SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2426. [1999: 172] *Proposed by Mohammed Aassila, Strasbourg, France.*

(a) Show that there are two polynomials, $p(x)$ and $q(x)$, both having three integer roots and such that $p(x) - q(x)$ is a non-zero constant.

(b)* Do there exist two polynomials, $p(x)$ and $q(x)$, both having $n > 3$ integer roots and such that $p(x) - q(x)$ is a non-zero constant?

Combination of solutions by Michael Lambrou, University of Crete, Crete, Greece, and Kenneth M. Wilke, Topeka, KS, USA.

This problem is essentially a special case of the Tarry–Escott or Prouhet–Tarry–Escott problem, which requires finding solutions of the system of equations

$$\begin{aligned} a_1 + a_2 + \cdots + a_n &= b_1 + b_2 + \cdots + b_n \\ a_1^2 + a_2^2 + \cdots + a_n^2 &= b_1^2 + b_2^2 + \cdots + b_n^2 \\ &\vdots \\ a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} &= b_1^{n-1} + b_2^{n-1} + \cdots + b_n^{n-1} \end{aligned} \tag{1}$$

in integers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$, where $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are not a permutation of each other. By setting

$$p(x) = (x - a_1)(x - a_2)\cdots(x - a_n) = x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n$$

and

$$q(x) = (x - b_1)(x - b_2)\cdots(x - b_n) = x^n + B_1x^{n-1} + \cdots + B_{n-1}x + B_n ,$$

we can use Newton's relations regarding the roots of an equation to determine the sums of the powers of the roots of $p(x)$; that is, putting

$$S_j = a_1^j + a_2^j + \cdots + a_n^j \qquad \text{for} \quad 1 \le j \le n ,$$

we have

$$\begin{aligned} S_1 + A_1 &= 0 \\ S_2 + A_1S_1 + 2A_2 &= 0 \\ &\vdots \\ S_n + A_1S_{n-1} + \cdots + A_{n-1}S_1 + nA_n &= 0 . \end{aligned}$$

[See for example Ed Barbeau's book *Polynomials*, page 199, exercise 6.—*Ed*.] Similarly we have for $q(x)$:

$$\begin{aligned} T_1 + B_1 &= 0 \\ T_2 + B_1T_1 + 2B_2 &= 0 \\ &\vdots \\ T_n + B_1T_{n-1} + \cdots + B_{n-1}T_1 + nB_n &= 0\,, \end{aligned}$$

where

$$T_j \;=\; b_1^j + b_2^j + \cdots + b_n^j \qquad \text{for} \quad 1 \le j \le n\,.$$

The condition that $p(x) - q(x)$ is a non-zero constant amounts to saying that $A_1 = B_1$, $A_2 = B_2$, ..., $A_{n-1} = B_{n-1}$. It is easy to see that this reduces to finding solutions of the equations $S_i = T_i$ for $1 \le i \le n-1$; that is, (1). Conversely if (1) holds, then so do $A_i = B_i$ for $1 \le i \le n-1$. Moreover $p \ne q$ requires that the roots a_1, ..., a_n of p and b_1, ..., b_n of q are not a permutation of each other.

An excellent summary of the known results regarding this problem may be found on the Internet at

http://mathworld.wolfram.com/Prouhet-Tarry-EscottProblem.html

(which also contains a large bibliography) or on Chen Shuwen's webpage at

http://member.netease.com/~chin/eslp/TarryPrb.htm

[Readers without access to the Internet could consult Hardy and Wright's *An Introduction to the Theory of Numbers*, 5th ed. (1979), § 21.9 and 21.10, pp. 328–332, 338.]

For example, to answer part (a), let

$$p(x) \;=\; (x-a)(x-b)(x+a+b) \quad \text{and} \quad q(x) \;=\; (x+a)(x+b)(x-a-b)\,,$$

where a and b are arbitrarily chosen non-zero integers such that $a \ne -b$. Then $p(x) - q(x) = 2ab(a+b)$, so that we have an infinite family of solutions. However, these are not the only solutions possible for part (a).

Solutions for (1), and hence the present problem, are known for all n from 3 to 10 (see the second website given above). For example, for $n = 6$ we could use

$$p(x) \;=\; x^2(x^2-49)^2\,, \qquad q(x) \;=\; (x^2-9)(x^2-25)(x^2-64)\,,$$

so that $p(x) - q(x) = 14400$. Recently (1999) Chen Shuwen gave an example with $n = 12$, from which we may write

$$\begin{aligned} p(x) \;=\;& x(x-11)(x-24)(x-65)(x-90)(x-129)(x-173) \\ &\times (x-212)(x-237)(x-278)(x-291)(x-302) \end{aligned}$$

and

$$q(x) \;=\; (x-3)(x-5)(x-30)(x-57)(x-104)(x-116)(x-186) \times (x-198)(x-245)(x-272)(x-297)(x-299)\,.$$

No example is known for $n = 11$ or for n greater than 12. However, it has been conjectured that there is a solution for any integer $n \geq 3$.

Also solved by FEDERICO ARBOLEDA and BERNARDO RECAMÁN SANTOS, Bogotá, Colombia; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE and TREY SMITH, Angelo State University, San Angelo, Texas, USA; DOUGLASS L. GRANT, University College of Cape Breton, Sydney, Nova Scotia; G.P. HENDERSON, Garden Hill, Campbellcroft, Ontario; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; PETER HURTHIG, Columbia College, Vancouver, BC; and DIGBY SMITH, Mount Royal College, Calgary, Alberta. All these gave a solution for at least $n = 3$ and 4. Part (a) *only was solved by HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.*

WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria, and MAX SHKARAYEV and MARK LYON, students, University of Arizona, Tuczon, AZ, USA, gave solutions in which $p(x)$ and $q(x)$ have degrees larger than n, which was not explicitly prohibited in the statement of the problem (but should have been). Their solutions are almost the same, namely they let $p(x)$ be a polynomial (which can be of degree $2n-1$) satisfying

$$p(0) \;=\; p(1) \;=\; \cdots \;=\; p(n{-}1) \;=\; 0\,, \quad p(n) \;=\; p(n{+}1) \;=\; \cdots \;=\; p(2n{-}1) \;=\; 1\,,$$

and let $q(x) = p(x) - 1$ (Diminnie and Smith also mention this interpretation). Can anyone give (for all $n \geq 3$) polynomials $p(x)$ and $q(x)$, of degree smaller than $2n-1$, each with (at least) n integer roots, and so that $p(x) - q(x)$ is a non-zero constant?

Many solvers pointed out that if there is a solution for any value of n, then there are infinitely many solutions, because the polynomials can be translated by an arbitrary integer: if $p(x)$ and $q(x)$ work, so do $p(x-c)$ and $q(x-c)$ for any integer c. Thus we could assume that $p(0) = 0$, say. Even with this restriction, infinite families of solutions are known for some values of n. The above solution for part (a) *is one such, and Henderson found such families for $n = 3, 4, 5, 6$. His solution for $n = 6$ contains the solution given above:*

$$p(x) \;=\; x^2[x^2-(3r^2+s^2)^2]^2\,,$$

$$q(x) \;=\; (x^2-16r^2s^2)[x^2-(3r^2-2rs-s^2)^2][x^2-(3r^2+2rs-s^2)^2]\,,$$

for which $p(x) - q(x) \;=\; 16r^2s^2(r^2-s^2)^2(9r^2-s^2)^2$. This is a solution for all distinct positive integers r, s for which $s \neq 3r$. The solution given above is the case $r = 1$, $s = 2$. Other solvers found infinite families for $n = 3$ and 4; for example, for $n = 4$ we could use

$$p(x) \;=\; x^2(x^2-a^2)\,, \qquad q(x) \;=\; (x^2-b^2)(x^2-c^2)$$

where (a, b, c) is any Pythagorean triple*; that is, positive integers satisfying $a^2 = b^2 + c^2$.*

2427. [1999: 172] *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Suppose that G is the centroid of triangle ABC, and that

$$\angle GAB + \angle GBC + \angle GCA \;=\; 90^\circ\,.$$

Characterize triangle ABC.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

Let $x = \cot A$, $y = \cot B$, $z = \cot C$. Since $A + B + C = 180°$, then $\cot(B + C) = \cot(180° - A)$, which can be written as

$$\frac{\cot B \cot C - 1}{\cot B + \cot C} \;=\; -\cot A\,,$$

or

$$xy + yz + zx \;=\; 1\,. \tag{1}$$

Let $\angle GAB = \alpha$, $\angle GBC = \beta$, $\angle GCA = \gamma$. Then $\alpha + \beta + \gamma = 90°$ implies $\cot(\beta + \gamma) = \cot(90° - \alpha)$, which can be written as

$$\frac{\cot\beta\cot\gamma - 1}{\cot\beta + \cot\gamma} \;=\; \frac{1}{\cot\alpha}\,,$$

or

$$\cot\alpha + \cot\beta + \cot\gamma \;=\; \cot\alpha\cot\beta\cot\gamma\,. \tag{2}$$

Let M be the mid-point of the segment BC. By the Law of Sines for $\triangle ABM$ and $\triangle ABC$,

$$\frac{BM}{\sin\alpha} \;=\; \frac{AB}{\sin(B+\alpha)} \quad\text{and}\quad \frac{BC}{\sin A} \;=\; \frac{AB}{\sin(B+A)}\,.$$

Since $BC = 2BM$ we obtain

$$\frac{\sin(B+\alpha)}{\sin B\sin\alpha} \;=\; \frac{2\sin(B+A)}{\sin B\sin A}\,.$$

This implies $\cot B + \cot\alpha = 2(\cot B + \cot A)$; that is, $\cot\alpha = 2x + y$. Similarly, $\cot\beta = 2y + z$, and $\cot\gamma = 2z + x$. Substitution in (2) yields

$$3(x + y + z) \;=\; (2x + y)(2y + z)(2z + x)\,.$$

Using the equality (1),

$$3(x + y + z)(xy + yz + zx) \;=\; (2x + y)(2y + z)(2z + x)\,,$$

which transforms to

$$(x - y)(y - z)(z - x) \;=\; 0\,.$$

It follows that $x = y$, or $y = z$, or $z = x$. Therefore, the triangle is isosceles.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; VEDULA N. MURTY, Dover, PA, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There were also two incorrect solutions submitted.

2428. [1999: 172] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given triangle ABC with $\angle BAC = 90^\circ$. The incircle of triangle ABC touches AB and AC at D and E respectively. Let M be the mid-point of BC, and let P and Q be the incentres of triangles ABM and ACM respectively. Prove that

1. $PD \parallel QE$;
2. $PD^2 + QE^2 = PQ^2$.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

Let the incircle of $\triangle ABC$ touch BC at the point F. Then $BD = BF$ and since BP bisects the angle ABC, $PD = PF$ and $\angle PDB = \angle PFB$. Similarly, $QE = QF$ and $\angle QEC = \angle QFC$.

1. Let PD meet AC at G. We shall prove that $\angle QEC = \angle DGA$. Since $\triangle ABM$ is isosceles, MP is the perpendicular bisector of the segment AB. Thus MP meets AB at the point K, where $AK = KB$, and we obtain

$$PK \;=\; BK\tan\frac{B}{2} \;=\; BM\tan\frac{B}{2}\cos B \;=\; BC\tan\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}\,.$$

We know that

$$BD \;=\; \frac{AB+BC-AC}{2}\,,$$

so that

$$KD \;=\; \frac{BC-AC}{2} \;=\; \frac{BC(1-\cos C)}{2} \;=\; BC\sin^2\frac{C}{2}\,.$$

Let $\omega = \angle PDB$ and $\varphi = \angle QEC$. Then

$$\tan\omega \;=\; \frac{PK}{KD} \;=\; \frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}\,.$$

Similarly,

$$\tan\varphi \;=\; \frac{\tan\frac{C}{2}}{\tan\frac{B}{2}}\,.$$

Then $\tan\omega\tan\varphi = 1$ and therefore $\omega + \varphi = 90^\circ$.

Finally, $\angle QEC = \varphi = 90^\circ - \omega = \angle DGA$, which shows that $PD \parallel QE$.

2. Since $\angle PFQ = 180^\circ - \angle PFM - \angle QFC = 180^\circ - \omega - \varphi = 90^\circ$, then

$$PD^2 + QE^2 \;=\; PF^2 + QF^2 \;=\; PQ^2\,,$$

which completes the proof.

Also solved by ŠEFKET ARSLANAGIĆ, *University of Sarajevo, Sarajevo, Bosnia and Herzegovina;* SAM BAETHGE, *Nordheim, TX, USA;* MICHEL BATAILLE, *Rouen, France;* FRANCISCO BELLOT ROSADO, *I.B. Emilio Ferrari, Valladolid, Spain;* CHRISTOPHER

J. BRADLEY, Clifton College, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2429. [1999: 172] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Suppose that D, E and F are points on the side AB (or its production) of triangle ABC. Suppose further that CD is a median, that CE is the bisector of $\angle ACB$, and that CF is its external bisector.

The circumcircle, Γ, of triangle EFC intersects CD again at P. Suppose that Γ_A and Γ_B are the circumcircles of triangles CPA and CPB respectively.

Show that Γ_A and Γ_B are tangent to AB at A and B respectively.

Solution by Toshio Seimiya, Kawasaki, Japan.

We may assume that $CA > CB$. [If $CA = CB$, then $D = E$, and F is at infinity.] Since CE and CF are the interior and exterior bisectors of $\angle ACB$ we have

$$AE : EB \;=\; CA : CB \;=\; AF : BF\,. \tag{1}$$

We put $AD = DB = x$, $DE = e$, and $DF = f$. In this notation (1) becomes

$$(x+e):(x-e) \;=\; (x+f):(f-x)\,;$$

that is, $(x+e)(f-x) = (x-e)(x+f)$. It follows that

$$x^2 = ef\,. \tag{2}$$

Since C, P, E, F are concyclic we get $DP \cdot DC = DE \cdot DF = ef$. Hence we have from (2)

$$DA^2 \;=\; DP \cdot DC\,, \tag{3}$$

and

$$DB^2 \;=\; DP \cdot DC\,. \tag{4}$$

From (3) $\Gamma_A = CPA$ touches AB at A, and from (4) $\Gamma_B = CPB$ touches AB at B.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2430. [1999: 172] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Points A and B lie outside circle Γ. Find a point C on Γ with the following property:

AC and BC intersect Γ again at D and E respectively, with $DE \parallel AB$.

Solution by Eckard Specht, Magdeburg, Germany.

Let u and v be the respective lengths of the tangential segments to Γ from A and from B. By Euclid III.36 we know that $u^2 = AC \cdot AD$ and $v^2 = BC \cdot BE$. It follows that

$$\frac{u^2}{v^2} = \frac{AC}{BC} \cdot \frac{AD}{BE} . \tag{1}$$

Since $DE||AB$ we have $\dfrac{AC}{BC} = \dfrac{AD}{BE}$, and with (1),

$$\frac{AC}{BC} = \frac{u}{v} := q .$$

A, B, and Γ are fixed, so that q is constant. The locus of points C whose distances to given points A, B are in the same ratio is known as the ***circle of Apollonius***. So we have to divide the segment AB internally and externally in the ratio q to get the points X and Y as a diameter of the Apollonian circle. The intersection of the circle with Γ determines the two possible locations of the desired point C.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; TOSHIO SEIMIYA, Kawasaki, Japan; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Several solvers noted that since C is easily seen to be on a circle through A and B that is tangent to Γ, our problem can quickly be reduced to that special case of the problem of Apollonius. *It was an amusing coincidence that our featured solution made use of another idea that is due to Apollonius. There is a vast literature on all aspects of the Apollonius problem, where a variety of solutions to our special case can be found.*

2432. [1999: 173] *Proposed by K.R.S. Sastry, Bangalore, India.*

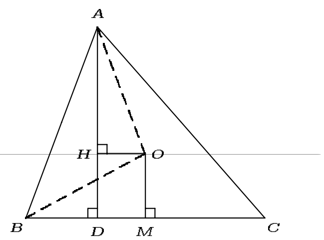
In $\triangle ABC$, we use the standard notation: O is the circumcentre, H is the orthocentre. Let M be the mid-point of BC, $OH = m$, $OM = n$ ($m, n \in \mathbb{N}$), and suppose that $OH \parallel BC$.

How many sides of $\triangle ABC$ can have integer lengths?

Solution by the proposer.

Answer: at most two sides.

It is known that $AH = 2OM$. If a, b, c are all integers, then $[ABC] = \frac{1}{2}BC \cdot AD$ is rational, so that all of $\sin A$, $\sin B$, $\sin C$, $\cos A$, $\cos B$ and $\cos C$ are rational; that is, $\triangle ABC$ is a Heron triangle.



$OH = m \quad OM = n$

$BO = AO = R = \sqrt{m^2 + 4n^2}$

$BM = \sqrt{m^2 + 3n^2}$

$AH = 2n$

Since $a = 2R\sin A$, the circumradius R is rational as well.

We have $R^2 = m^2 + (2n)^2$. Therefore $R = u^2 + v^2$, $2n = 2uv$ and $m = u^2 - v^2$ by the well-known Pythagorean solution.

But then, $BM^2 = m^2 + 3n^2 = (u^2 - v^2)^2 + 3u^2v^2 = u^4 + u^2v^2 + v^4$ must be the square of an integer. This is known to be impossible. See, for example, L.E. Dickson, *History of the Theory of Numbers*, Vol. II, Chelsea, NY (1971) p. 636.

Hence all three side lengths cannot be integers.

However, two of these can be integral. To see this, set $m = 23$, $n = 40$. Then $BM = 73$, $BC = 146$, $AB^2 = AD^2 + BD^2 = 120^2 + (73 - 23)^2 = 130^2$, so that $AB = 130$.

Also solved by MICHAEL LAMBROU, University of Crete, Crete, Greece; AND GERRY LEVERSHA, St. Paul's School, London, England. There was one incomplete solution.

Lambrou's solution was essentially the same as the proposer's, but took twice as long. Lambrou remarked that a parametric family of triangles can be taken with

$$m = 14(st)^2 - s^4 - t^4, \qquad n = 4(s^2 - t^2)st,$$

giving $a = 2(s^4 + 10(st)^2 + t^4)$ *and* $b = 12st(s^2 + t^2)$. *Again, we have* $OH \| BC$.

WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria, commented on a characterization of triangles ABC *satisfying* $OH \| BC$, *using the (unfortunately not too well-known) fact:*

> *If* O *is the centre of the circumcircle of* $\triangle ABC$ *and also the origin of a rectangular system of coordinates, then* H *is given by* $H = A + B + C$.

Thus, $OH \| BC$ *is equivalent to the existence of a real number* λ *such that* $\overrightarrow{OH} = \lambda\overrightarrow{BC}$, *which, in turn, is equivalent to* $A + B + C = \lambda(C - B)$, *or*

$$A = (\lambda - 1)C - (\lambda + 1)B. \qquad \star$$

Therefore, if $\triangle ABC$ *is in "general position", we have to replace* A, B, C *in* $(\star)$ *by* $A - O$, $B - O$, $C - O$ *respectively, yielding* $OH \| BC$ *if and only if there exists a* $\lambda \in \mathbb{R}$ *such that*

$$A = (\lambda - 1)C - (\lambda + 1)B + 3(O).$$

2433. [1999: 173] *Proposed by K.R.S. Sastry, Bangalore, India.*

In $\triangle ABC$, let e denote the length of the segment of the Euler line between the orthocentre and the circumcentre.

Prove or disprove that $\triangle ABC$ is right angled if and only if e equals one half of the length of one of the sides of $\triangle ABC$.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

We use the well-known formula

$$e^2 = OH^2 = R^2(1 - 8\cos A \cos B \cos C)\,.$$

[Ed. O is the circumcentre, H is the orthocentre and R is the circumradius.]

If $A = 90^\circ$, then $\cos A = 0$, $\sin A = 1$ and $e = R = \frac{a}{2\sin A} = \frac{a}{2}$.

Similarly, if either B or $C = 90^\circ$, then $e = \frac{b}{2}$ or $\frac{c}{2}$ respectively.

On the other hand, if $e = \frac{a}{2}$, then $e^2 = \frac{a^2}{4} = R^2 \sin^2 A$.

Hence, $\sin^2 A = 1 - 8\cos A \cos B \cos C = 1 - \cos^2 A$.

Therefore $\cos A = 0$ or $\cos B \cos C = \frac{1}{8}\cos A$.

Thus, the right angle is sufficient but not necessary for e to be equal to one half of the length of one of the sides of triangle ABC.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

The proposer notes that the claim would be true if we insist that triangle ABC has integer sides. He provides a proof, but the editor would like to see if another reader can solve this — see problem **2533**.

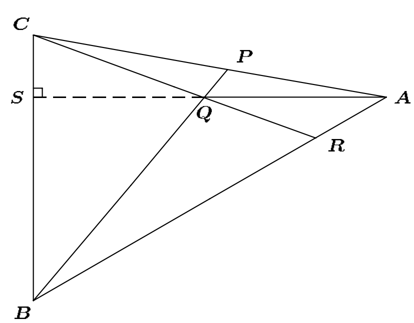
2434. [1999: 173] *Proposed by K.R.S. Sastry, Bangalore, India.*

In $\triangle ABC$, let $\angle ABC = 60^\circ$. Point P is on the line segment AC such that $\angle CBP = \angle BAC$. Point Q is on the line segment BP such that $BQ = BC$.

Prove that Q lies on the altitude through A of $\triangle ABC$ if and only if $\angle BAC = 40^\circ$.

I. *Solution by the proposer.*

Extend CQ to meet AB at R. Let AS be the altitude from A. We will show that AS contains the point Q.



We calculate the following angles:

$$\angle BAS = 30^\circ\,,\quad \angle SAC = A - 30^\circ\,,\quad \angle ABP = 60^\circ - A\,,\quad \angle BCR = 90^\circ - \tfrac{A}{2}\,.$$

Now, using $[XYZ]$ to denote the area of $\triangle XYZ$, etc.,

$$\frac{BS}{SC} = \frac{[ABS]}{[ASC]} = \frac{\frac{1}{2}AB\cdot AS\sin 30^\circ}{\frac{1}{2}AC\cdot AS\sin(A-30^\circ)} = \frac{AB\sin 30^\circ}{AC\sin(A-30^\circ)}\,.$$

Likewise,

$$\frac{CP}{PA} = \frac{BC\sin A}{AB\sin(60^\circ - A)}\,,\qquad\text{and}\qquad \frac{AR}{RB} = \frac{AC\sin(30^\circ-\frac{A}{2})}{BC\cos\frac{A}{2}}\,.$$

By Ceva's Theorem, AS, BP and CR are concurrent if and only if $\dfrac{BS}{SC}\cdot\dfrac{CP}{PA}\cdot\dfrac{AR}{RB} = 1$. That is,

$$(\sin 30^\circ)(\sin A)\left(\sin(30^\circ-\tfrac{A}{2})\right) = (\sin(A-30^\circ))(\sin(60^\circ-A))(\cos\tfrac{A}{2})\,,$$

which becomes

$$(\tfrac{1}{2})(2\sin\tfrac{A}{2}\cos\tfrac{A}{2}) = \sin(A-30^\circ)\left(2\sin(30^\circ-\tfrac{A}{2})\cos(30^\circ-\tfrac{A}{2})\right)\cos\tfrac{A}{2}\,.$$

On simplification, this becomes

$$\sin\tfrac{A}{2} = 2\sin(A-30^\circ)\cos\left(30^\circ-\tfrac{A}{2}\right) = \sin\tfrac{A}{2} + \sin\left(\tfrac{3}{2}A - 60^\circ\right)\,.$$

Hence we conclude that the lines AS, BP and CR are concurrent if and only if $\sin(\frac{3}{2}A - 60^\circ) = 0$; that is, if and only if $\frac{3}{2}A - 60^\circ = 0^\circ$ or 180°; that is, if and only if $\angle BAC = 40^\circ$ or 160°. However, $\angle ABC = 60^\circ$, forcing $\angle BAC = 40^\circ$.

II. *Solution by D.J. Smeenk, Zaltbommel, the Netherlands.*

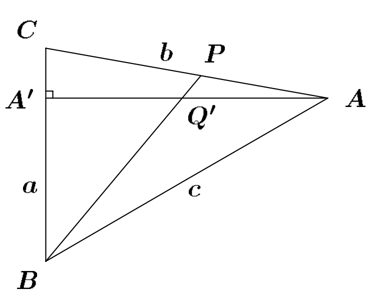


Figure 1.

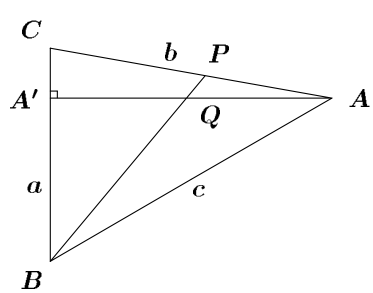


Figure 2.

First assume that $\angle BAC = 40^\circ$ (Figure 1.). Since $\angle ABC = 60^\circ$, we know that $\angle ACB = 80^\circ$. We denote by A' the foot of the perpendicular from A to BC, and let Q' be the intersection of AA' and BP.

Now, in $\triangle BA'Q'$, we have $BA' = \frac{1}{2}c$ and $BQ' = \frac{c}{2\cos 40^\circ}$.

Also, in $\triangle ABC$, we have $\frac{c}{a} = \frac{\sin 80^\circ}{\sin 40^\circ}$, which implies that $c = 2a\cos 40^\circ$. Hence $BQ' = a$, so that $Q' = Q$, telling us that the point Q lies on AA'.

Now assume that Q lies on AA' (Figure 2). We need to show that $\angle BAC = 40$.

From $BA' = \frac{1}{2}c$ and $BQ = \frac{c}{2\cos A}$, we obtain, using the Law of Sines, that $\sin C = 2\sin A\cos A = \sin 2A$.

Hence either $C = 2A$, in which case (since $B = 60^\circ$), we obtain $A = 40^\circ$, or $C = 180^\circ - 2A$ (in which case we obtain $A = B = C = 60^\circ$).

Editor's comment.

In the original proposal of this problem by SASTRY, the wording describing the position of point P was different: "The point P is in AC ... ". This clearly suggests that the proposer thought of P as lying strictly between A and C. The wording of 2434 was changed to read "Point P is on the line segment AC ... ", which allows for P to assume either end-point position $P = A$ or $P = B$. Unfortunately, the wording in CRUX, makes the problem incorrect, for in this case, when $BAC = 60^\circ$ (thus making the triangle equilateral), it happens that $P = Q = A$, and so the altitude through A naturally passes through Q.

Both MICHAEL LAMBROU, University of Crete, Crete, Greece, and NIKOLAOS DERGIADES, Thessaloniki, Greece, were explicit in their solution in pointing out this error in the wording. As well, SMEENK, WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria, SAM BAETHGE, Nordheim, TX, USA, and GERRY LEVERSHA, St. Paul's School, London, England, provided correct solutions in which they recognized that the equilateral triangle provided a trivial case in which the result of the problem held.

The solutions submitted by HEINZ-JÜRGEN SEIFFERT, Berlin, Germany, ÀNGEL JOVAL ROQUET, La Seu d'Urgell, Spain, KEE-WAI LAU, Hong Kong, and TOSHIO SEIMIYA, Kawasaki, Japan, correctly solved the problem as it had been intended by the proposer, even though they either failed to recognize the degenerate case, or dismissed it out of hand. Partial solutions were submitted by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK, VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA, and PETER Y. WOO, Biola University, La Mirada, CA, USA.

Most of the solutions used the Law of Sines in one form or another. The exception to this was the solution by the proposer, which uses Ceva's Theorem.

2435. [1999: 173] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Show that, for $x > 0$, the following functions are increasing:

$$f(x) = \frac{\left(1+\frac{1}{x}\right)^x}{(1+x)^{\frac{1}{x}}} \quad \text{and} \quad g(x) = \left(1+\frac{1}{x}\right)^x - (1+x)^{\frac{1}{x}}.$$

Composite of the solutions by Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and by Phil McCartney, Northern Kentucky University, Highland Heights, KY, USA.

Define $h(x) := \left(1+\frac{1}{x}\right)^x$ and $d(x) := \ln\left(1+\frac{1}{x}\right) - \frac{1}{1+x}$ for $x > 0$. Note that $h'(x) = h(x)\cdot d(x)$ and $h(x) > 0$. But $d'(x) = \frac{-1}{x(1+x)^2} < 0$ so

$d(x)$ decreases on $(0, \infty)$. However, $\lim\limits_{x\to\infty} d(x) = 0$, and so $d(x) > 0$ for all $x \in (0, \infty)$ which implies that $h'(x) > 0$ on $(0, \infty)$. Therefore, $h(x)$ is increasing and so $h\left(\frac{1}{x}\right)$ is decreasing on $(0, \infty)$. Since $f(x) = \frac{h(x)}{h\left(\frac{1}{x}\right)}$ and $g(x) = h(x) - h\left(\frac{1}{x}\right)$, both conclusions following immediately.

Also solved by MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; VEDULA N. MURTY, Visakhapatnam, India; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; MAX SHKARAYEV and MARK LYON, students, University of Arizona, Tucson, AZ; and the proposer.

Of course, it is a well-known fact in calculus that the function $(1 + \frac{1}{x})^x$ *is strictly increasing on* $(0, \infty)$. *This was explicitly pointed out by Lambrou.*

2436. [1999: 173] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Find all real solutions of

$$2\cosh(xy) + 2^y - \left[(2\cosh(x))^y + 2\right] \;=\; 0\,.$$

Solution by Nikolaos Dergiades, Thessaloniki, Greece; and by Heinz-Jürgen Seiffert, Berlin, Germany (combined by the editor).

Let $t = e^x$ and $p = y$. Then the given equation becomes

$$t^p + t^{-p} + 2^p \;=\; t + t^{-1} + 2\,.$$

This equality occurs in Michael Lambrou's solution of **2329**★ [1999 : 241].

Hence the only real solutions of the given equation are given by

1. $x = 0$, y arbitrary;
2. $y = 0$, x arbitrary;
3. $y = 1$, x arbitrary; and
4. $y = 2$, x arbitrary.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; and the proposer.

Lambrou, in fact, gave a complete self-contained proof, as well as referring to his previous solution.

2437. [1999: 173] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Let P be a point in the plane of triangle ABC. If the mid-points of the line segments AP, BP, CP all lie on the nine-point circle of triangle ABC, prove that P must be the orthocentre of triangle ABC.

Solution by Jeremy Young, student, Nottingham High School, Nottingham, UK.

P is the external centre of similitude of the circumcircle and the nine-point circle of $\triangle ABC$ (since A, B, C lie on the circumcircle while the mid-points of AP, BP, CP are assumed to lie on the nine-point circle). H (the orthocentre of $\triangle ABC$) is also the external centre of similitude of these two circles. (See, for example, H.S.M. Coxeter, *Introduction to Geometry*, section 5.2, p. 72; it is also implicit in Coxeter and S.L. Greitzer, *Geometry Revisited*, section 1.8, p. 20.) Because a pair of circles can have at most one external centre of similitude (*Introduction to Geometry*, p. 70), it follows that $P = H$, as desired.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.