BOOK REVIEWS

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The Sensual (Quadratic) Form by John Horton Conway, assisted by Francis Y.C. Fung,
published by the Mathematical Association of America, 1997, The Carus Mathematical Monographs, Number **26**.
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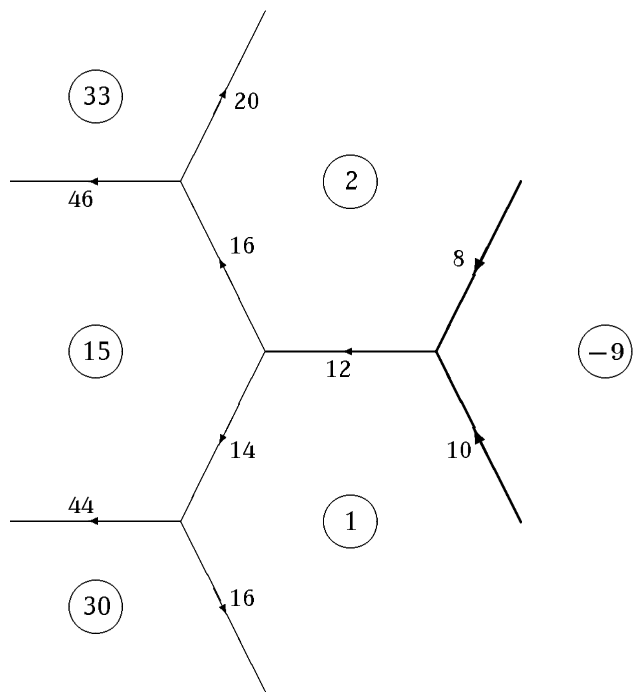
I am proud that during my service on the Hedrick Lecturers Committee, we secured the services, along with those of Sir Michael Atiyah and Ron Graham, of John Horton Conway. The lectures had to be heard, seen, felt, smelled and tasted in order to appreciate them fully, but now we have the next best thing, a written (and thankfully expanded) record.

If you know nothing about quadratic forms, or if you are an expert, you should read this book. There is no shortage of books on quadratic forms [1, 2, 3, 5, 6, 7, 9, 10, 12], so something unusual is needed to excuse another. As we have come to expect, Conway provides us with the unusual. He gives us new insights, insounds, inscents; he puts us in touch while writing tastefully. To do justice to it, we would need to copy out the whole book. I content myself with describing one item, Conway's river, and leave you to read about conorms and vonorms, Farey fractions and $PSL_2(\mathbb{Z})$, isospectral lattices (why you cannot hear their shape), gluing, the little Methuselah form, the fifteen theorem, the quadratic form as a bouquet of flowers – each flower from a finite field, and much more, in this remarkably informative yet concise little book.

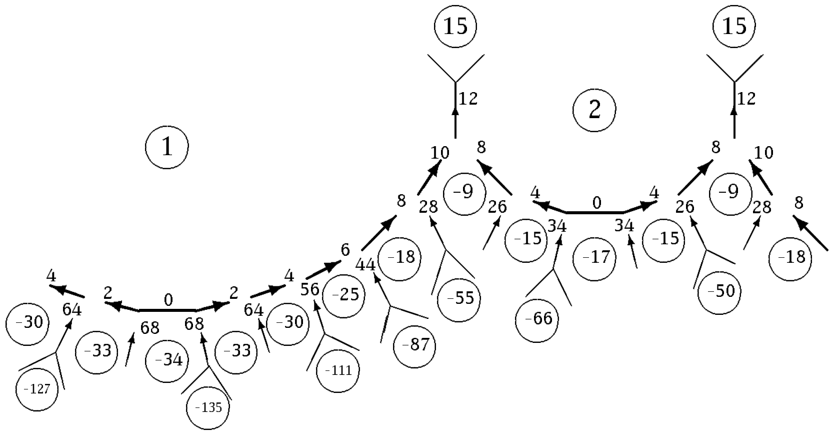
Binary quadratic forms have been largely understood since the work of Legendre [8] and Gauss [4]. But Conway can always make things clearer. His **river**, which is just a simple or double **well** for a definite binary quadratic form, separates the positive and negative values of an indefinite form, and is periodic unless the form represents zero, in which case it ends in **lakes** – there is a special case in which the river is of zero length and degenerates into a **weir** between the two lakes.

Start from your favourite form, say $2x^2 + 12xy + y^2$, and draw a **topograph**, a trivalent graph in which the edges represent **bases** and the vertices **superbases**. The regions will represent the values taken by the form and by equivalent forms. For example, $(1, 0)$ and $(0, 1)$ form a familiar base, and we throw in their sum, $(1, 1)$, and get a superbase. Draw a trivalent vertex, and label the three regions with the values of your form at these three vectors, 2, 1 and 15. To extend the topograph, make the other ends of the edges trivalent, and calculate the numbers to put on the edges and to label the new regions by the **Arithmetic Progression Rule**. In going from region 1 along the edge between regions 2 and 15 we increase from 1 to $2+15=17$; that is, by 16, which we write on the edge. Continue into the new region by adding another 16 to give 33. This is the value of our form for $(x,y)=(2,1)$, in fact for $(\pm 2, \pm 1)$. Similarly, from region 2, between $1+15$, we increase by 14, and a further 14 gives 30. When we go from 15 between $2+1$, we *decrease* by 12, and a further decrease gives -9, a negative value, taken by our form at $(\pm 1, \mp 1)$, and we have discovered two bits of the river, the heavy lines on the right of the figure.

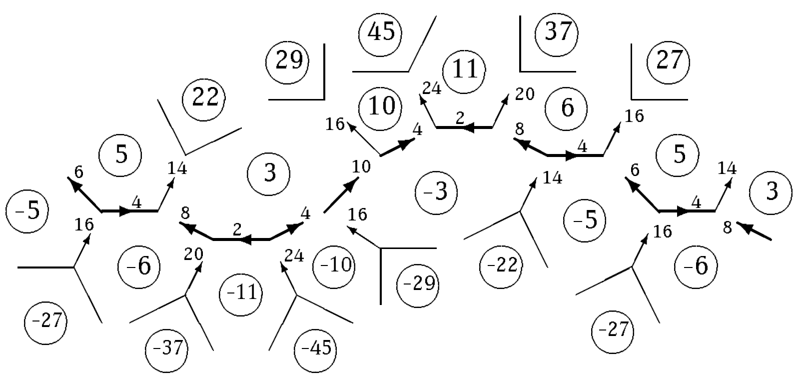


A useful check is that, although Kirchhoff's first law does not hold, the algebraic sum of the flows from a vertex is the sum of the values of the three regions, $16+14-12=1+2+15$, $12-8-10=-9+1+2$. Notice also that as you go to each next edge round a region, the flow increases by $2v$, where v is the value of the region.

Continue to extend the topograph along the river. If you wander away from the river, the values in the regions increase in size, so the smallest values are always next to the river. After a while you will discover that the river is periodic. In the picture, we have only drawn a bit more than half the period, because it is not only periodic, but is its own reflection in the perpendicular bisectors of the horizontal, zero, edges.



Gaze at it and learn! If you go from region a across edge b into region c, then you have the form $ax^2 + bxy + cy^2$, for example, $-9x^2 + 8xy + 2y^2$, which takes the values 1, 2, -9 and -15 at $(1, 1)$, $(0, 1)$, $(1, 0)$ and $(1, -1)$. All the equivalent quadratic forms are there, if you care to wander far enough away from the river. Are all the quadratic forms of discriminant 136 there? No, because the class number is 2, and the topograph illustrates only one class. You can easily find a form which is not there. There is no edge 14 and $136 = 14^2 - 4 \times 3 \times 5$, so we can draw another topograph starting from $3x^2 + 14xy + 5y^2$.



This is also periodic, but the extra symmetry this time is by rotation through $180°$ about the mid-points of the 6–edges and the 10–edges, while changing the signs of the values of the regions. Exercises for the reader: pick out the 26 forms (reduced in the narrow sense) with $a > 0$, $c > 0$, $b > a + c$, and the 2×26 simple forms, with $a > 0 > c$. Hint: 16 from the first river, 10 from the second; wander along the banks, or keep jumping across.

This is just one of the many delights in this book. Get it and read it. Mathematics is a difficult subject, and as usual, you must work as you read, but Conway makes the work unusually untedious.

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