THE OLYMPIAD CORNER

No. 205

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We begin this number with a whirlwind tour of contests from the four corners of the globe. My thanks go to Richard Nowakowski, Canadian Team Leader at the IMO in Buenos Aires, who collected all four sets for our use. We start the tour in Finland.

FINNISH HIGH SCHOOL MATHEMATICS CONTEST

Final Round

Varkaus, January 25, 1997

1. Determine all numbers a for which the equation

$$a3^x + 3^{-x} = 3$$

has a unique solution x.

2. Two circles, of radii R and r, $R > r$, are externally tangent. Consider the common tangent of the circles, not passing through their common point. Determine the maximal radius of a circle drawn in the domain bounded by this tangent line and the circles.

3. Twelve knights sit around a round table. Every knight hates the two knights sitting next to him, but none of the other nine knights. A task group of five knights is to be sent to save a princess in trouble. No two knights who hate each other can be included in the group. In how many ways can the group be selected?

4. Determine the sum of all four-digit numbers, all the digits of which are odd.

5. Let $n \geq 3$. Find a configuration of n points in the plane such that the mutual distance of no pair of points exceeds one and exactly n pairs of points have a mutual distance equal to one.

Now we fly east to Georgia for XI and X form contests.

GEORGIAN MATHEMATICAL OLYMPIAD
May 1997, Final Round
XI Form

1. Consider the following sequence of functions:

$$f_1(x) = \log_{\sqrt{5}}(x)\,,\ f_2(x) = f_1(f_1(x))\,,\ \ldots\,,\ f_{n+1}(x) = f_1(f_n(x))\,,\ \ldots\,.$$

Find the smallest natural value of k such that $f_k(k)$ is not defined.

2. Two positive numbers are written on a board. At each step you must perform one of the following:

(i) choose one of the numbers, say a, already written on the board and write down either a^2 or $\frac{1}{a}$ on the board;

(ii) choose two numbers, say a and b, on the board and write down either $a+b$ or $|a-b|$ on the board.

Obviously, after each step the quantity of numbers on the board increases. How should you proceed in order that the product of the two initial numbers will eventually be written on the board?

3. Given a convex quadrilateral with sides not exceeding 20, prove that the distance from any interior point to the nearest vertex does not exceed 15.

4. We say that there is an algebraic operation defined on the closed interval $[0,1]$ if there is a rule that corresponds to every pair (a,b) of numbers from this interval a new number c from the same interval. We denote it by $c = a \otimes b$. Find all positive k with the property that there exists an algebraic operation defined on $[0,1]$ such that for any x, y, z from $[0,1]$ the following equalities hold:

(i) $x \otimes 1 = 1 \otimes x = x$,

(ii) $x \otimes (y \otimes z) = (x \otimes y) \otimes z$,

(iii) $(zx) \otimes (zy) = z^k (x \otimes y)$.

For all such k define the corresponding algebraic operation.

5. In the rectangular parallelepiped $ABCDA'B'C'D'$ the points N and P are the centres of the faces $ABB'A'$ and $ADD'A'$ respectively. Let M be a point on the diagonal $A'C$, such that $A'M = \frac{1}{3}A'C$. Prove that $MN \perp AB'$ and $MP \perp AD'$ if and only if the given parallelepiped is a cube.

X Form

1. Find all triples (x,y,z) of integers satisfying the inequality:

$$x^2 + y^2 + z^2 + 3 \ < \ xy + 3y + 2z\,.$$

2. Determine whether or not it is possible to fill an $n \times n$ table with entries equal to 0, -1, or 1 so that when calculating the sums of the entries along the rows and the columns, one gets 20 different numbers.

3. See XI.2.

4. The area of a given trapezoid is 2 cm^2 and the sum of its diagonals equals 4 cm. Find the altitude of the trapezoid.

5. Prove that in any triangle the following inequality holds: $pR \geq 2S$, where p, R, S are respectively the semiperimeter, the radius of the circumcircle and the area of the triangle.

Continuing east (but backing up in time!) we catch the three parts of the 6$^{\text{th}}$ Republic of China Mathematical Olympiad.

6$^{\text{th}}$ ROC (TAIWAN) MATHEMATICAL OLYMPIAD

Part I

April 14, 1997 (Time: 4.5 hours)

Each problem is worth 7 points.

1. Let a be a rational number, b, c, d be real, and the function $f : R \to [-1, 1]$ satisfy

$$f(x+a+b) - f(x+b) \;=\; c \cdot \left[x + 2a + [x] - 2[x+a] - [b]\right] + d$$

for each $x \in \mathbb{R}$, where $[t]$ denotes the largest integer that is less than or equal to t. Show that f is a periodic function (that is, there is a positive number p such that $f(x+p) = f(x)\ \forall x \in \mathbb{R}$).

2. Let AB be a given line segment. Find all possible points C in the plane such that in $\triangle ABC$, the height from the vertex A and the length of the median from the vertex B are equal.

3. Let $n \geq 3$. Suppose that the sequence a_1, a_2, $\ldots$, a_n are positive real numbers satisfying $a_{i-1} + a_{i+1} = k_i a_i$, $\forall i = 1, 2, \ldots, n$, where each k_i is a positive integer, $a_0 = a_n$, $a_{n+1} = a_1$. Show that

$$2n \leq k_1 + k_2 + \cdots + k_n \;\leq\; 3n\,.$$

Part II

May 11, 1997 (Time: 4.5 hours)

Each problem is worth 7 points.

1. Let $k = 2^{2^n} + 1$ for some positive integer n. Show that k is a prime if and only if k is a factor of $3^{(k-1)/2} + 1$.

2. Let $ABCD$ be a tetrahedron. Show that:

(i) if $AB = CD$, $AD = BC$, $AC = BD$, then $\triangle ABC$, $\triangle ACD$, $\triangle ABD$, and $\triangle BCD$ are acute triangles;

(ii) if the area of $\triangle ABC$, $\triangle ACD$, $\triangle ABD$, and $\triangle BCD$ are the same, then $AB = CD$, $AD = BC$, $AC = BD$.

3. Let X be the set consisting of elements of the form

$$a_{2k} \cdot 10^{2k} + a_{2k-2} \cdot 10^{2k-2} + \cdots + a_2 \cdot 10^2 + a_0 \,,$$

where $k = 0, 1, 2, \ldots$, and each $a_{2i} \in \{1, 2, 3, \ldots, 9\}$. Show that every integer of the form $2^p \cdot 3^q$, where p, q are non-negative integers, is a factor of some element in X.

Part III

June 25, 1997 (Time: 4.5 hours)

Each problem is worth 7 points.

1. Determine all the possible integers k such that there is a function $f : \mathbb{N} \longrightarrow \mathbb{Z}$ such that

(i) $f(1997) = 1998$,

(ii) $f(ab) = f(a) + f(b) + k \cdot f(d(a, b))$, $\forall a, b \in \mathbb{N}$, where $d(a, b)$ denotes the greatest common divisor of a and b.

2. Let $\triangle ABC$ be an acute triangle with circumcentre O and circumradius R. Show that if AO meets the circle OBC again at D, BO meets the circle OCA again at E, and CO meets the circle OAB again at F, then $OD \cdot OE \cdot OF \geq 8R^3$.

3. Let $X = \{1, 2, 3, \ldots, n\}$, $n \geq k \geq 3$, and let F_k be a family of subsets of X with k elements, so that any two subsets in F_k have at most $k - 2$ common elements. Show that for each $k \geq 3$ there exists a subset M_k of X with at least $\lfloor \log_2 n \rfloor + 1$ elements such that it does not contain any subset in F_k.

Finally we give the problems of the 11^{th} Iberoamerican Mathematical Olympiad.

11^{th} IBEROAMERICAN MATHEMATICAL OLYMPIAD

September 24–25, 1996, Costa Rica

First Day — Time Allowed: 4.5 hours

1. (Brazil): Let n be a natural number. A cube of side n can be split into 1996 cubes. The sides of these cubes are also natural numbers. Determine the minimum possible value of n.

2. (Spain): Let M be the mid-point of the median AD of the triangle ABC (D belongs to the side BC). The line BM meets the side AC at the point N. Show that AB is tangent to the circumcircle of the triangle NBC if and only if the equality

$$\frac{BM}{MN} = \frac{BC^2}{BN^2}$$

holds.

3. (Spain): We have a chessboard of size $(k^2 - k + 1) \times (k^2 - k + 1)$, with $k = p + 1$, p being a prime number.

For each prime number p, give a method of distribution of the numbers 0 and 1, one number in each square of the chessboard, in such a way that in each row, there are exactly k zeros; in each column, there are exactly k zeros; and moreover, no rectangle with sides parallel to the sides of the chessboard has a number 0 on the vertices.

Second Day — Time Allowed: 4.5 hours

4. (Brazil): Given a natural number $n \geq 2$, all the fractions of the form $\frac{1}{ab}$, with a and b natural numbers, coprime, and such that

$$a < b \leq n, \qquad a + b > n,$$

are considered. Show that the sum of all these fractions equals $\frac{1}{2}$.

5. (Peru): Three coins, A, B and C are situated one at each vertex of an equilateral triangle of side n. The triangle is divided into little equilateral triangles of side 1 by lines parallel to the sides.

At the beginning, all the lines of the figure are blue. The coins move along the lines, painting in red their trajectory, following the two rules:

(i) First coin to move is A, then B, then C, then again A, and so on. At each turn, each coin paints exactly one side of one of the little triangles.

(ii) No one coin can move along a side of a triangle which is already painted red; but that coin can stay at the end of a painted segment, alone or with another coin waiting its turn at moving.

Show that, for all integers $n > 0$, it is possible to paint all the sides of all the little triangles red.

6. (Spain): We have n distinct points $A_1, \dots, A_n$ in the plane. To each point A_i a real number $\lambda_i \neq 0$ is assigned, in such a way that

$$\overline{A_iA_j}^2 = \lambda_i + \lambda_j\,, \quad \text{for all } i,\, j \text{ with } i \neq j\,.$$

Show that:

(a) $n \leq 4$;

(b) if $n = 4$, then

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4} = 0\,.$$

Next we turn to readers' solutions to problems given in the November 1998 number of the *Corner*.

3. [1998: 385] *18th Austrian-Polish Mathematics Competition*

Let $P(x) = x^4 + x^3 + x^2 + x + 1$. Show that there exist polynomials $Q(y)$ and $R(y)$ of positive degrees, with integer coefficients, such that $Q(y) \cdot R(y) = P(5y^2)$ for all y.

Solutions by Mohammed Aassila, Strasbourg, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, whose solution we give.

Since $P(5y^2) = 5^4y^8 + 5^3y^6 + 5^2y^4 + 5y^2 + 1$, we try factors of the form

$$(25y^4 + ay^3 + by^2 + cy + 1)(25y^4 - ay^3 + by^2 - cy + 1)\,.$$

On expanding out, these are factors: $a = 25$, $b = 15$, and $c = 5$.

Now we turn to solutions to four problems of the Georgian Mathematical Olympiad 1995, Final Round (see [1998: 388]).

3. (Grade IX). Prove that if the product of three positive numbers is 1 and their sum is more than the sum of their reciprocals, then only one of these numbers can be more than 1.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let a, b, and c denote the three numbers and let $m = \max\{a,\, b,\, c\}$. Since $abc = 1$, clearly $m \geq 1$ and $m = 1$ if and only if $a = b = c = 1$, in which case we have

$$a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\,,$$

a contradiction. Thus $m > 1$. Suppose two of the a, b, c are greater than 1, say, a and b. Then substituting $c = \frac{1}{ab}$ into $a+b+c > \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ we get

$$a+b+\frac{1}{ab} \;>\; \frac{1}{a}+\frac{1}{b}+ab \;=\; \frac{a+b}{ab}+ab$$

or

$$ab(a+b)+1 \;>\; a+b+a^2b^2$$

or

$$(a+b)(ab-1) > (ab-1)(ab+1).$$

Thus $a+b > ab+1$ or $(a-1)(b-1) < 0$, which is clearly a contradiction, and our proof is complete.

Remark: In fact, the proof above shows that if $m = a$, then b and c must both be *strictly* smaller than 1 since, for example, if $b = 1$, then $ac = 1$ would imply $c = \frac{1}{a}$ and so $a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$, contradicting the assumption.

5. (Grade IX). A set M of integers has the following property: if the numbers a and b are in M, then $a+2b$ also belongs to M. It is known that the set contains positive as well as negative numbers. Prove that if the numbers a, b and c are in M, then $a+b-c$ is also in M.

Comment by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The problem, as stated, is clearly *incorrect* unless $a = b$. A simple counterexample is $M = \{-1, 1, 3, 5, 7, \ldots\}$.

1. (Grade X).

(a) Five different numbers are written in one line. Is it always possible to choose three of them placed in increasing or decreasing order?

(b) Is it always possible to do the same, if we have to choose four numbers from nine?

Comment by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The answer to (a) is YES. In fact, these are special cases of a well-known theorem due to Erdős and Szekeres, which states that if $m, n \in \mathbb{N}$, then any sequence of $mn+1$ real numbers contains a monotonically increasing subsequence of $m+1$ terms or a monotonically decreasing subsequence of $n+1$ terms, or both. Here, $5 = 2\cdot 2+1$. (See for example, *Introduction to Combinatorics* by Martin J. Erickson, Wiley-Interscience Series; p. 41.) [*Ed*. The answer to (b) is NO, as shown by the sequence 321654987.]

Since $5 = 2\times 2+1$ and $9 = 2\times 4+1$, both answers follow immediately.

3. (Grade X). Prove that for any natural number n, the average of all its factors lies between the numbers $\sqrt{n}$ and $\frac{n+1}{2}$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let $d_1, d_2, \ldots, d_k$ denote the positive divisors of n where $k = \tau(n)$ is the number of (positive) divisors of n. We are to show that

$$\sqrt{n} \;\le\; \frac{1}{k}\sum_{i=1}^{k} d_i \;\le\; \frac{n+1}{2}\,. \tag{1}$$

To establish the right inequality in (1) we first show that

$$d + \frac{n}{d} \;\le\; n+1 \tag{2}$$

for all divisors d of n.

This is clearly true if $d = 1$. If $d > 1$, then

$$\begin{aligned} d + \frac{n}{d} \le n+1 \;&\Longleftrightarrow\; d^2 + n \le d(n+1) \\ &\Longleftrightarrow\; d(d-1) \le (d-1)n \;\Longleftrightarrow\; d \le n\,, \end{aligned}$$

which clearly holds. Since $d \mid n \Longleftrightarrow \frac{n}{d} \mid n$ we have, from (2) that

$$2\sum_{i=1}^{k} d_i \;=\; \sum_{i=1}^{k}\left(d_i + \frac{n}{d_i}\right) \;\le\; k(n+1)$$

from which $\frac{1}{k}\sum_{i=1}^{k} d_i \le \frac{n+1}{2}$ follows.

If equality holds, then for *any* divisor d of n, either $d = 1$ or $\frac{n}{d} = 1$. Thus $k = 2$ and n must be a prime. Conversely, if n is a prime, then $k = 2$ and $\frac{1}{k}\sum_{i=1}^{k} d_i = \frac{n+1}{2}$. To show the left inequality in (1) we use the Arithmetic-Geometric-Mean Inequality. Note that if n is not a perfect square, then for all divisors d of n we have $d \ne \frac{n}{d}$ and so k must be even. Pairing off d with $\frac{n}{d}$, we obtain $\prod_{i=1}^{k} d_i = n^{k/2}$. If $n = q^2$ is a perfect square, then k is odd. Again, pairing off d with $\frac{n}{d}$ for all $d \ne q$, we find that

$$\prod_{i=1}^{k} d_i \;=\; q \cdot n^{(k-1)/2} \;=\; n^{1/2} \cdot n^{(k-1)/2} \;=\; n^{k/2}\,,$$

which is the same as above.

Hence in both cases, we have

$$\frac{1}{k}\sum_{i=1}^{k} d_i \;\ge\; \left(\prod_{i=1}^{k} d_i\right)^{1/k} \;=\; (n^{k/2})^{1/k} \;=\; \sqrt{n}\,.$$

Clearly, equality holds in this inequality if and only if $n = 1$.

Now we turn to the December 1998 *Corner* and readers' comments and solutions regarding the problems of the Bi-National Israel-Hungary Competition 1995, [1998: 452].

BI-NATIONAL ISRAEL-HUNGARY COMPETITION 1995

1. Denote the sum of the first n prime numbers by S_n. Prove that there exists a whole square between S_n and S_{n+1}.

Comment by Mohammed Aassila, Strasbourg, France.

The problem appeared in ***CRUX*** in 1984 as problem 874. It was proposed by the COPS of Ottawa and solved by Walther Janous.

Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Courdimanche, France. We give Bataille's write-up.

We have $S_1 < 2^2 < S_2 < 3^2 < S_3 < 4^2 < S_4 < 5^2 < S_5$.

Suppose that $n \geq 5$, so that $p_n \geq 11$. Let a_n be the integer ≥ 6 defined by $p_n = 2a_n - 1$, and S'_n the sum of all odd integers from 1 to p_n, inclusive. It is well known that $S'_n = a_n^2 = \left(\frac{p_n+1}{2}\right)^2$ and easily seen that $S'_n > S_n$, because $n \geq 5$. Now let us assume that there is no square between S_n and S_{n+1}. Then there would exist an integer k such that $k^2 \leq S_n < S_{n+1} \leq (k+1)^2$ and we would have

$$S_{n+1} - S_n \;\leq\; 2k+1\,; \quad \text{that is } p_{n+1} \;\leq\; 2k+1\,.$$

From this we could write successively:

$$p_n \;\leq\; 2k-1, \quad k \;\geq\; \frac{p_n+1}{2} \quad \text{and} \quad S_n \;\geq\; \left(\frac{p_n+1}{2}\right)^2$$

which contradicts $S'_n > S_n$. The result follows.

2. Let P, P_1, P_2, P_3, P_4 be five points on a circle. Denote the distance of P from the line P_iP_k by d_{ik}. Prove that $d_{12}d_{34} = d_{13}d_{24}$.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Geoffrey A. Kandall, Hamden, CT, USA. We give Kandall's solution.

Let D denote the diameter of the circle.

We have $d_{ik} \cdot P_iP_k = 2[PP_iP_k] = PP_i \cdot PP_k \cdot \sin\angle P_iPP_k$. Hence

$$d_{ik} \;=\; \frac{PP_i \cdot PP_k \cdot \sin\angle P_iPP_k}{P_iP_k} \;=\; \frac{PP_i \cdot PP_k}{D}\,.$$

It is now obvious that

$$d_{12}d_{34} \;=\; d_{13}d_{24} \;=\; (d_{14}d_{23}) \;=\; \frac{1}{D^2}\prod_{i=1}^{4} PP_i\,.$$

3. Consider the polynomials $f(x) = ax^2 + bx + c$ which satisfy $|f(x)| \leq 1$ for all $x \in [0, 1]$. Find the maximal value of $|a| + |b| + |c|$.

Comment by Mohammed Aassila, Strasbourg, France. Solved by Pierre Bornsztein, Courdimanche, France. We give the remark of Aassila.

The solution to this problem appeared in ***CRUX*** in 1984. It was given at the 1980 Leningrad Mathematical Olympiad [1983: 304]. It was corrected and solved by M.S. Klamkin [1984: 287].

As a last solution set this issue we give readers' comments and solutions to problems of the 31st Spanish Mathematical Olympiad, First Round, given in [1998: 452–453].

1. Let a, b, c be distinct real numbers and $P(x)$ a polynomial with real coefficients. If:

- the remainder on division of $P(x)$ by $x - a$ equals a,
- the remainder on division of $P(x)$ by $x - b$ equals b,
- and the remainder on division of $P(x)$ by $x - c$ equals c;

determine the remainder on division of $P(x)$ by $(x - a)(x - b)(x - c)$.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Geoffrey A. Kandall, Hamden, CT, USA; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bataille's solution.

As is well known, the remainder on division of $P(x)$ by $x - a$ is $P(a)$. So, the hypotheses imply: $P(a) = a$, $P(b) = b$, $P(c) = c$.

Let $R(x)$ be the remainder on division of $P(x)$ by $(x-a)(x-b)(x-c)$, so that the degree of $R(x)$ is ≤ 2 and $P(x) = (x - a)(x - b)(x - c)Q(x) + R(x)$ for a polynomial $Q(x)$.

We remark that $R(a) = P(a) = a$ and similarly $R(b) = b$ and $R(c) = c$. From this observation, we may conclude through one of the three following ways:

(1) the polynomial $R(x) - x$ has degree ≤ 2 and three distinct zeros a, b, c. Hence $R(x) - x$ is the zero polynomial and $R(x) = x$.

(2) $R(x)$ has the form $ux^2 + vx + w$ where (u, v, w) is the solution of the system

$$\begin{cases} ua^2 + va + w = a \\ ub^2 + vb + w = b \\ uc^2 + vc + w = c\,. \end{cases} \qquad \text{(S)}$$

The determinant of (S) is a Vandermonde determinant and is not zero (since a, b, c are distinct), so (S) has a unique solution, which clearly is $u = 0$, $v = 1$, $w = 0$. Thus $R(x) = x$ again.

(3) $R(x)$ is the Lagrange's interpolation polynomial:

$$R(x) \;=\; a\,\frac{(x-b)(x-c)}{(a-b)(a-c)} + b\,\frac{(x-a)(x-c)}{(b-a)(b-c)} + c\,\frac{(x-a)(x-b)}{(c-a)(c-b)}\,.$$

Multiplying out and grouping similar terms, a lengthy but easy calculation provides $R(x) = x$ again.

2. Show that, if $(x+\sqrt{x^2+1})(y+\sqrt{y^2+1}) = 1$, then $x+y=0$.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Geoffrey A. Kandall, Hamden, CT, USA; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We first give Aassila's solution and generalization.

We prove, more generally, that if $(x+\sqrt{x^2+1})(y+\sqrt{y^2+1}) = p$, then $x+y = \frac{p-1}{\sqrt{p}}$.

Set $z = x+\sqrt{x^2+1}$. Then $z > 0$ and $x = \frac{z^2-1}{2z}$. Consequently, we obtain

$$y+\sqrt{y^2+1} \;=\; \frac{p}{z}$$

and then

$$y \;=\; \frac{(p/z)^2-1}{(2p/z)} \;=\; \frac{p^2-z^2}{2pz}\,.$$

Hence,

$$\begin{aligned} x+y \;&=\; \frac{z^2-1}{2z} + \frac{p^2-z^2}{2zp} \;=\; \frac{p-1}{2p}\left(z+\frac{p}{z}\right) \\ &\geq\; \frac{p-1}{p}\sqrt{z\cdot\frac{p}{z}} \;=\; \frac{p-1}{\sqrt{p}}\,. \end{aligned}$$

Equality occurs for $x = y = \frac{p-1}{2\sqrt{p}}$.

Next we give a nice solution of Bataille.

Taking logarithms, the hypothesis implies

$$\ln(x+\sqrt{x^2+1}) \;+\; \ln(y+\sqrt{y^2+1}) \;=\; 0 \quad \text{or}$$

$$\sinh^{-1}(x) \;+\; \sinh^{-1}(y) \;=\; 0\,.$$

Since the function $\sinh^{-1}$ is odd we obtain

$$\sinh^{-1}(x) \;=\; \sinh^{-1}(-y) \quad \text{and} \quad x = -y\,,$$

as required.

And for yet another method we turn to Wang's solution.

Since

$$\left(x-\sqrt{x^2+1}\right)\left(x+\sqrt{x^2+1}\right)\left(y-\sqrt{y^2+1}\right)\left(y+\sqrt{y^2+1}\right) = (-1)^2 = 1\,,$$

the given condition implies that

$$\left(x-\sqrt{x^2+1}\right)\left(y-\sqrt{y^2+1}\right) \;=\; 1\,.$$

Hence

$$x+\sqrt{x^2+1} \;=\; \frac{1}{y+\sqrt{y^2+1}} \;=\; -\left(y-\sqrt{y^2+1}\right)$$

and

$$x-\sqrt{x^2+1} \;=\; \frac{1}{y-\sqrt{y^2+1}} \;=\; -\left(y+\sqrt{y^2+1}\right)\,.$$

Adding, we get

$$2x \;=\; -2y \quad \text{or} \quad x+y \;=\; 0\,.$$

3. The squares of the sides of a triangle ABC are proportional to the numbers 1, 2, 3.

(a) Show that the angles formed by the medians of ABC are equal to the angles of ABC.

(b) Show that the triangle whose sides are the medians of ABC is similar to ABC.

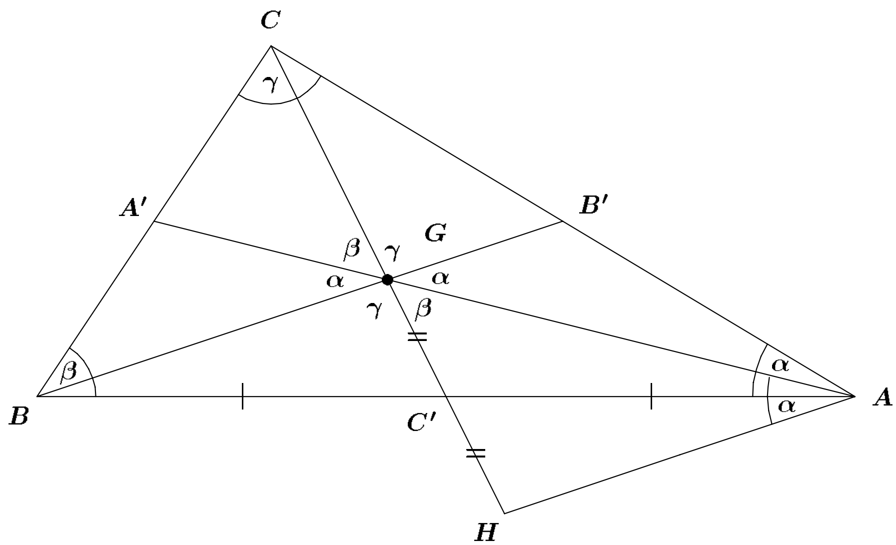
Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Geoffrey A. Kandall, Hamden, CT, USA; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Kandall's solution.

(b) Let us say that $a^2 : b^2 : c^2 = 1 : 2 : 3$; that is, $a^2 = t$, $b^2 = 2t$, $c^2 = 3t$. Let m_a, m_b, m_c denote the medians passing through A, B, C respectively.

Then $4m_a^2 = 2b^2 + 2c^2 - a^2 = 9t$, $4m_b^2 = 2a^2 + 2c^2 - b^2 = 6t$, $4m_c^2 = 2a^2 + 2b^2 - c^2 = 3t$. Thus, $m_c^2 : m_b^2 : m_a^2 = 1 : 2 : 3 = a^2 : b^2 : c^2$. Hence $m_c : m_b : m_a = a : b : c$. This proves (b).

(a) Let A', B', C' be the mid-points of BC, AC, AB, and let G be the centroid of ABC. Extend GC' to H so that $GC' = C'H$.

Then $AG = \frac{2}{3}m_a$, $AH = BG = \frac{2}{3}m_b$, $GH = \frac{2}{3}m_c$. Hence, by what was proved in (b), $ABC \sim AGH$. Consequently, $\angle GAH = \angle BAC \equiv \alpha$, $\angle HGA = \angle CBA \equiv \beta$, $\angle AHG = \angle ACB \equiv \gamma$. We can now fill in our diagram as follows:



This proves (a).

4. Find the smallest natural number m such that, for all natural numbers $n \geq m$, we have $n = 5a + 11b$, with a, b integers ≥ 0.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's more general presentation of the solution.

We prove the more general result that if $p, q \in \mathbb{N}$ are relatively prime, then the smallest integer m such that, for all $N \geq m$, n can be written as a non-negative integer linear combination of p and q is $m = pq - p - q + 1$. For the present problem $\{p, q\} = \{5, 11\}$ and so the answer is $m = 40$.

For convenience, call an integer n "*expressible*" if $n = ap + bq$ for some non-negative integers a and b. We prove the following result which was first obtained by J.J. Sylvester in 1884 in response to a problem proposed earlier by G. Frobenius.

Theorem. The smallest integer m such that n is expressible for all $n \geq m$ is

$$m = pq - p - q + 1 .$$

Proof. We first show that $m - 1$ is not expressible. Suppose, to the contrary, that $pq - p - q = ap + bq$ where a, b are non-negative integers. Then we have $pq = (1+a)p + (1+b)q$. Hence $p \mid 1+b$ and $q \mid 1+a$. Letting $1 + a = a'q$ and $1 + b = b'p$ where $a' \geq 1$, $b' \geq 1$ we get $pq = (a' + b')pq$ and so $a' + b' = 1$, which is clearly a contradiction.

Next we show that m is expressible. Since $(p, q) = 1$, there exist integers x and y such that $xp + yq = 1$ and thus $(x - kq)p + (y + kp)q = 1$

for all integers k. Since clearly $p \not| y$, we can choose an appropriate k so that $-p < y + kp < 0$. Then clearly $x - kq > 0$. Let $x_0 = x - kq$ and $y_0 = y + kp$. Then we have $-p < y_0 < 0 < x_0$, $x_0p + y_0q = 1$ and so

$$m \;=\; pq - p - q + (x_0p + y_0q) \;=\; (x_0 - 1)p + (p + y_0 - 1)q\,,$$

showing that m is indeed expressible.

Now we show by induction that n is expressible for all $n \geq m$. Suppose n_0 is expressible for some $n_0 \geq m$. Then $n_0 = \alpha p + \beta q$ for some non-negative integers α and β, and so

$$n_0 + 1 \;=\; (\alpha + x_0)p + (\beta + y_0)q\,.$$

If $\beta + y_0 \geq 0$, then we are done. Suppose $\beta + y_0 < 0$. Then we write $n_0 + 1 = (\alpha + x_0 - q)p + (\beta + y_0 + p)q$. Since $\beta + y_0 + p > 0$, it remains to show that $\alpha + x_0 - q \geq 0$. If $\alpha + x_0 - q < 0$, then $\alpha + x_0 - q \leq -1$ and thus

$$\alpha + x - kq - q + 1 \;\leq\; 0\,. \qquad (1)$$

On the other hand, since $\beta + y_0 < 0$, we have $\beta + y_0 \leq -1$ and thus

$$\beta + y + kp + 1 \;<\; 0\,. \qquad (2)$$

From $p \times (1) + q \times (2)$ we get

$$\alpha p + \beta q + xp + yq - pq + p + q \;\leq\; 0$$

or

$$n_0 + 1 \;\leq\; pq - p - q \;=\; m - 1\,,$$

a contradiction. Therefore, $\alpha + x_0 - q \geq 0$ and thus $n_0 + 1$ is expressible, completing our proof.

5. A subset $A \subseteq M = \{1, 2, 3, \ldots, 11\}$ is good if it has the following property:

"If $2k \in A$, then $2k - 1 \in A$ and $2k + 1 \in A$".

(The empty set and M are good). How many good subsets has M?

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bornsztein's solution.

We will prove that the number N of good subsets is $N = 233$.

Let $n(A)$ be the number of even numbers which belong to A.

Case 0. $n = 0$. We only have to determine the odd numbers in A. There are 6 odd numbers in M. For each we have two possibilities, so we have 2^6 good subsets A with $n(A) = 0$.

Case 1. $n = 1$. There are 5 possibilities for the choice of the even number. For each choice, 2 odd numbers are necessarily in A. The remaining odd numbers can be determined in 2^4 ways. Thus we have $5 \times 2^4 = 80$ good sets A with $n(A) = 1$.

Case 2. $n = 2$.

Subcase (i). The even numbers in good subsets are consecutive. We have 4 choices for the two consecutive even numbers. Each choice decides 3 odd numbers, leaving 2^3 choices. There are thus 4×2^3 good subsets A under this subcase.

Subcase (ii). The two even numbers in A are not consecutive. We have $\binom{5}{2} - 4 = 6$ choices for even numbers. Since for each choice, 4 odd numbers are decided, this leaves 2^2 choices. The total here is then 6×2^2.

The total number of good subsets A with $n(A) = 2$ is 56.

Case 3. $n = 3$.

Subcase (i). The even numbers in A are consecutive. This gives 3 possibilities, each deciding 4 odd numbers and leaving 2^2 choices.

Thus we have 3×2^2 such good sets.

Subcase (ii). No two even numbers of A are consecutive. This gives only 1 choice for even numbers 2, 6, 10. Then $A = \{1, 2, 3, 5, 6, 7, 9, 10, 11\}$, a unique choice.

Subcase (iii). Exactly two of the 3 even numbers of A are consecutive. We have $\binom{5}{3} - 4 = 6$ choices for the even numbers. For each one, 5 odd numbers are fixed. This leaves 2 choices for each of $6 \times 2 = 12$ good subsets.

Thus we have $3 \times 2^2 + 1 + 12 = 25$ good subsets A with $n(A) = 3$.

Case 4. $n = 4$.

Subcase (i). $2 \notin A$ or $10 \notin A$. This gives 2 possibilities for the even numbers, and each leaves a choice for only 1 odd number, a total of 4 good sets.

Subcase (ii). $2 \in A$ and $10 \in A$.

We have 3 choices for the even number which is not an element of A. For each, the odd numbers are all in A. This gives 3 good sets.

Thus we have $4 + 3 = 7$ good subsets A with $n(A) = 4$.

Case 5. $n = 5$, then $A = M$, 1 possibility.

Finally the total number of good subsets is

$$2^6 + 5 \times 2^4 + 56 + 25 + 7 + 1 \;=\; 233\,.$$

That completes the *Corner* for this issue. It is Olympiad Season. Send me your contest materials along with your nice solutions!