MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

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Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from this issue be submitted in time for issue 4 of 2001.

High School Problems

Editor: Adrian Chan, 1195 Harvard Yard Mail Center, Cambridge, MA, USA 02138-7501 <ahchan@fas.harvard.edu>

H269. Find the lengths of the sides of a triangle with 20, 28, and 35 as the lengths of its altitudes.

H270. Find all triangles ABC that satisfy

$$\sin(A-B)+\sin(B-C)+\sin(C-A) \;=\; 0\,.$$

H271. *Proposed by Ho-Joo Lee, student, Kwangwoon University, Seoul, South Korea.*

Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Let $n = \lfloor 1/(a - \lfloor a \rfloor) \rfloor$ for some positive real number a.

Show that $\lfloor (n+1)a \rfloor \equiv 1 \pmod{n+1}$.

H272. *Proposed by Ho-Joo Lee, student, Kwangwoon University, Seoul, South Korea.*

Let $\{a_1, a_2, a_3, \ldots, a_n\}$ be a set of real numbers. Let $s = a_1 + a_2 + \cdots + a_n$. Show that $\displaystyle\sum_{i=1}^{n}\sum_{k=1}^{n}(a_k - a_i)(s - a_i) \ \geq\ 0$.

Advanced Problems

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A245. Show that a polygon with fixed side lengths has maximal area when it can be inscribed in a circle.

A246. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

Given a triangle with angles A, B, C, circumradius R, and inradius r, prove that

$$1 + \frac{r}{R} \leq \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \ \leq\ \frac{17}{12} + \frac{r}{6R}\,.$$

A247. There are n flat convex planar surfaces in 3–space, and the sum of their areas is 1. Prove that there exists a plane so that the sum of the areas of their projections to the plane is less than $1/2$.

A248.

(a) Prove that in every sequence of 79 consecutive positive integers written in the decimal system, there is a positive integer whose sum of digits is divisible by 13.

(1997 Baltic Way)

(b) Give a sequence of 78 consecutive positive integers each with a sum of digits not divisible by 13.

Challenge Board Problems

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C98. Let H be a subset of the positive integers with the property that if x, $y \in H$, then $x + y \in H$. Define the *gap sequence* G_H of H to be the set of positive integers not contained in H.

(a) Prove that if G_H is a finite set, then the arithmetic mean of the integers in G_H is less than or equal to the number of elements in H.

(b) Determine all sets H for which equality holds in part (a).

C94. *Proposed by Edward Crane and Russell Mann, graduate students, Harvard University, Cambridge, MA, USA.*

Suppose that V is a k–dimensional vector subspace of the Euclidean space $\mathbb{R}^n$ which is defined by linear equations with coefficients in $\mathbb{Q}$. Let Λ be the lattice in V given by the intersection of V with the lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$, and let $\Lambda^\perp$ be the lattice given by the intersection of the perpendicular vector space $V^\perp$ with $\mathbb{Z}^n$. Show that the (k–dimensional) volume of Λ is equal to the ($(n-k)$-dimensional) volume of $\Lambda^\perp$.

Problem of the Month

Jimmy Chui, student, University of Toronto

Problem. Let x_1, x_2, ... , x_{n+1} be positive real numbers such that

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_{n+1}} \;=\; 1\,.$$

Show that

$$x_1 x_2 \cdots x_{n+1} \;\geq\; n^{n+1}\,.$$

(1998 Canadian IMO Training)

Solution. Let us make the substitution $y_i = 1/(1+x_i)$ for $i = 1$, 2, ..., $n+1$. Then, we want to show that

$$\prod_{i=1}^{n+1} \left(\frac{1-y_i}{y_i}\right) \;\geq\; n^{n+1}\,,$$

where $y_1 + y_2 + \cdots + y_{n+1} = 1$.

Let

$$s_i = \sum_{\substack{j=1 \\ j \neq i}}^{n+1} y_j \,, \quad \text{and} \quad p_i = \prod_{\substack{j=1 \\ j \neq i}}^{n+1} y_j \,.$$

Observe that

$$\frac{1 - y_i}{y_i} = \frac{s_i}{y_i} \geq \frac{n \sqrt[n]{p_i}}{y_i}$$

by the AM-GM Inequality.

Note that

$$\prod_{i=1}^{n+1} \sqrt[n]{p_i} = \prod_{i=1}^{n+1} y_i$$

since each y_i appears exactly n times in the product. Then we have that

$$\prod_{i=1}^{n+1} \left(\frac{1 - y_i}{y_i} \right) \geq \prod_{i=1}^{n+1} \frac{n \sqrt[n]{p_i}}{y_i} = n^{n+1} \cdot \frac{\prod_{i=1}^{n+1} y_i}{\prod_{i=1}^{n+1} y_i} = n^{n+1} \,,$$

QED.

The key step in this solution is the very clever substitution $1 - y_i = s_j$. Literally, this is a very neat five-line solution to an otherwise difficult problem. Now, as a follow-up, try to solve Problem 3 of the 1998 USAMO:

Let $a_0, a_1, \ldots, a_n$ be numbers from the interval $(0, \pi/2)$ such that

$$\tan \left(a_0 - \frac{\pi}{4} \right) + \tan \left(a_1 - \frac{\pi}{4} \right) + \cdots + \tan \left(a_{n+1} - \frac{\pi}{4} \right) \geq n - 1 \,.$$

Prove that

$$\tan a_0 \tan a_1 \cdots \tan a_{n+1} \geq n^{n+1} \,.$$

J.I.R. McKnight Problems Contest 1995

1. Solve for x, given

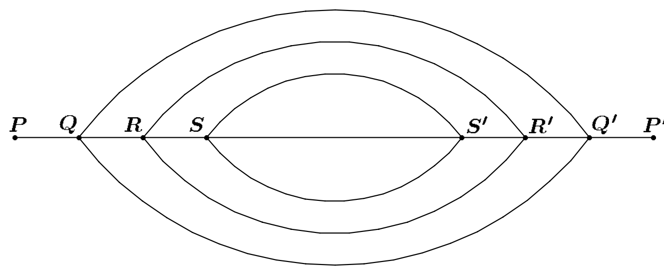
   (a) $(\log_2 x)(\log_2 x^2) - \log_2 x^3 - 9 = 0$,

   (b) $(\log_2(35 - x^3))/(\log_2(5 - x)) = 3$.

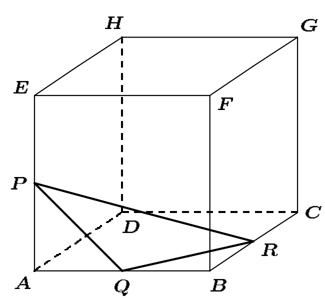
2. Solve for x, given the following equation:

$$\cos^{-1} x - \sin^{-1} x = \frac{\pi}{6} \,.$$

3. In the diagram, $PQ = P'Q'$, $QR = Q'R'$, and $RS = R'S'$. Albert Mouse leaves from P to go to P' while Betty Mouse leaves from P' to go to P. Both mice start out at the same time and proceed at the same speed. If the chances are even of picking any route at each intersection, then what is the probability that Albert and Betty will not meet? (Back tracking is not permitted).



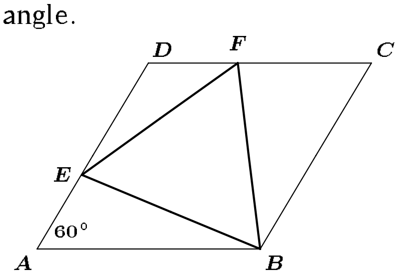
4. Given P, Q, R are the mid-points of the edges of a cube of side 2, as shown.



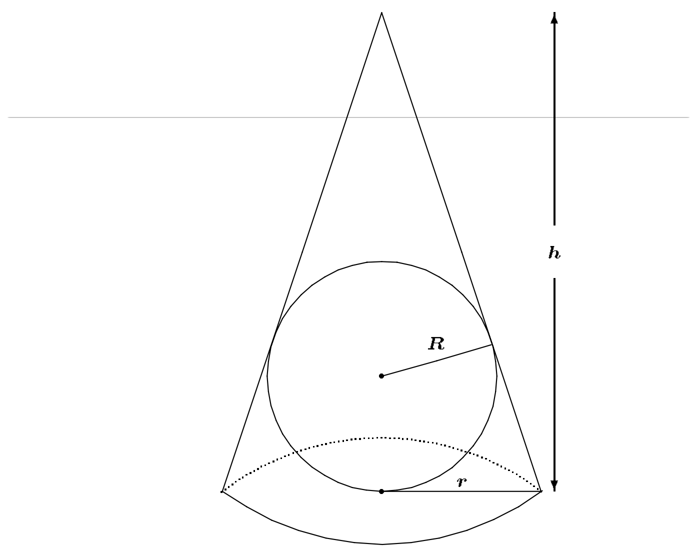
(a) Find the angle between the two planes determined by triangle PQR and square $ABCD$, and express it in degree measure rounded to 1 digit after the decimal point.

(b) Find the area of triangle PQR.

5. Given a rhombus $ABCD$, such that angle $DAB = 60°$. Point E is on AD and point F on DC such that $AE = DF$. Show that triangle BEF is an equilateral triangle.



6. Find the radius r and the height h of the right circular cone of minimum volume which can be circumscribed about a sphere of fixed radius R. Express both quantities r and h in terms of R.

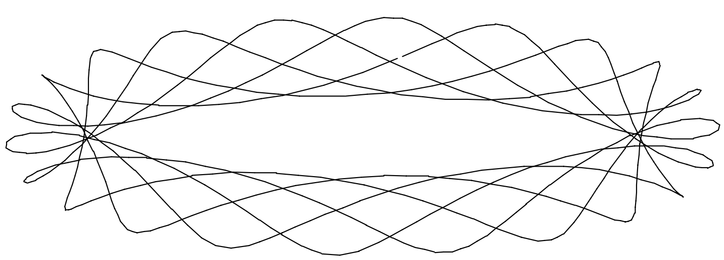


7. Find the sum of the following:

$$S_n = \frac{2^2+1}{2^2-1} + \frac{3^2+1}{3^2-1} + \frac{4^2+1}{4^2-1} + \cdots + \frac{(n+1)^2+1}{(n+1)^2-1}\,.$$

8. Find $f(x)$ such that:

$$f(x) + f\left(\frac{x-1}{x}\right) = 1 + x\,.$$



How to Solve the Cubic

Naoki Sato

The simple and usual method of solving the quadratic polynomial is "completing the square"; however, as anyone who has tried knows, this is woefully inadequate for the cubic polynomial. And, unlike the quadratic, the solution of the cubic is not a part of the standard high-school curriculum, but is still quite accessible to young students. All that is required is some background in basic polynomial algebra, which we will cover. So now, you too can solve a cubic.

Polynomials: Roots and Coefficients

Let the roots of the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be $r_1, r_2, \ldots, r_n$. By the Factor Theorem,

$$q(x) \;=\; (x - r_1)(x - r_2)\cdots(x - r_n)$$

divides $p(x)$. However, both p and q have degree n, so one is a constant multiple of the other. The leading coefficients (the coefficients of x^n) in p and q are a_n and 1 respectively, so $p(x) = a_n q(x)$. Let s_k be the sum of the products of the r_i taken k at a time, so that

$$\begin{aligned} s_1 &= r_1 + r_2 + \cdots + r_n \,, \\ s_2 &= r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n \,, \\ &\cdots \\ s_n &= r_1 r_2 \cdots r_n \,. \end{aligned}$$

Then

$$\begin{aligned} & a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \\ &\quad= a_n (x - r_1)(x - r_2)\cdots + (x - r_n) \\ &\quad= a_n \big[x^n - (r_1 + r_2 + \cdots + r_n)x^{n-1} \\ &\qquad\quad +(r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n)x^{n-2} \\ &\qquad\quad - \cdots + (-1)^n r_1 r_2 \cdots r_n\big] \\ &\quad= a_n (x^n - s_1 x^{n-1} + s_2 x^{n-2} - \cdots + (-1)^n s_n) \,. \end{aligned}$$

Equating coefficients, we conclude that

$$s_k \;=\; (-1)^k \cdot \frac{a_{n-k}}{a_n}$$

for $k = 1, 2, \ldots, n$. The s_k are called the *elementary symmetric polynomials* in the r_i, and by the derivation above, they are determined by the coefficients of p.

For example, let a and b be the roots of $x^2 - 4x + 2 = 0$. Then we need not solve for the roots to tell that $a + b = 4$ and $ab = 2$; we can simply read off the coefficients. As a simple exercise, calculate $a^2 + b^2$ and $a^3 + b^3$. We are now ready to attack the cubic polynomial.

An Algebraic Approach

We begin with the general cubic polynomial

$$a_3x^3 + a_2x^2 + a_1x + a_0 \;=\; 0\,.$$

Divide by a_3 to get the monic cubic polynomial

$$x^3 + ax^2 + bx + c \;=\; 0\,.$$

We perform the substitution $x = t + s$, and express the cubic in terms of the new variable t:

$$\begin{aligned}
&x^3 + ax^2 + bx + c\\
&\quad= (t+s)^3 + a(t+s)^2 + b(t+s) + c\\
&\quad= t^3 + 3st^2 + 3s^2t + s^3 + at^2 + 2ast + as^2 + bt + bs + c\\
&\quad= t^3 + (a+3s)t^2 + (2as + b + 3s^2)t + s^3 + as^2 + bs + c\,. \qquad (1)
\end{aligned}$$

The correct value of s will turn (1) into a cubic with no t^2 term, called a *depressed* cubic. Solving: $a + 3s = 0$, or $s = -a/3$. Let our new cubic be

$$t^3 + pt + q\,, \qquad (2)$$

where p and q are functions of a, b, and c.

Let ω be a primitive cube root of unity; that is, $\omega^3 = 1$, but $\omega \neq 1$ and $\omega^2 \neq 1$. These imply that $\omega^3 - 1 = (\omega - 1)(\omega^2 + \omega + 1) = 0$, so $\omega^2 + \omega + 1 = 0$. If we solve this quadratic, then we obtain the roots

$$\omega = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\, i\,.$$

Check that $\omega^3 = 1$. We will see that it does not matter which root we take ω to be.

We now utilize the wonderful identity

$$\begin{aligned}
&(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)\\
&\quad= (x+y+z)(x^2+y^2+z^2-xy-xz-yz)\\
&\quad= x^3+y^3+z^3-3xyz\,.
\end{aligned}$$

Verify this identity. Note that the right-hand side is a cubic in x with no term in x^2; we wish to emulate our depressed cubic with this expression. Substitute $x = t$, $y = -\alpha$ and $z = -\beta$, to obtain

$$(t - \alpha - \beta)(t - \omega\alpha - \omega^2\beta)(t - \omega^2\alpha - \omega\beta) = t^3 - 3\alpha\beta t - \alpha^3 - \beta^3 . \quad (3)$$

Note that the roots of (3) are $\alpha+\beta$, $\omega\alpha+\omega^2\beta$, and $\omega^2\alpha+\omega\beta$. Our depressed cubic is $t^3 + pt + q$, so we want to find (solve for) α and β such that

$$\alpha\beta = -\frac{p}{3} \quad \text{and} \quad \alpha^3 + \beta^3 = -q .$$

Then $\alpha^3\beta^3 = -p^3/27$. We conclude that α^3 and β^3 are the roots of

$$u^2 + qu - \frac{p^3}{27} = 0 .$$

Therefore, α and β are

$$\sqrt[3]{\frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}} = \sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} .$$

The roots of (2) are $\alpha + \beta$, $\omega\alpha + \omega^2\beta$, and $\omega^2\alpha + \omega\beta$, as stated above. That, technically, does it.

But if $q^2/4 + p^3/27 < 0$, then we run into difficulties and the formula above is not very practical (see Problem 3). In such a case, we must try another mode of attack.

A Trigonometric Approach

Assume that $q^2/4 + p^3/27 < 0$. Then $p < 0$. This time, we utilize the trigonometric identity

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta .$$

Note that the right-hand side is a cubic in $\cos\theta$, with no quadratic term. We begin with the depressed cubic in (2), and make the substitution $t = r\cos\theta$, where r is a constant to be determined:

$$t^3 + pt + q = r^3\cos^3\theta + rp\cos\theta + q .$$

We wish the expression $r^3\cos^3\theta + rp\cos\theta$ to mimic the expression $4\cos^3\theta - 3\cos\theta$. We can achieve this by setting r to be a value which makes the two proportional, so we solve

$$\frac{r^3}{4} = -\frac{rp}{3} \Longrightarrow r^2 = -\frac{4}{3}p \Longrightarrow r = \pm 2\sqrt{-\frac{p}{3}} .$$

Then

$$\begin{aligned} t^3 + pt + q &= r^3 \cos^3\theta + pr\cos\theta + q \\ &= r(r^2\cos^3\theta + p\cos\theta) + q \\ &= r\left(-\frac{4}{3}\,p\cos^3\theta + p\cos\theta\right) + q \\ &= -\frac{rp}{3}(4\cos^3\theta - 3\cos\theta) + q \\ &= -\frac{rp}{3}\cos 3\theta + q \\ \Longrightarrow \quad \cos 3\theta &= \frac{3q}{rp}\,. \end{aligned} \tag{4}$$

Let θ_0 be a solution of (4). Then $\theta_1 := \theta_0 + 120°$ and $\theta_2 := \theta_0 + 240°$ are also solutions of (4). Then the roots of (2) are $r\cos\theta_0$, $r\cos\theta_1$, and $r\cos\theta_2$.

Problems.

1. In (1), express p and q in terms of a, b, and c.

2. Solve the following cubics. You may be required to use different methods.

   (a) $x^3 - 7x - 6 = 0$.

   (b) $2x^3 - 30x^2 + 162x - 350 = 0$.

   (c) $x^3 - 3x^2 + 3x - 2 = 0$.

   (d) $x^3 + x^2 + x + 1/3 = 0$.

3. According to the algebraic approach, the "real root" of $x^3 - 15x - 4 = 0$ is
$$\sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}\,.$$
   However, we can see that the real root is 4. Explain.

4. Find a geometrical interpretation of the trigonometric approach.

5. Solve the following system of equations:
$$\begin{aligned} x + y + z &= 1\,, \\ x^2 + y^2 + z^2 &= 7\,, \\ x^3 + y^3 + z^3 &= 13\,. \end{aligned}$$
   Hint: Find the values of $xy + xz + yz$ and xyz.

6. Let r_1, r_2, and r_3 be the roots of the depressed cubic $t^3 + pt + q = 0$. The *discriminant* of this cubic is defined to be

$$D := [(r_1 - r_2)(r_1 - r_3)(r_2 - r_3)]^2 .$$

   (a) Show that $D = -4p^3 - 27q^2$.
   Hint: Remember that $r_1 + r_2 + r_3 = 0$ and $r_i^3 + pr_i + q = 0$ for all i.

   (b) Assume that r_1 is non-real, say $u + vi$, where $u, v \in \mathbb{R}$, $v \neq 0$. Then a result in algebra states that $u - vi$ must also be a root, so let $r_2 = u - vi$. Since the sum of the roots is zero, $r_3 = -2u$. Show that $D = -4v^2(9u^2 + v^2)^2 < 0$.

   (c) Conclude that

$$D \begin{cases} > 0 & \text{if and only if the roots are real and distinct,} \\ = 0 & \text{if and only if the roots are real and at least two are equal, and} \\ < 0 & \text{if and only if one root is real and the other two are non-real.} \end{cases}$$

   Thus, D indicates the nature of the roots.

7. The real numbers α, β satisfy the equations

$$\begin{aligned} \alpha^3 - 3\alpha^2 + 5\alpha - 17 &= 0 \\ \beta^3 - 3\beta^2 + 5\beta + 11 &= 0 . \end{aligned}$$

   Find $\alpha + \beta$.

   (1993 Irish Mathematical Olympiad)

8. Prove that if c is a rational number, then the equation

$$x^3 - 3cx^2 - 3x + c = 0$$

   has at most one rational solution.

9. Prove that $\sin(\pi/14)$ is a root of the equation

$$8x^3 - 4x^2 - 4x + 1 = 0 .$$

   What are the other two roots?

10. Let the cubic equation $x^3 + ax^2 + bx + c = 0$ have the three real roots r_1, r_2, and r_3, such that $r_1 \leq r_2 \leq r_3$.

   (a) Show that $a^2 \geq 3b$.

   (b) Show that

$$\frac{-a - 2\sqrt{a^2 - 3b}}{3} \leq r_1 \leq \frac{-a - \sqrt{a^2 - 3b}}{3},$$

   and that
$$\frac{-a + \sqrt{a^2 - 3b}}{3} \leq r_3 \leq \frac{-a + 2\sqrt{a^2 - 3b}}{3}.$$

(c) Show that

$$\sqrt{a^2 - 3b} \le r_3 - r_1 \le \frac{2\sqrt{3}}{3}\sqrt{a^2 - 3b}\,.$$

11. Let $f(x)$ be a cubic polynomial in x with roots r_1, r_2, and r_3. If

$$\frac{f(\frac{1}{2}) + f(-\frac{1}{2})}{f(0)} = 997\,,$$

then find

$$\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}\,.$$

(***Mathematical Mayhem***, **H46**)

12. (a) Let x, y, and z be real numbers, such that $x+y+z$, $xy+xz+yz$, and xyz are all positive. Prove that x, y, and z are also positive.

    (b) Does the assertion in (a) still hold if x, y, and z are allowed to be complex?

(***Mathematical Mayhem***, **A119**)

13. How many triples a, b, c of real numbers are there such that a, b, c are the roots of the equation $x^3 + ax^2 + bx + c = 0$?

(1993 Descartes Contest, Problem 9)

14. Find necessary and sufficient conditions on the coefficients of the cubic $x^3 + ax^2 + bx + c = 0$ for the roots to be in arithmetic progression, and in geometric progression.

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