1	–	0	0	0	1	1	0
D	1	1	0	–	1	0	1	0	1
E	0	1	0	1	–	0	1	0	0
F	0	1	0	0	0	–	0	0	1
G	1	1	1	1	1	0	–	0	0
H	0	1	1	0	0	0	0	–	0
I	0	1	0	1	0	1	0	0	–

Councillor A recently started a rumour. It was heard by each councillor once and only once. Each councillor heard it from and passed it to a councillor with whom he or she was on speaking terms. Counting councillor A as zero, councillor E was the eighth and last to hear it. Who was the fourth councillor to hear the rumour?

Solution. The answer is **(a)**. We need to find a sequence of all nine councillors beginning with A and ending with E such that each pair of consecutive councillors are 'on speaking terms' with each other. When one first looks at the table provided, it looks a little daunting. However, a first observation is that among the councillors other than A and E (who need to appear at the ends of the sequence) councillors F and H are only 'on speaking terms' with two others, one of which is councillor B. Thus, councillor F must receive the rumour from B and pass it to I, or vice versa. Similarly, councillor H must hear the rumour from B and pass it to C, or vice versa. Thus, we must have either I–F–B–H–C or C–H–B–F–I as consecutive councillors in the sequence. Since A and E lie on the ends, and neither of them are 'on speaking terms with' either councillors C or I, we see that councillors D and G must be placed one on either end of the above subsequence of five councillors. This leaves us with either D–I–F–B–H–C–G or G–C–H–B–F–I–D. Councillors A and E can be placed on the front and rear of either of these sequences to gives the final sequence as either A–D–I–F–B–H–C–G–E or A–G–C–H–B–F–I–D–E. In either case the fourth person after councillor A (who started the rumour) to hear the rumour was councillor B.

9. A, B, and C are thermometers with different scales. When A reads 10° and 34°, B reads 15° and 31°, respectively. When B reads 30° and 42°, C reads 5° and 77°, respectively. If the temperature drops 18° using A's scale, how many degrees does it drop using C's scale?

Solution. The answer is **(e)**. Let us examine the temperature differences on the respective pairs of thermometers. A difference of 24° on A corresponds to a difference of 16° on B; thus, they are in the ratio of $3 : 2$. A difference of 12° on B corresponds to a difference of 72° on C; thus, they are in the ratio of $1 : 6$. Now a temperature drop of 18° on A means a drop of 12° on B (using the ratio $3 : 2$). This results in a temperature drop of 72° on C (using the ratio $1 : 6$).

10. The number of positive integers between 200 and 2000 that are multiples of 6 or 7 but not both is:

Solution. The answer is **(b)**. The number of positive integers less than or equal to n which are multiples of k is the integer part of n/k (that is, perform the division and discard the decimal fraction, if any). This integer is commonly denoted $\lfloor n/k \rfloor$. Thus, the number of positive integers between 200 and 2000 which are multiples of 6 is

$$\left\lfloor \frac{2000}{6} \right\rfloor - \left\lfloor \frac{200}{6} \right\rfloor = 333 - 33 = 300 .$$

Similarly, the number of positive integers between 200 and 2000 which are multiples of 7 is

$$\left\lfloor \frac{2000}{7} \right\rfloor - \left\lfloor \frac{200}{7} \right\rfloor = 285 - 28 = 257 .$$

In order to count the number of positive integers between 200 and 2000 which are multiples of 6 or 7 we could add the above numbers. This, however, would count the multiples of both 6 and 7 twice; that is, the multiples of 42 would be counted twice. Thus, we need to subtract from this sum the number of positive integers between 200 and 2000 which are multiples of 42. That number is

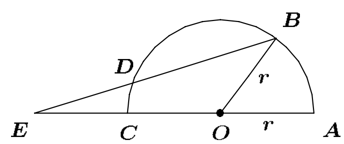
$$\left\lfloor \frac{2000}{42} \right\rfloor - \left\lfloor \frac{200}{42} \right\rfloor = 47 - 4 = 43 .$$

Therefore, the number of positive integers between 200 and 2000 which are multiples of 6 or 7 is $300 + 257 - 43 = 514$. But we are asked for the number of positive integers which are multiples of 6 or 7, but NOT BOTH. Thus, we need to again subtract the number of multiples of 42 in this range, namely 43. The final answer is $514 - 43 = 471$.

Part B — Final Round — May 5, 2000

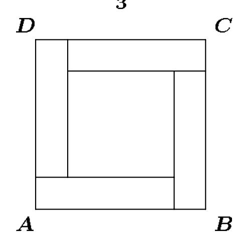
1. In the diagram O is the centre of a circle with radius r, and $ED = r$.

The angle $\angle DEC = k\angle BOA$. Find k.



Solution. First draw the radius OD. Let $\angle AEB = \alpha$. Since $DE = r = OD$, $\triangle DOE$ is isosceles. Therefore, $\angle DOE = \alpha$. Since $\angle BDO$ is an exterior angle to $\triangle DOE$, it is equal in measure to the sum of the opposite interior angles of the triangle; that is, $\angle BDO = 2\alpha$. Now, $\triangle BOD$ is isosceles, since two of its sides are radii of the circle. Thus, $\angle DBO = \angle BDO = 2\alpha$. Since $\angle BOA$ is an exterior angle to $\triangle BOE$, it is equal in measure to the sum of $\angle AEB = \alpha$ and $\angle EBO = 2\alpha$. Thus, $\angle BOA = 3\alpha$, which means that the value k in the problem is $\frac{1}{3}$.

2. The square $ABCD$, whose area is 180 square units, is divided into five rectangular regions of equal area, four of which are congruent as shown. What are the dimensions of one of the rectangular regions which is not a square?



Solution. Since there are five regions of equal area which sum to 180 square units, each region has area 36 square units. The dimensions of the inner square are clearly 6 units on a side, and of the outer square are $\sqrt{180} = 6\sqrt{5}$ units on a side. Let x and y be the dimensions of one of the four congruent regions, where $x < y$. Then $x + y = 6\sqrt{5}$ and $y - x = 6$. On adding these and dividing by 2 we get $y = 3(\sqrt{5} + 1)$, and then it easily follows that $x = 3(\sqrt{5} - 1)$.

3. An integer i *evenly divides* an integer j if there exists an integer k such that $j = ik$; that is, if j is an integer multiple of i.

(a) Recall $n! = (n)(n-1)(n-2)\cdots(2)(1)$. Find the largest value of n such that 25 evenly divides $n! + 1$.

(b) Show that if 3 evenly divides $x + 2y$, then 3 evenly divides $y + 2x$.

Solution. (a) Note that $5! = 5 \cdot 4 \cdot 3 \cdot 2 = 120$, which ends in the digit 0. Thus, $n!$, where $n > 5$, must also end in the digit 0, since $5!$ evenly divides $n!$, for $n > 5$. Thus, $n! + 1$ ends in the digit 1 whenever $n \geq 5$, which means that 25 can never evenly divide $n! + 1$ when $n \geq 5$. We are left to examine the cases $n = 4$, 3, 2, and 1. Since $4! + 1 = 24 + 1 = 25$, we see that 25 divides evenly $4! + 1$. Thus, $n = 4$ is the largest value of n such that 25 evenly divides $n! + 1$.

(b) Note first that $(x + 2y) + (y + 2x) = 3x + 3y = 3(x + y)$. Thus, three evenly divides the sum of the two numbers. This can be rewritten as $y + 2x = 3(x + y) - (x + 2y)$. Suppose that 3 evenly divides $x + 2y$. This means that $x + 2y = 3k$ for some integer k. Thus, $y + 2x = 3(x + y) - 3k = 3(x + y - k)$, which means that 3 evenly divides $y + 2x$.

4. A circular coin is placed on a table. Then identical coins are placed around it so that each coin touches the first coin and its other two neighbours. It is known that exactly six coins can be so placed.

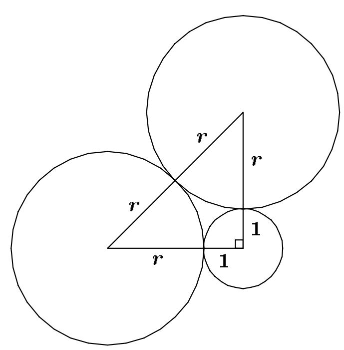
(a) If the radius of all seven coins is 1, find the total area of the spaces between the inner coin and the six outer coins.

(b) If the inner coin has radius 1, find the radius of a larger coin, so that exactly four such larger coins fit around the outside of the coin of radius 1.

Solution. (a) Same as Problem #5(b) on the Junior Paper (Part B) given on [2001: 33].

(b) Let r be the radius of each of the four larger coins which surround the coin of radius 1. Then by considering two such neighbouring coins and the coin of radius 1 (as in the diagram below) we have a right-angled triangle when we connect the three centres. The Theorem of Pythagoras then implies that

$$\begin{aligned} (2r)^2 &= (r+1)^2 + (r+1)^2\,, \\ 4r^2 &= 2r^2 + 4r + 2\,, \\ 2r^2 - 4r - 2 &= 0\,, \\ r^2 - 2r - 1 &= 0\,. \end{aligned}$$

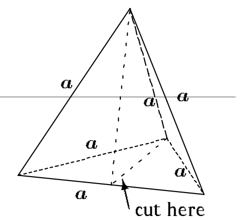


This quadratic has solutions

$$r = \frac{2 \pm \sqrt{4 - 4(1)(-1)}}{2} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}\,.$$

When we reject the negative root, we are left with $r = 1 + \sqrt{2}$.

5. The 6 edges of a regular tetrahedron are of length a. The tetrahedron is sliced along one of its edges to form two identical solids.



(a) Find the perimeter of the slice.

(b) Find the area of the slice.

Solution. (a) The slice is in the shape of an isosceles triangle with two sides equal to the altitude of the equilateral triangular faces of side a and the third side of length a. By using the Theorem of Pythagoras on half of one equilateral triangular face we see that its altitude is given by $a\sqrt{3}/2$. Thus, the perimeter of the triangular slice is $a + 2(a\sqrt{3}/2) = a(1+\sqrt{3})$.

(b) We must now find the area of the triangular slice whose sides we found in part (a) above. Consider the altitude h which splits the isosceles slice into two congruent halves. Each half is a right-angled triangle with hypotenuse $a\sqrt{3}/2$ and one side of length $a/2$. The third side is h, which can be found by the Theorem of Pythagoras:

$$\begin{aligned} h^2 &= \left(\frac{a\sqrt{3}}{2}\right)^2 - \left(\frac{a}{2}\right)^2 = \frac{3a^2}{4} - \frac{a^2}{4} = \frac{a^2}{2} \\ h &= \frac{a}{\sqrt{2}} = \frac{a\sqrt{2}}{2}\,. \end{aligned}$$

Thus, the area of the slice is

$$\frac{1}{2}\cdot a \cdot \frac{a\sqrt{2}}{2} = \frac{a^2\sqrt{2}}{4}\,.$$

That completes the *Skoliad Corner* for this issue.

Who wrote this?

The mathematician is fascinated with the marvellous beauty of the forms he constructs, and in their beauty he finds everlasting truth.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 975 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z5 (NEW!)**. The electronic address is

NEW! mayhem-editors@cms.math.ca NEW!

The Assistant Mayhem Editor is Chris Cappadocia (University of Waterloo). The rest of the staff consists of Adrian Chan (Harvard University), Jimmy Chui (University of Toronto), Donny Cheung (University of Waterloo), and David Savitt (Harvard University).

Problem of the Month

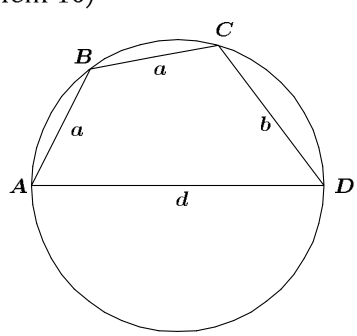
Jimmy Chui, student, University of Toronto

Problem. $ABCD$ is a cyclic quadrilateral, as shown, with side $AD = d$, where d is the diameter of the circle. $AB = a$, $BC = a$, and $CD = b$. If a, b, and d are integers, $a \neq b$,

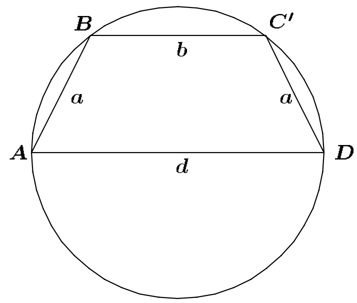
(a) prove that d cannot be a prime number.

(b) determine the *minimum* value of d.

(1999 Euclid, Problem 10)



Solution. Consider the cyclic quadrilateral $ABC'D$, where C' is the point in the interior of minor arc BD, such that $BC' = b$ and $C'D = a$. We are guaranteed a point C' exists since we are just reflecting C in the perpendicular bisector of BD.



Note that the cyclic quadrilateral $ABC'D$ is a trapezoid. Then, let $c = BD = AC'$.

By Ptolemy's Theorem, $c^2 = a^2 + bd$. (Note: This relationship can also be obtained without the construction of point C'. Take the cosine of $\angle BAD$ and the cosine of $\angle BCD$ using the Cosine Law. These cosines are negatives of each other, since the angles add up to $180°$. Adding the cosines and simplifying yields the relationship.)

Also, since AD is the diameter of a circle, we also have $\angle ABD = 90°$. Then $a^2 + c^2 = d^2$.

We know that a, b, and d are integers, therefore, let us eliminate c. We get $2a^2 = d^2 - bd = d(d - b)$.

If d is 2, then we get $a^2 = 2 - b$, and the only positive integer solution is $a = b = 1$. This contradicts $a \neq b$, so that d cannot be 2.

If d is an odd prime, then $d|a$. But that means the left side is divisible by d^2, so that the right side must be divisible by d^2. This leads to $d|(d - b)$, which is clearly impossible since b is positive.

Hence, d cannot be a prime, and so, (a) is proven.

Let us find the least possible value for d by brute force. We know d is not prime.

If $d = 4$, then from $2a^2 = d(d - b)$, we have $a^2 = 2(4 - b)$ and there is no integer solution other than $a = b = 2$.

If $d = 6$, then we have $a^2 = 3(6 - b)$ and there is no integer solution other than $a = b = 3$.

If $d = 8$, then we have $a^2 = 4(8 - b)$. There are two integer solutions: $a = b = 4$ and $a = 2$, $b = 7$. Here we have a solution with $a \neq b$.

We should check to see if this solution does indeed form a cyclic quadrilateral. We find the diagonal lengths of the quadrilateral by using the Pythagorean Theorem. From Ptolemy's Theorem, because we have an equality, the quadrilateral is cyclic with d as the diameter. (We can apply this to either $ABCD$ or $ABC'D$.)

Hence, the least possible value for d is 8.

Note: Let a quadrilateral have sides of length a_1, a_2, a_3, and a_4, *in that order*, and let it have diagonals of length d_1 and d_2. Ptolemy's Theorem states that the quadrilateral is cyclic if and only if $d_1 d_2 = a_1 a_3 + a_2 a_4$.

A related inequality is Ptolemy's Inequality. This inequality states that in *any* quadrilateral, $d_1 d_2 \leq a_1 a_3 + a_2 a_4$, with equality holding if and only if the quadrilateral is cyclic.

Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from this issue be submitted in time for issue 4 of 2002.

High School Problems

Editor: Adrian Chan, 1195 Harvard Yard Mail Center, Cambridge, MA, USA 02138-7501 <ahchan@fas.harvard.edu>

H285. Four people, A, B, C, D, are on one side of a river. To get across the river they have a rowboat, but it can only fit two people at a time. A, B, C, D, could each row across the river in the boat individually in 1, 2, 5, and 10 minutes respectively. However, when two people are on the boat, the time it takes them to row across the river is the same as the time necessary to row across for the slower of the two people. Assuming that no one can cross without the boat, and everyone is to get across, what is the minimum time for all four people to get across the river?

H286. A mouse eats his way through a $3 \times 3 \times 3$ cube of cheese by tunnelling through all of the 27 $1 \times 1 \times 1$ subcubes. If he starts at one of the corner sub-cubes and always moves onto an uneaten adjacent sub-cube, can he finish at the centre of the cube? (Assume that he can tunnel through walls but not edges or corners.)

H287. Suppose we want to construct a solid polyhedron using just n pentagons and some unknown number of hexagons (none of which need be regular), so that exactly three faces meet at every vertex on the polyhedron. For what values of n is this feasible?

H288. *Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain.*

If x, y, z are positive real numbers, show that

$$\frac{1}{\sinh(x+z)}\left(\frac{\cosh y\cosh z}{\cosh(x+y+z)}-\cosh x\right) = \frac{1}{\sinh(y+z)}\left(\frac{\cosh x\cosh z}{\cosh(x+y+z)}-\cosh y\right)$$

Advanced Problems

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A261. Let $P_k(x) = 1 + x + x^2 + \cdots + x^{k-1}$. Show that

$$\sum_{k=1}^{n}\binom{n}{k}P_k(x) \;=\; 2^{n-1}P_n\left(\frac{1+x}{2}\right)$$

for every real number x and every positive integer n.

(1998 Baltic Way)

A262. *Proposed by Mohammed Aassila, Strasbourg, France.*

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for any real number a, the sequence $f(a), f(2a), f(3a), \ldots$ converges to zero. Prove that $\lim_{x\to\infty} f(x) = 0$.

A263. Find all real numbers x for which there exists a closed convex region on the Euclidean plane with both area and perimeter equal to x.

A264. *Proposed by Mohammed Aassila, Strasbourg, France.*

Prove that for any integer a and natural number m,

$$a^m \;\equiv\; a^{m-\phi(m)} \pmod{m}\,.$$

(This is a generalization of Euler's Theorem.)

Challenge Board Problems

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

History of the Konhauser Problemfest

Stan Wagon (Macalester College)

Joe Konhauser, a geometer and avid problemist, came to Macalester College in 1968. He had been invited by Sy Schuster (Carleton College) to work on some geometry movies for the Mathematical Association of America, and he must have liked Minnesota, since he moved from Pennsylvania to the University of Minnesota and then Macalester shortly thereafter. He started the Problem of the Week tradition at Macalester, posing a problem every week and offering 50 cents (since raised to $1) for correct solutions. When I came to Macalester in 1990 I was very impressed by the fact that students would work on these problems just for the fun of it. Part of the reason was that Joe had a knack for posing simply stated and compelling problems, often with surprising twists.

During his career at Macalester, Joe served as editor of the Pi Mu Epsilon journal and also served on the many problems committees, including those of the USA Math Olympiad and the Putnam competition.

When Joe died in 1992, I took over the PoW program, adding an e-mail component. Joe posed problems 1–682 and I and my colleagues (notably Tom Halverson) have continued the tradition and are now into the 900s. The MAA has published a book, "Which Way Did the Bicycle Go?" (by Konhauser, D. Velleman, and me), containing the best of the first 800 problems.

In 1993, partially inspired by the Lower Michigan Math Contest, Mark Krusemeyer (Carleton College), Loren Larson (St. Olaf) and I started the Konhauser Problemfest, a three-hour contest with ten problems for teams of three students working together. The event is now in its ninth year, and the University of St. Thomas and Gustavus Adolphus College also take part. The winning team gets a cash prize and their school gets the travelling trophy, a small granite sculpture by Helaman Ferguson illustrating a proof of a geometrical fact known as the pizza theorem. Carleton has won five times, including the 2001 contest, Macalester has won twice, these two tied one year, and Gustavus Adolphus won the event in Feb. 2000.

At the URL

http://www.macalester.edu/~mathcs/potw.html

you can find past Konhauser problems, an archive of past Problems of the Week, and information on how to receive the Problem of the Week automatically by e-mail.

The Ninth Annual Konhauser Problemfest

Carleton College, February 24, 2001
Problems by David Savitt and Russell Mann (Harvard University)

This contest is held annually in memory of Professor Joseph Konhauser (1924-1992) of Macalester College, who posted nearly 700 Problems of the Week at Macalester over a 25-year period. Joe died in February of 1992, and the contest was started the following year.

INSTRUCTIONS: Each team must hand in all work to be graded at the same time (at the end of the three-hour period). Only one version of each problem will be accepted per team. Calculators of any sort are allowed (although they may not be all that helpful). Justifications and/or explanations are expected for all problems, but, in view of the time constraint, rigorous proofs are only required when the wording of the problem makes that clear ("show that" or "prove that"). All ten problems will be weighted equally, and partial credit will be given for substantial progress toward a solution. Good luck!

1. Last season, the Minnesota Timberwolves won five times as many games as they lost, in games in which they scored 100 or more points. On the other hand, in games in which their opponents scored 100 or more points, the Timberwolves lost 50% more games than they won. Given that there were exactly 34 games in which either the Timberwolves or their opponents scored 100 or more points, what was the Timberwolves' win-loss record in games in which *both* they and their opponents scored 100 or more points?

2. Three circles are drawn in chalk on the ground. To begin with, there is a heap of n pebbles inside one of the circles, and there are "empty heaps" (containing no pebbles) in the other two circles. Your goal is to move the entire heap of n pebbles to a different circle, using a series of moves of the following type. For any non-negative integer k, you may move exactly 2^k pebbles from one heap (call it heap A) to another (heap B), provided that heap B begins with fewer than 2^k pebbles, and that after the move, heap A ends up with fewer than 2^k pebbles. Naturally, you want to reach your goal in as few moves as possible. For what values of $n \leq 100$ would you need the largest number of moves?

3. (a) Begin with a string of 10 `A`'s, `B`'s, and `C`'s, for example

`A B C C B A B C B A`

and underneath, form a new row, of length 1 shorter, as follows: between two consecutive letters that are different, you write the third letter, and between two letters that are the same, you write that same letter again. Repeat this process until you have only one letter in the new row. For example, for the string above, you will now have:

```
A B C C B A B C B A
 C A C A C C A A C
  B B B B C B A B
   B B B A A C C
    B B C A B C
     B A B C A
      C C A B
       C B C
        A A
         A
```

Prove that the letters at the corners of the resulting triangle are always either all the same or all different.

(b) For which positive integers n (besides 10) is the result from part (a) true for all strings of n A's, B's, and C's?

4. When Mark climbs a staircase, he ascends either 1, 2, or 3 stair steps with each stride, but in no particular pattern from one foot to the next. In how many ways can Mark climb a staircase of 10 steps? (Note that he must finish on the top step. Two ways are considered the same if the number of steps for each stride is the same; that is, it does not matter whether he puts his best or his worst foot forward first.) Suppose that a spill has occurred on the 6^{th} step and Mark wants to avoid it. In how many ways can he climb the staircase without stepping on the 6^{th} step?

5. Number the vertices of a cube from 1 to 8. Let A be the 8×8 matrix whose (i, j) entry is 1 if the cube has an edge between vertices i and j, and is 0 otherwise. Find the eigenvalues of A, and describe the corresponding eigenspaces.

6. Let $f(x)$ be a twice-differentiable function on the open interval $(0, 1)$ such that

$$\lim_{x \to 0+} f(x) = -\infty, \text{ and } \lim_{x \to 1^-} f(x) = +\infty.$$

Show that $f''(x)$ takes on both negative and positive values.

7. Three stationary sentries are guarding an important public square which is, in fact, square, with each side measuring 8 rods (recall that one rod equals 5.5 yards). (If any of the sentries see trouble brewing at any location on the square, the sentry closest to the trouble spot will immediately cease to be stationary and dispatch to that location. And like all good sentries, these three are continually looking in all directions for trouble to occur.) Find the maximum value of D, so that no matter how the sentries are placed, there is always some spot in the square that is at least D rods from the closest sentry.

8. The Union Atlantic Railway is planning a massive project: a rail road track joining Cambridge, Massachusetts and Northfield, Minnesota. However, the

funding for the project comes from the will of Orson Randolph Kane, the eccentric founder of the U.A.R., who has specified some strange conditions on the railway; thus the sceptical builders are unsure whether or not it is possible to build a railway subject to his unusual requirements.

Kane's will insists that there must be exactly 100 stops (each named after one of his great grand-children) between the termini, and he even dictates precisely what the distance along the track between each of these stops must be. (Unfortunately, the tables in the will *do not* list the order in which the stops are to appear along the railway.) Luckily, it is clear that Kane has put some thought into these distances; for any three distinct stops, the largest of the three distances among them is equal to the sum of the smaller two, which is an obvious necessary condition for the railway to be possible. (Also, all the given distances are shorter than the distance along a practical route from Cambridge to Northfield!)

U.A.R.'s engineers have pored over the numbers and noticed that for any four of Kane's stops, it would be possible to build a railway with these four stops and the distances between them as Kane specifies. Prove that, in fact, it is possible to complete the entire project to Kane's specifications.

9. Gail was giving a class on triangles, and she was planning to demonstrate on the blackboard that the three medians, the three angle bisectors, and the three altitudes of a triangle each meet at a point (the centroid, incentre, and orthocentre of the triangle, respectively). Unfortunately, she got a little careless in her example, and drew a certain triangle ABC with the median from vertex A, the altitude from vertex B, and the angle bisector from vertex C. Amazingly, just as she discovered her mistake, she saw that the three segments met at a point anyway! Luckily it was the end of the period, so no one had a chance to comment on her mistake. In recalling her good fortune later that day, she could only remember that the side across from vertex C was 13 inches in length, that the other two sides also measured an integral number of inches, and that none of the lengths were the same. What were the other two lengths?

10. An infinite sequence of digits "1" and "2" is determined uniquely by the following properties:

> (i) The sequence is built up by stringing together pieces of the form "12" and "112".
>
> (ii) If we replace each "12" piece with a "1" and each "112" piece with a "2", then we get the original sequence back.

(a) Write down the first dozen digits in the sequence. At which place will the 100^{th} "1" occur? What is the 1000^{th} digit?

(b) Let A_n be the number of "1"s among the first n digits of the sequence. Given that the ratio A_n/n approaches a limit, find that limit.

(c) (Tiebreaker) Show that the limit from part (b) actually exists.

The Volume of a Tetrahedron and Areas of its Faces

C.-S. Lin

A classical result states that if a tetrahedron $ABCD$ is rectangular at the vertex B, so that all the edges meeting at B are mutually perpendicular (See Figure 1), then the relation $b^2 = a^2 + c^2 + d^2$ is called the S–dimensional version of Pythagorean Theorem, where x denotes the area of the face opposite to the vertex X of the tetrahedron $ABCD$. The classical Heron's Formula of plane geometry says that for a triangle having sides of length a, b and c, and area K, we have

$$K = \frac{1}{4}\left(2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)\right)^{\frac{1}{2}}, \text{ or}$$

$$K = (s(s-a)(s-b)(s-c))^{\frac{1}{2}} \text{ for } s = \tfrac{1}{2}(a+b+c).$$

Although a well known classical result indicates that the volume of a tetrahedron can be expressed in terms of its six edges, and the formula may be found, for example, in [2, p.13 and 1731], I presented in [1] a relation which I called the S–dimensional analogue of Heron's Formula. Indeed, if V denotes the volume of an H–tetrahedron (attributed to Heron) $ABCD$; that is, it can be formed by gluing together two tetrahedra $ACDE$ and $BCDE$ both rectangular at the vertex E, and on a common congruent face $\triangle CDE$ (see figure 2) so that $\triangle ABC \perp \triangle ABD$, then

$$(*) \qquad V^2 = \frac{2cd\left(4(s-a)(s-b)(s-c)(s-d) - c^2d^2 - 2abcd\right)^{\frac{1}{2}}}{9(c^2+d^2)^{\frac{1}{2}}}$$

for $s = (a+b+c+d)$, and the small letters denote areas of faces as described before.

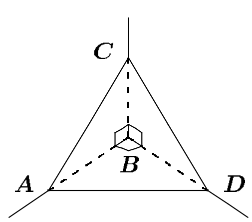


Figure 1

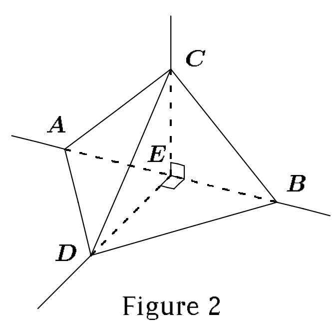


Figure 2

The identity $(*)$ leads naturally to the following question: Can we establish a formula which expresses the volume of an arbitrary tetrahedron in terms of its four faces? In this article I shall construct two formulas for an L–tetrahedron (See definition below). Each one shows the volume of an L–tetrahedron in terms of areas of its faces and some constant.

We begin with a definition first. A tetrahedron is said to be L–shaped, and called an L–tetrahedron if any two of its four faces are perpendicular. In Figure 3 or 4, we have arranged a tetrahedron $ABCD$ located in the first octant for which $\triangle ABC$ is on the xz–plane and $\triangle ABD$ on the xy–plane, so that $\triangle ABC \perp \triangle ABD$. More precisely, we assume that the vertex B is the origin such that $\overline{CE} \perp \overline{AB}$, $\overline{DF} \perp \overline{AB}$, and $\angle BED = \alpha$ with $0 < \alpha < \frac{\pi}{2}$ as in Figure 3, or $\frac{\pi}{2} < \alpha < \pi$ as in Figure 4. If $\alpha = \frac{\pi}{2}$, then $\overline{BF} = \overline{BE}$ and the L–tetrahedron becomes an H–tetrahedron as shown in Figure 2.

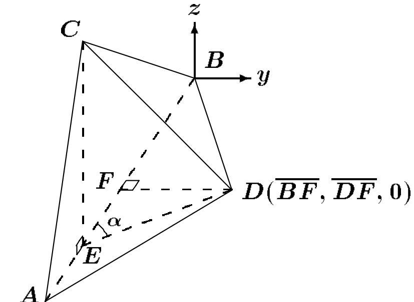


Figure 3

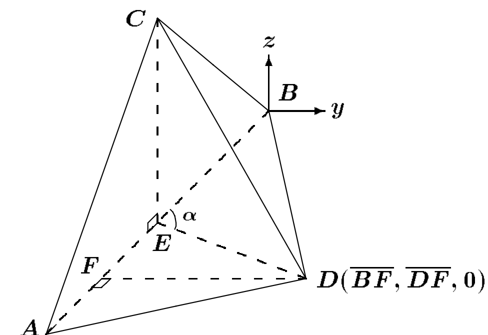


Figure 4

In what follows we shall consider an L–tetrahedron only. Of course it is possible that some L–tetrahedrons may be geometrically deformed, and are not analogously shown in Figure 3 or 4. For instance, the vertices C and D could be located anywhere on the xz–plane, and xy–plane, respectively, but not on the x–axis. However, the derivation of the volume (of a tetrahedron) formula, or the area (of a triangle) formula does not depend upon the location of the object concerned. For those people who enjoy doing long computations and simplifications the next result is a good one.

Theorem 1. Let V denote the volume of an L–tetrahedron $ABCD$ as in Figure 3 or 4, and let $r = \overline{CE}$, $q = \overline{BE}$, $u = \overline{BF}$, $m = \overline{BA}$, and $p = \overline{DF}$. If a, b, c, and d stand for areas of the four faces as described before, then we have the following relations.

(1) $V^2 = \frac{1}{9}cdrp$

(2) $a^2 = \frac{1}{4}\big((p^2 + r^2)q^2 + p^2r^2 + (u-q)^2(r^2+q^2)$
$\quad -2r^2q(p^2 + (u-q)^2)^{\frac{1}{2}}\cos\alpha - q^2(p^2+(u-q)^2)\cos^2\alpha\big)$

(3) $b^2 = \frac{1}{4}\big((p^2 + r^2)(m-q)^2 + p^2r^2 + (u-q)^2(r^2+(m-q)^2)$
$\quad +2r^2(m-q)(p^2+(u-q)^2)^{\frac{1}{2}}\cos\alpha$
$\quad -(m-q)^2(p^2+(u-q)^2)\cos^2\alpha\big)$

(4) $c^2 = \frac{1}{4}p^2m^2$

(5) $d^2 = \frac{1}{4}r^2m^2$

(6) $\cos\alpha = \frac{q-u}{\sqrt{p^2+(q-u)^2}}$

Proof:

(1) Because the volume of a tetrahedron is one-third of the product of its base area with its height, the result follows.

(2) Clearly, $\overline{BC} = (q^2 + r^2)^{\frac{1}{2}}$ and $\overline{CD} = \left(r^2 + p^2 + (u-q)^2\right)^{\frac{1}{2}}$. From $\triangle BDE$ we have $\overline{DB} = \left(q^2 + p^2 + (u-q)^2 - 2q(p^2 + (u-q)^2)^{\frac{1}{2}}\cos\alpha\right)^{\frac{1}{2}}$ by the Cosine Law. In order to find a^2 we use Heron's Formula:

$$a^2 = \frac{1}{16}\left(2\left(\overline{BC}^2\overline{CD}^2 + \overline{CD}^2\overline{DB}^2 + \overline{DB}^2\overline{BC}^2\right) - \left(\overline{BC}^4 + \overline{CD}^4 + \overline{DB}^4\right)\right).$$

Long substitutions and simplifications show the identity (2), and we shall omit the details.

(3) Note that $\overline{CA} = \left(r^2 + (m-q)^2\right)^{\frac{1}{2}}$, and by $\triangle AED$ we get

$$\overline{AD} = \left(p^2 + (u-q)^2 + (m-q)^2 + 2(m-q)(p^2 + (u-q)^2)^{\frac{1}{2}}\cos\alpha\right)^{\frac{1}{2}},$$

since $\cos\angle AED = -\cos\alpha$. We also notice as in (2) that

$$b^2 = \frac{1}{16}\left(2\left(\overline{CA}^2\overline{CD}^2 + \overline{CA}^2\overline{AD}^2 + \overline{AD}^2\overline{CD}^2\right) - \left(\overline{CA}^4 + \overline{AD}^4 + \overline{CD}^4\right)\right).$$

Thus, the desired equation may be similarly obtained as equation (2).

The relations (4) and (5) are obvious by assumptions.

(6) $\cos\alpha = -\cos\angle AED = \dfrac{q-u}{\overline{ED}} = \dfrac{q-u}{\sqrt{p^2 + (q-u)^2}}$ if $\frac{\pi}{2} \le \alpha < \pi$,

and

$$\cos\alpha = \frac{q-u}{\overline{ED}} = \frac{q-u}{\sqrt{p^2 + (q-u)^2}} \quad \text{if } 0 < \alpha \le \tfrac{\pi}{2}.$$

We are now ready to state and prove the main result.

Theorem 2. Let V denote the volume of an L–tetrahedron $ABCD$ as in Figure 3 or 4, and let $r = \overline{CE}$, $q = \overline{BE}$, $u = \overline{BF}$, $m = \overline{BA}$, and $p = \overline{DF}$. If a, b, c, and d stand for areas of the four faces as described before, then we have the following relations.

(I) If $q-u=k_1p$ for some constant k_1, then

$$V^4=\frac{c^2d^2\left(2\left(a^2c^2+a^2d^2+a^2b^2+b^2c^2+b^2d^2-c^2d^2\right)-\left(a^4+b^4+c^4+d^4\right)\right)}{81\left(c^2+k_1^2c^2+d^2\right)}.$$

(II) If $u=k_2q$ for some constant $k_2\geq 0$, then

$$V^4=\frac{c^2d^2\left(4a^2\left(c^2+d^2k_2\right)^2-\left(c^2+d^2k_2^2\right)\left(a^2+c^2+d^2-b^2\right)^2\right)}{81\left(c^2+d^2k_2\right)^2}.$$

Notice that both constants k_1 and k_2 determine different shapes of an L–tetrahedron.

Proof. (I) Since $q-u=k_1p$, and $\cos\alpha=\frac{q-u}{\sqrt{p^2+(q-u)^2}}$, by (6) in Theorem 1, we find $\cos\alpha=\frac{k_1}{\sqrt{1+k_1^2}}$. Thus, identities (2) and (3) in Theorem 1 become

(2′) $a^2=\frac{1}{4}\left(\left(p^2+r^2\right)q^2+p^2r^2+k_1^2p^2r^2-2k_1r^2pq\right)$,

(3′) $b^2=\frac{1}{4}\left(\left(p^2+r^2\right)(m-q)^2+p^2r^2+p^2k_1^2r^2+2k_1r^2(m-q)p\right)$.

Now , the whole idea is to eliminate p, q, r, and m from the five equations (1), (2′), (3′), (4), and (5) in Theorem 1, and to express V in terms of a, b, c, d, and k_1. We shall start with the next identity which follows from (1), (4) and (5).

(7) $V^4=\frac{16c^4d^4}{81m^4}$.

Subtracting equation (3′) from equation (2′), we have

$4(a^2-b^2)=(p^2+r^2)(2mq-m^2)-2k_1r^2mp$,

whence we get

$a^2-b^2=\frac{1}{m^2}[(c^2+d^2)(2mq-m^2)-4k_1cd^2]$

by (4) and (5), so that

(8) $q=\frac{1}{2m(c^2+d^2)}[m^2(a^2+c^2+d^2-b^2)+4k_1cd^2]$.

Using identities (2′), (4) and (5) we get

$m^4a^2 = m^2q^2(c^2+d^2)+4c^2d^2(1+k_1^2)-4k_1cd^2mq\,.$

Replacing the last q above by (8), and expressing q^2 in terms of the others, we get

(9) $q^2=\frac{m^2}{c^2+d^2}\Big(a^2-\frac{4c^2d^2(1+k_1^2)}{m^4}$
$\qquad+\frac{2k_1cd^2}{m^4(c^2+d^2)}\left(m^2(a^2+c^2+d^2-b^2)+4k_1cd^2\right)\Big)$.

By eliminating q from equations (8) and (9) we find that

$$
\begin{aligned}
4m^4\left(c^2+d^2\right)&\left(a^2-\frac{4c^2d^2\left(1+k_1^2\right)}{m^4}\right.\\
&\left.+\frac{2k_1cd^2}{m^4\left(c^2+d^2\right)}\left(m^2\left(a^2+c^2+d^2-b^2\right)+4k_1cd^2\right)\right)\\
&=\left(m^2\left(a^2+c^2+d^2-b^2\right)+4k_1cd^2\right)^2 .
\end{aligned}
$$

From the above, we express m^4 in terms of a, b, c, d, and k_1 as follows.

(10) $m^4=\dfrac{16c^2d^2\left(c^2+k_1^2c^2+d^2\right)}{2\left(a^2b^2+b^2c^2+b^2d^2+a^2c^2+a^2d^2-c^2d^2\right)-\left(a^4+b^4+c^4+d^4\right)}$.

Finally, substituting m^4 in equation (7) yields the required formula (I).

(II) First we remark that the process of the proof of this formula is exactly the same as in (I) above. From the condition $u=k_2q$, identities (2) and (3) become

(2″) $a^2=\frac{1}{4}\left(\left(p^2+r^2\right)q^2+p^2r^2+q^2\left(k_2-1\right)^2r^2+2r^2q^2\left(k_2-1\right)\right)$,

(3″) $b^2=\frac{1}{4}\left(\left(p^2+r^2\right)(m-q)^2+p^2r^2+q^2\left(k_2-1\right)^2r^2-2r^2(m-q)q\left(k_2-1\right)\right)$.

Subtracting equation (3″) from equation (2″), we get

$4\left(a^2-b^2\right)=\left(p^2+r^2\right)m(2q-m)+2r^2mq\left(k_2-1\right)$,

and using identities (4) and (5), we obtain

$$
\begin{aligned}
a^2-b^2&=\frac{1}{m}\left(c^2+d^2\right)(2q-m)+\frac{1}{m}2d^2q\left(k_2-1\right)\\
&=\frac{1}{m}2q\left(c^2+d^2k_2\right)-\left(c^2+d^2\right)
\end{aligned}
$$

Consequently,

(8′) $q=\dfrac{m(a^2+c^2+d^2-b^2)}{2(c^2+d^2k_2)}$.

With the aids of identities (4) and (5), (2″) becomes

$$
\begin{aligned}
m^4a^2&=q^2m^2(c^2+d^2)+4c^2d^2+q^2(k_2-1)^2d^2m^2+2q^2(k_2-1)d^2m^2\\
&=q^2(m^2c^2+k_2^2d^2m^2)+4c^2d^2 ,
\end{aligned}
$$

and hence

(9′) $q^2=\dfrac{m^4a^2-4c^2d^2}{m^2(c^2+d^2k_2^2)}$.

The next step is immediate from (8′) and (9′).

$$
m^4(c^2+d^2k_2^2)(a^2+c^2+d^2-b^2)^2 = 4(c^2+d^2k_2)^2(m^4a^2-4c^2d^2),
$$

and further,

$$(10') \ m^4 = \frac{16c^2d^2(c^2 + d^2k_2)^2}{4a^2(c^2 + d^2k_2)^2 - (c^2 + d^2k_2^2)(a^2 + c^2 + d^2 - b^2)^2}.$$

Substituting m^4 in (7) yields the desired formula (II), and the proof of the theorem is completed.

Remarks. Since the shape of an L–tetrahedron changes as the constant k_i, $i = 1, 2$, varies, let us consider two special cases in Theorem 2.

(A) If $k_1 = 0$ in (I), or if $k_2 = 1$ in (II); that is, the case when $\alpha = \frac{\pi}{2}$, we have, for $k_1 = 0$, that

$$V^4 = \frac{c^2d^2[2(a^2b^2+b^2c^2+b^2d^2+a^2c^2+a^2d^2-c^2d^2)-(a^4+b^4+c^4+d^4)]}{81(c^2+d^2)},$$

and, for $k_2 = 1$, that $V^4 = \frac{c^2d^2[4a^2(c^2+d^2)^2-(c^2+d^2)(a^2+c^2+d^2-b^2)^2]}{81(c^2+d^2)}$.

Rewriting the above two identities by using the expression $s = \frac{1}{2}(a+b+c+d)$ one should have the equation $(*)$ for an H–tetrahedron, and we shall omit the details.

(B) It is immediate, from Figure 3 or 4, that if $u = q = 0$, then we get a new tetrahedron rectangular at the vertex B as shown in Figure 1, which has volume

$$V = \frac{1}{6}(\overline{BA} \cdot \overline{BC} \cdot \overline{BD}) = \frac{1}{3}\sqrt{2acd}.$$

Since $u = q = 0$ implies $k_1 = 0$ in (I), and $k_2 \geq 0$ in (II), we have

$$\left(\frac{1}{3}\sqrt{2acd}\right)^4 = \frac{c^2d^2\big(2(a^2c^22+a^2d^2+a^2b^2+b^2c^2+b^2d^2-c^2d^2)-(a^4+b^4+c^4+d^4)\big)}{81(c^2+d^2)}$$

and $\left(\frac{1}{3}\sqrt{2acd}\right)^4 = \frac{c^2d^2\big(4a^2(c^2+d^2k_2)^2-(c^2+d^2k_2^2)(a^2+c^2+d^2-b^2)^2\big)}{81(c^2+d^2k_2)^2}$,

respectively. By simplifying both relations we arrive uniquely at $b^2 = a^2 + c^2 + d^2$. This is indeed an alternative proof of the 3–dimensional version of the Pythagorean Theorem.

References

[1] C.-S. Lin, *Heron's Formula in 3–dimensional space*, Math. Gaz. 80 (1996), 370–372.

[2] G. Polya, *Patterns of Plausible Inference*, Princeton University Press, Princeton, NJ, USA 1954.

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PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}$"×11" or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 November 2001**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in LaTeX format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2620. Correction. *Proposed by Bill Sands, University of Calgary, Calgary, Alberta, dedicated to Murray S. Klamkin, on his 80th birthday.*

Three cards are handed to you. On each card are three non-negative real numbers, written one below the other, so that the sum of the numbers on each card is 1. You are permitted to put the three cards in any order you like, then write down the first number from the first card, the second number from the second card, and the third number from the third card. You add these three numbers together.

Prove that you can always arrange the three cards so that your sum lies in the interval $\left[\frac{1}{3}, \frac{3}{2}\right]$.

2633 . *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that $\dfrac{n(n+1)}{2e} < \displaystyle\sum_{k=1}^{n}(k!)^{\frac{1}{k}} < \dfrac{31}{20} + \dfrac{n(n+1)}{4}$.

2626 ★. *Proposed by Achilleas Sinefakopoulos, student, University of Athens, Greece.*

Let $a_n = 2n + \lfloor n\sqrt{2}\rfloor$ for $n = 1, 2, \ldots$. Suppose that k and m are positive integers such that a_m is a multiple of 10 and $a_k = a_m + 10j$ for some positive integer j. Prove or disprove that if $j \le 4$, then $k = m + 3j$.

2627. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $x_1, \ldots, x_n$ be positive real numbers and let $s_n = x_1 + \cdots + x_n$ ($n \geq 2$). Let $a_1, \ldots, a_n$ be non-negative real numbers. Determine the optimum constant $C(n)$ such that

$$\sum_{j=1}^{n} \frac{a_j(s_n - x_j)}{x_j} \geq C(n) \left(\prod_{j=1}^{n} a_j \right)^{\frac{1}{n}} .$$

2628. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Four points, X, Y, Z and W are taken inside or on triangle ABC. Prove that there exists a set of three of these points such that the area of the triangle formed by them is less than $\frac{3}{8}$ of the area of the given triangle.

2629. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

In triangle ABC, the symmedian point is denoted by S. Prove that

$$\frac{1}{3}\left(AS^2 + BS^2 + CS^2\right) \geq \frac{BC^2AS^2 + CA^2BS^2 + AB^2CS^2}{BC^2 + CA^2 + AB^2} .$$

2630. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n} \sum_{j=1}^{n} \frac{1}{j} = \frac{\pi^2}{12}$.

2631. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Find the exact value of $\displaystyle\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} \binom{n-k}{k} 2^{n-2k}$.

2632. *Proposed by Mihály Bencze, Brasov, Romania.*

Let $\displaystyle S_m = \lim_{n\to\infty} \frac{1}{n^2} \sum_{1 \leq j \leq k \leq n} \cos\left(\frac{(j + (-1)^m k)x}{n}\right)$, where $x \in \mathbb{R}$.

Find exact expressions for S_0 and S_1.

2634. *Proposed by Mihály Bencze, Brasov, Romania.*

Let $\displaystyle P(x) = 1 + \sum_{k=1}^{n} a_k x^k$, where $a_k \in [0, 2]$ ($k = 1, 2, \ldots, n$).

Prove that $P(x)$ is never zero in $(1 - \sqrt{2}, 0]$.

2635. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Consider triangle ABC, and three squares $BCDE$, $CAFG$ and $ABHI$ constructed on its sides, outside the triangle. Let XYZ be the triangle enclosed by the lines EF, DI and GH.

Prove that $[XYZ] \leq (4-2\sqrt{3})[ABC]$, where $[PQR]$ denotes the area of $\triangle PQR$.

2636. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that $A_1A_2 \ldots A_n$ is a convex n–gon with $n \geq 5$, and that the angle at each vertex is divided into $(n-2)$ equal angles by the $(n-3)$ diagonals through that vertex. Prove that $A_1A_2 \ldots A_n$ is a regular n–gon.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

There was a typographical error in the statement of Peter Woo's theorem in the second solution to 2516 — the theorem should read:

Theorem. Let CD, BE be cevians of a triangle ABC where $BD \geq CD$. Let CD intersect BE at P. Then $AD + DP > AE + EP$.

2513. [2000 : 114] *Proposed by Waldemar Pompe, University of Darmstadt, Darmstadt, Germany; dedicated to Prof. Toshio Seimiya on his 90^{th} birthday.*

A circle is tangent to the sides BC, AD of convex quadrilateral $ABCD$ in points C, D, respectively. The same circle intersects the side AB in points K and L. The lines AC and BD meet in P. Let M be the mid-point of CD. Prove that if $CL = DL$, then the points K, P, M are collinear.

I. *Solution by Michel Bataille, Rouen, France.*

We suppose first that AD is parallel to BC. In this case, M is the centre of the circle, say Γ, defined in the statement of the problem, L is the mid-point of AB, and $PC/PA = PB/PD = CB/AD$. We also assume $K \neq L$ (otherwise $ABCD$ is a rectangle and the conclusion is obvious), and we call I the intersection of lines MP and BC.

From Menelaus' Theorem applied to the transversal MP of $\triangle BCD$, and since $MC/DM = 1$ and $PD/BP = -AD/CB$, we get

$$\frac{MC}{DM} \cdot \frac{PD}{BP} \cdot \frac{IB}{CI} = -1,$$

so that $IB/CI = CB/AD$. Expressing the powers of B and A with respect to Γ, we obtain $BK \cdot BL = (BC)^2$ and $AK \cdot AL = (AD)^2$, which yields $KA/BK = (AD)^2/(BC)^2$ (since $BL = -AL$). Now

$$\frac{PC}{AP} \cdot \frac{KA}{BK} \cdot \frac{IB}{CI} = -\frac{CB}{AD} \cdot \frac{(AD)^2}{(BC)^2} \cdot \frac{CB}{AD} = -1,$$

and, from the converse of Menelaus' Theorem, the points P, K, I are collinear. The collinearity of P, K, M follows immediately.

As for the general case, it can be reduced to the previous one by means of a central projection transforming the circle Γ (with its interior point M) into a circle Γ' whose centre is the image M' of M. The whole figure is then transformed into that of the particular case, and the collinearity proved above implies the collinearity of M, P, K in the general case.

Note. For the use of a central projection, we refer to *Geometric Transformations III* by I. M. Yaglom, MAA (NML 21), Random House (1973), page 54, Theorem 1. See also Chapter 26 of *The Enjoyment of Mathematics*, Rademacher and Toeplitz, Princeton Univ. Press, 1957.

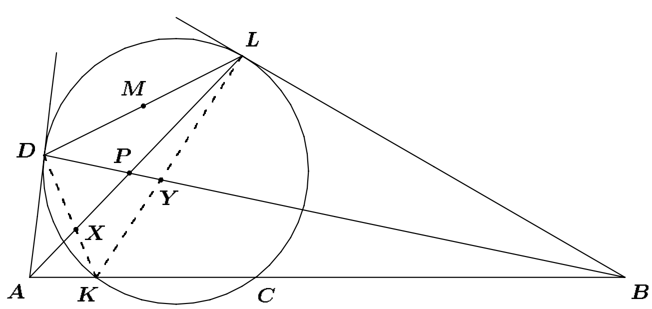
II. *Solution by the proposer*.

Let $X = DK \cap AC$, $Y = CK \cap BD$ (see the figure below). Since $CL = DL$,

$$\angle AKD = \angle BKC. \tag{1}$$

[Since arc CL = arc DL, $\angle CKL = 180° - \angle DKL = \angle DKA$. — *Ed*.] The given circle is tangent to the sides BC and AD, so we get

$$\angle ADK = \angle DCK \qquad \text{and} \qquad \angle BCK = \angle CDK. \tag{2}$$



According to Ceva's Theorem [and because $DM = MC$], the points K, P, M are collinear if and only if

$$\frac{KX}{XD} = \frac{KY}{YC}.$$

Denote by $[TUW]$ the area of triangle TUW. Then the above equality can be rewritten as

$$\frac{[AKC]}{[ADC]} = \frac{[BKD]}{[BCD]} \quad \text{or} \quad \frac{[AKC]}{[BKD]} = \frac{[ADC]}{[BCD]}\,.$$

Using the equalities (1) and (2), we see that the last equality is equivalent to

$$\frac{AK \cdot KC}{BK \cdot KD} = \frac{AD}{BC}\,. \tag{3}$$

We need to prove (3). In order to do this, we start with the following equality

$$\frac{[AKD]}{[DCK]} \cdot \frac{[DCK]}{[BKC]} = \frac{[AKD]}{[BKC]}\,.$$

Thus, using once again the equalities (1) and (2), gives

$$\frac{AD \cdot KD}{DC \cdot KC} \cdot \frac{KD \cdot DC}{KC \cdot BC} = \frac{AK \cdot KD}{BK \cdot KC} \quad \text{or} \quad \frac{AD \cdot KD}{KC \cdot BC} = \frac{AK}{BK}\,,$$

which is the same as (3). Thus the points K, P, M are collinear.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; TOSHIO SEIMIYA, Kawasaki, Japan; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

Seimiya's solution treated the cases $AD \not\parallel BC$ and $AD \parallel BC$ separately, as did the solution of Benito and Fernández. Two other solvers seemed to consider only the case $AD \not\parallel BC$.

Woo sent in two solutions, one similar to Bataille's solution above, and one using more projective geometry which, Woo points out, is valid even if the circle is a general conic. On being sent this latter solution for expert analysis, Chris Fisher interpreted it to arrive at the following generalization.

> *A conic is tangent to the sides BC and AD of convex quadrilateral $ABCD$ at C and D. The same conic intersects the side AB in points K and L. The lines AC and BD meet in P. If R is the point where the given tangents intersect, and M is where RL intersects CD, then the points K, P, M are collinear.*

(The proposer has recently and independently sent in the same generalization.)

Chris's proof is a supercharged version of Solution I, using a transformation that immediately reduces the problem to the case when $ABCD$ is a rectangle. In his words: "Just define Q to be where KL meets CD and note that, if QR misses the conic (which happens if $ABCD$ is convex), there is a projective transformation that simultaneously takes the given conic to a circle and the line QR to infinity. The picture becomes a circle whose diameter is CD with LK parallel to it, and the figure is so symmetric that the statement 'P is on MK' is clear from the picture."

For "projective transformation" one may consult the Yaglom reference again. Readers may like to attempt to prove this general statement (with "conic" specialized to a circle again) by Euclidean means only. It amounts to the proposer's original problem but without the condition that $CL = DL$ and with M defined as the intersection of CD and RL, where R is the intersection of the tangents.

2520. [2000 : 115] *Proposed by Paul Bracken, CRM, Université de Montréal, Montréal, Québec.*

Let a, b be real numbers such that $b \leq 0$ and $1 + ax + bx^2 \geq 0$ for every $x \in [0, 1]$. Define

$$F_n(a, b) \;=\; \int_0^1 \left(1 + ax + bx^2\right)^n \, dx \,.$$

Show that the following asymptotic expressions are valid for $F_n(a, b)$ as $n \to \infty$:

1. If $a < 0$ and $b \leq 0$, then

$$F_n(a, b) \;=\; -\frac{1}{an} + \frac{1}{n^2 a}\left(1 - \frac{2b}{a^2}\right) + \mathrm{O}\left(n^{-3}\right) \,.$$

2. If $a \geq 0$ and $b < 0$, then

$$F_n(a, b) \;\sim\; \sqrt{\frac{\pi}{n|b|}}\left(1 - \frac{a^2}{4b}\right)^{n+\frac{1}{2}} \,.$$

Editor's comment: there were a couple of typographical errors in the original statement. These were spotted by all solvers.

Solution to part 1 by Manuel Benito and Emilio Fernández, I.B. Praxedes Mateo Sagasta, Logroño, Spain.

If $a < 0$ and $b \leq 0$, then $y(x) = 1 + ax + bx^2$ is, by the hypothesis, a positive decreasing function on the interval $[0, 1)$; since $\ln y(0) = 0$, $\ln y(x) < 0$ in $(0, 1)$ and also decreases. Let δ be the positive solution of the equation $1 + ax + bx^2 = 1 + a$; that is, $\delta = \frac{1}{2b}\left(-a - \sqrt{a^2 + 4ab}\right)$. Also, we have $\delta < 1$ and

$$\ln(1 + ax + bx^2) \leq \ln(1 + a) \;<\; 0 \qquad \forall\, x \geq \delta > 0\,,$$

and so,

$$F_n(a, b) = \int_0^1 e^{n\ln(1+ax+bx^2)}\, dx = \int_0^\delta e^{n\ln(1+ax+bx^2)}\, dx + \mathrm{o}\left(\frac{1}{n^3}\right), \qquad (1)$$

given that

$$\int_\delta^1 e^{n\ln(1+ax+bx^2)}\, dx \leq (1 - \delta)e^{n\ln(1+a)} \;=\; \mathrm{O}\big((1 + a)^n\big) \;=\; \mathrm{o}\left(\frac{1}{n^3}\right)$$

as $n \to \infty$.

Let us propose, for the integral on the right side of (1), the change of variable

$$\ln(1 + ax + bx^2) \;=\; au\,;$$

this equation implicitly defines, near 0, a differentiable $x(u)$. It is possible to obtain the first terms of the powers of u series development of $x(u)$ by undetermined coefficients. Let $x = Bu + Cu^2 + Du^3 + \cdots$, and make the identification

$$1 + ax + bx^2 = 1 + aBu + (aC + bB^2)u^2 + (aD + 2bBC)u^3 + \ldots$$
$$\equiv e^{au} = 1 + au + \frac{1}{2}a^2u^2 + \frac{1}{6}a^3u^3 + \ldots .$$

It follows that

$$B = 1\,, \quad C = \frac{1}{a}\left(\frac{a^2}{2} - b\right)\,, \quad D = \frac{1}{a}\left(\frac{a^3}{6} - ab + \frac{2b^2}{a}\right)\,, \ldots .$$

And thus, near $u = 0$, we have

$$\frac{dx}{du} = B + 2Cu + 3Du^2 + \ldots = 1 + \left(a - \frac{2b}{a}\right)u + 3Du^2 + \ldots ,$$

and by following the calculus on (1), by developing the change on the integral and integrating by parts,

$$F_n(a,b) = \int_0^{\frac{\ln(1+a)}{a}} e^{nau}\left(1 + \left(a - \frac{2b}{a}\right)u + 3Du^2 + \ldots\right)du + \mathrm{o}\left(\frac{1}{n^3}\right)$$
$$= \mathrm{o}\big((1+a)^n\big) - \frac{1}{an} + \frac{1}{n^2a^2}\left(a - \frac{2b}{a}\right) - \frac{6D}{n^3a^3} + \ldots$$
$$= -\frac{1}{an} + \frac{1}{n^2a}\left(1 - \frac{2b}{a^2}\right) + \mathrm{O}(n^{-3})\,,$$

as desired.

Editor's comment. There was disagreement amongst solvers about part 2. We shall leave it for the nonce.

Also solved by MICHEL BATAILLE, Rouen, France; and the proposer.

2528. [2000 : 177] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Prove that every rectifiable centrosymmetric curve on a unit sphere in $\mathbb{E}^3$ has length greater than or equal to 2π.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

Let A and A' be two centrosymmetric points on the curve. The result follows since the shortest distance on the sphere between A and A' is π.

Also solved by AUSTRIAN IMO-TEAM 2000 (whose solution was essentially the same as Klamkin's); MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Klamkin commented that a related known result is that any closed curve of length less than 2π lies in an open hemisphere. His open related problem is:

> *if one has a closed curve on the ellipsoid of semi-axes a, b, c, with $a \le b \le c$, and whose length is less than that of an ellipse of semi-axes a, b, then it lies in an open semi-ellipsoid.*

2529. [2000 : 178] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $G = \{A_1, A_2, \ldots, A_n\}$ be a set of points on a unit hemisphere. Let $\widehat{A_iA_j}$ be the spherical distance between the points A_i and A_j. Suppose that $\widehat{A_iA_j} \ge d$. Find $\max d$.

Solution by the proposer.

Let g be the spherical convex cover of G. It is obvious that for the perimeter L of g we have: $L(g) \ge nd$.

The condition on g, to be in the same hemisphere, is $L(g) \le 2\pi$. Therefore, g will be in the same hemisphere when $nd \le 2\pi$. So we must have

$$\max d \le \frac{2\pi}{n}.$$

There were no other solutions submitted for this problem.

2530. [2000 : 178] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let F be a compact convex set in $\mathbb{E}^3$, let T be the translation along a vector $\vec{a}$, and let $F' = T(F)$.

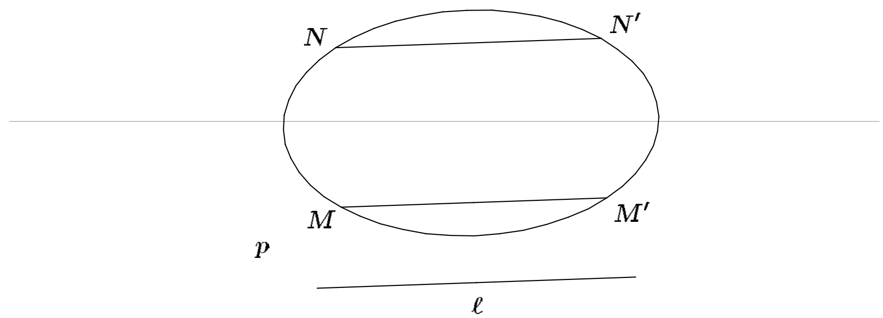
Prove that the intersection of the boundary of F and the boundary of F' is connected.

Solution by the proposer.

Let M, $M' \in F$ and $\overrightarrow{MM'} = \vec{a}$. Our problem is equivalent to proving that the locus of M' is connected.

We consider the line $l \parallel \vec{a}$ outside of F, and the support planes p_1, p_2 through ℓ to F. The dihedral angle of p_1, p_2 is denoted by ϑ; that is $\angle(p_1, p_2) = \vartheta$.

We also consider the plane p through ℓ moving into the dihedral angle $\angle(p_1, p_2)$. There are two positions of p so that the maximum diameters AB, $A'B'$ of c are parallel to ℓ and equal to $|\vec{a}|$. Between these two positions, the plane p intersects F along c', and there are two chords of c', MM' and NN' so that $\overrightarrow{MM'} = \overrightarrow{NN'} = \vec{a}$.



Suppose MM' is between NN' and ℓ. Denote the boundary of F by $\partial(F)$. Then the function $f : \vartheta \to \partial(F)$, so that for $0 \leq \vartheta \leq \vartheta_0$, $\vartheta \to M$ and $p = p(\vartheta)$ is continuous, so the point M describes a curve k_1 with endpoints A, A' on $\partial(F)$. Similarly, the point N describes a curve k_2 so that $k_1 \cap k_2 = \{A, A'\}$. We also easily see that M', N' describe curves k_1', k_2' so that $k_1' = T(k_1)$, $k_2' = T(k_2)$, $k_1' \cup k_2' = T(k_1 \cup k_2)$ and $k_1' \cup k_2' = \partial(F) \cap \partial(F')$.

The interval $[0, \vartheta]$ is connected (and compact), and f is a continuous function. Hence, according to a well-known theorem of topology,

$$k_1' \cup k_2' = \partial(F) \cap \partial(F') \text{ is connected.}$$

Comment: Using the same technique we can prove the following problem:

Suppose that η is the support line of F that is parallel to $\vec{a}$ and that $G = \{M | M \in F \cap \eta\}$. Then the point set G is connected.

There were no other solutions submitted.

2532. [2000 : 178] *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Suppose that a, b and c are positive real numbers satisfying $a^2 + b^2 + c^2 = 1$. Prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq 3 + \frac{2(a^3 + b^3 + c^3)}{abc}.$$

I. *Solution by Richard B. Eden, Ateneo de Manila University, Manila, The Philippines.*

$$\begin{aligned}
&\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-3-\frac{2\left(a^3+b^3+c^3\right)}{abc}\\
&\quad= \frac{a^2+b^2+c^2}{a^2}+\frac{a^2+b^2+c^2}{b^2}+\frac{a^2+b^2+c^2}{c^2}\\
&\qquad -3-2\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right)\\
&\quad= a^2\left(\frac{1}{b^2}+\frac{1}{c^2}\right)+b^2\left(\frac{1}{c^2}+\frac{1}{a^2}\right)+c^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\\
&\qquad -2\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right)\\
&\quad= a^2\left(\frac{1}{b}-\frac{1}{c}\right)^2+b^2\left(\frac{1}{c}-\frac{1}{a}\right)^2+c^2\left(\frac{1}{a}-\frac{1}{b}\right)^2 \;\ge\; 0\,,
\end{aligned}$$

with equality if and only if $a=b=c$.

II. *Solution by Goran Conar, student, University of Zagreb, Croatia.*

The given inequality is equivalent in sequence to

$$\frac{1-a^2}{a^2}+\frac{1-b^2}{b^2}+\frac{1-c^2}{c^2} \;\ge\; \frac{2\left(a^3+b^3+c^3\right)}{abc}$$

or

$$bc\left(\frac{b^2+c^2}{a}\right)+ca\left(\frac{c^2+a^2}{b}\right)+ab\left(\frac{a^2+b^2}{c}\right) \;\ge\; 2\left(a^3+b^3+c^3\right)$$

or

$$a^3\left(\frac{b}{c}+\frac{c}{b}-2\right)+b^3\left(\frac{c}{a}+\frac{a}{c}-2\right)+c^3\left(\frac{a}{b}+\frac{b}{a}-2\right) \;\ge\; 0\,,$$

which is true since $\frac{x}{y}+\frac{y}{x}\ge 2$ for all positive numbers x and y.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; the AUSTRIAN IMO Team 2000 (2 solutions); MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; JONATHAN CAMPBELL, Chapel Hill High School, Chapel Hill, NC, USA; MIGUEL CARRIÓN ÁLVAREZ, Universidad Complutense de Madrid, Spain; MAUREEN P. COX and ALBERT WHITE, Bonaventure University, St. Bonaventure, NY, USA; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES DIMINNIE and TREY SMITH, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; OLEG IVRII, student (grade 8), Cummer Valley Middle School, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; PHIL McCARTNEY, Northen Kentucky University, Highland Heights, KY, USA; VEDULA N. MURTY, Visakhapatnam, India; JUAN-BOSCO ROMERO MÁRQUEZ, *Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer (2 solutions)*

About half of the submitted solutions use the AM-GM Inequality in one way or another. The two solutions featured above are deviations from this. Solution I shows that, in fact, the inequality holds for all non-zero a, b, and c. This was explicitly pointed out only by the Austrian IMO Team 2000, Leversha, and Murty.

2536. [2000 : 179] *Proposed by Cristinel Mortici, Ovidius University of Constanta, Romania.*

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and periodic function such that for all positive integers n the following inequality holds:

$$\frac{|f(1)|}{1} + \frac{|f(2)|}{2} + \dots + \frac{|f(n)|}{n} \leq 1\,.$$

Prove that there exists $c \in \mathbb{R}$ such that $f(c) = 0$ and $f(c+1) = 0$.

I. *Solution by Michel Bataille, Rouen, France.*

Let $T > 0$ be a period of f. Since the function $|f|$ is continuous on $[0, T]$, there exists M such that $|f(x)| \leq M$ for all $x \in [0, T]$. By periodicity, we even have $|f(x)| \leq M$ for all $x \in \mathbb{R}$. Consider now the continuous, T-periodic function g defined by

$$g(x) \;=\; |f(x)| + |f(x+1)|\,.$$

We are required to show that there exists c such that $g(c) = 0$. Assume, for the purpose of contradiction, that $g(x) \neq 0$ for all x. Then, since g is a positive continuous function, it achieves a minimum $m > 0$ on $[0, T]$ and, by periodicity, we get $g(x) \geq m$ for all $x \in \mathbb{R}$. For $n = 1, 2, \dots$, let us define

$$S_n \;=\; \sum_{k=1}^{n} \frac{g(k)}{k}\,.$$

For any positive integer n we have, on the one hand,

$$S_n \;\geq\; m \sum_{k=1}^{n} \frac{1}{k}\,,$$

and, on the other hand,

$$\begin{aligned} S_n \;&=\; \sum_{k=1}^{n} \frac{|f(k)|}{k} + \sum_{k=1}^{n} \frac{|f(k+1)|}{k} \\ &=\; \sum_{k=1}^{n} \frac{|f(k)|}{k} + \sum_{k=1}^{n} \frac{|f(k+1)|}{k+1} + \sum_{k=1}^{n} \frac{|f(k+1)|}{k(k+1)} \\ &\leq\; 1 + 1 + M \sum_{k=1}^{n} \frac{1}{k(k+1)} \;\leq\; M + 2\,. \end{aligned}$$

(We have used the hypothesis and $\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1} \leq 1$.) Thus, we obtain $\sum_{k=1}^{n} \frac{1}{k} \leq \frac{M+2}{m}$ for all positive integers n, a clear contradiction to the well-known divergence of the harmonic series. The desired result follows.

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Case (i): Let $p \in \mathbb{Q}^+$ be the period of f; that is, $p = \alpha/\beta$ with $\alpha, \beta \in \mathbb{N}$, and $f(j) = f(j+\beta p) = f(j+\alpha)$ for all $j \in \mathbb{N}$ (even for all $j \in \mathbb{R}$). Assume there is a number $j \in \mathbb{N}$ such that $|f(j)| = \lambda > 0$. Then, because of the boundary condition of f, we have

$$\sum_{k=0}^{n} \frac{\lambda}{j+k\alpha} \;\leq\; \sum_{\ell=1}^{k\alpha+j} \frac{|f(\ell)|}{\ell} \;\leq\; 1\,,$$

a contradiction to $\sum\limits_{k=1}^{\infty} \frac{1}{j+k\alpha} = \infty$. Therefore, $f(x) = 0$ for all $x \in \mathbb{N}$ and we are done.

Case (ii): Let $p \in \mathbb{R}^+ \backslash \mathbb{Q}^+$ be the period of f. Then, the fractional part of the sequence $(np)_{n\geq 1}$ (that is, the sequence $\{a_n\}_{n\geq 1}$ where $a_n = np - \lfloor np \rfloor$) lies dense and equally distributed in the interval $[0,1)$. Therefore, for any interval of length ε (and contained in $[0,1)$), roughly speaking, every $1/\varepsilon^{\text{th}}$ member of the sequence a_n defined above lies in this interval. Assume that $f(c) \neq 0$ for some $c \in \mathbb{N}$. Then, $|f(c)| = \alpha > 0$. By continuity, there is an $\varepsilon > 0$ such that $|f(x)| > \alpha/2$ whenever $x \in \left(c - \frac{\varepsilon}{2}, c + \frac{\varepsilon}{2}\right)$. But then, the sums defined by

$$\sum_{j=0}^{n} \frac{|f(c_j)|}{c_j}\,, \qquad n \to \infty\,,$$

with $c_0 = c$, $c_j = c + jp$ (rounded), have to diverge. Thus, $f(c) = 0$ for all $c \in \mathbb{N}$. This in turn implies that $f(x) \equiv 0$ for $x \in \mathbb{R}$.

Also solved by the AUSTRIAN IMO TEAM 2000; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KEE-WAI LAU, Hong Kong; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.