PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 November 2000**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

For the information of proposers who submitted problems in 1998, we have either used them, or will not be using them.

2525. *Proposed by Antreas P. Hatzipolakis, Athens, Greece, and Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

In $\triangle ABC$, we have $B = 135^\circ - \frac{A}{2}$ and $C = 45^\circ - \frac{A}{2}$. Show that

(a) the centre V of the nine-point circle of $\triangle ABC$ lies on the side BC;

(b) if $A = 60^\circ$, then AV bisects angle A.

2526. *Proposed by K.R.S. Sastry, Dodballapur, India.*

In a triangle, prove that an internal angle bisector trisects an altitude if and only if the bisected angle has the measure $\pi/3$ or $2\pi/3$.

2527. *Proposed by K.R.S. Sastry, Dodballapur, India.*

Let AD, BE and CF be concurrent cevians of $\triangle ABC$. Assume that:

(a) AD is a median; (b) BE bisects $\angle ABC$; (c) BE bisects AD.

Prove that $CF > BE$.

2528. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Prove that every rectifiable centrosymmetric curve on a unit sphere in $\mathbb{E}^3$ has length greater than or equal to 2π.

2529. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $G = \{A_1, A_2, \ldots, A_n\}$ be a set of points on a unit hemisphere. Let $\widehat{A_iA_j}$ be the spherical distance between the points A_i and A_j. Suppose that $\widehat{A_iA_j} \geq d$. Find $\max d$.

2530. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let F be a compact convex set in $\mathbb{E}^3$, let T be the translation along a vector $\vec{a}$, and let $F' = T(F)$.

Prove that the intersection of the boundary of F and the boundary of F' is connected.

2531. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let F be a convex plane set and AB its diameter. The points A and B divide the perimeter of F into two parts, L_1 and L_2, say. Prove that

$$\frac{1}{\pi - 1} < \frac{L_1}{L_2} < \pi - 1\,.$$

2532. *Proposed by Ho-joo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Suppose that a, b and c are positive real numbers satisfying $a^2 + b^2 + c^2 = 1$. Prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq 3 + \frac{2(a^3 + b^3 + c^3)}{abc}\,.$$

2533. *Proposed by K.R.S. Sastry, Bangalore, India.*

In the integer sided $\triangle ABC$, let e denote the length of the segment of the Euler line between the orthocentre and the circumcentre.

Prove that $\triangle ABC$ is right angled if and only if e equals one half of the length of one of the sides of $\triangle ABC$.

Compare problem 2433. [1999 : 173, 2000 : 187]

2534. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that a is an integer and x and y natural numbers. Define $z_a(x, y) = \dfrac{x^2 + y^2 + a}{xy}$.

1. Show that there exist infinitely many values of a such that $z_a(x, y)$ is an integer for infinitely many pairs $(x, y) \in \mathbb{N}^2$.

2.* Is the set $E(a)$ of integers $z_a(x, y)$ as obtained above necessarily infinite? If the answer is "no", determine those a's which determine finite sets $E(a)$.

2535*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

1. Prove that neither of the integers $a(n) = 3n^2 + 3n + 1$ and $b(n) = n^2 + 3n + 3$ $(n \geq 1)$ has a divisor k such that $k \equiv 2 \pmod{3}$.
2. Prove or disprove that both of the sequences $\{a(n)\}$ and $\{b(n)\}$ $(n \geq 1)$ contain infinitely many primes.

2536. *Proposed by Cristinel Mortici, Ovidius University of Constanta, Romania.*

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and periodic function such that for all positive integers n the following inequality holds:

$$\frac{|f(1)|}{1} + \frac{|f(2)|}{2} + \ldots + \frac{|f(n)|}{n} \leq 1 .$$

Prove that there exists $c \in \mathbb{R}$ such that $f(c) = 0$ and $f(c+1) = 0$.

2537. *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

Find the exact value of $\cot\left(\frac{\pi}{7}\right) + \cot\left(\frac{2\pi}{7}\right) - \cot\left(\frac{3\pi}{7}\right)$.

2538. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

On a recent calculus test, students were asked to compute the arc length of a curve represented by a certain function $f(x)$, for $x = a$ to $x = b$, $a < b$. One of the students, a Mr. Fluke, simply calculated $f'(b) - f'(a)$, and obtained the correct answer.

Determine all real functions, $f(x)$, differentiable on some open interval I, such that, for all a, b satisfying $(a, b) \subset I$, the arc length of the curve $y = f(x)$, from $x = a$ to $x = b$ is equal to $f'(b) - f'(a)$.

2488. [1999: 431] **ADDENDUM**

The proposer, G. Tsintsifas, Thessaloniki, Greece, observes that we need to point out the necessary condition for a convex cover: $\sum_{k=1}^{n+1} \mu_k = 1$.