THE ACADEMY CORNER

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We present readers' solutions to some of the questions of the 1999 Atlantic Provinces Council on the Sciences Annual Mathematics Competition, held this year at Memorial University, St. John's, Newfoundland [1999 : 452].

3. Prove that $\sin^2(x+\alpha)+\sin^2(x+\beta)-2\cos(\alpha-\beta)\sin(x+\alpha)\sin(x+\beta)$ is a constant function of x.

   Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

   Let $f(x)$ denote the given function. Differentiating and using the familiar formula: $\sin a+\sin b=2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$, we have

$$\begin{aligned} f'(x) &= \sin(2(x+\alpha))+\sin(2(x+\beta)) \\ &\quad -2\cos(\sin(x+\alpha)\cos(x+\beta)+\cos(x+\alpha)\sin(x+\beta)) \\ &= 2\sin(2x+\alpha+\beta)\cos(\alpha-\beta) \\ &\quad -2\cos(\alpha-\beta)\sin(2x+\alpha+\beta) \\ &= 0\,. \end{aligned}$$

   Hence $f(x)=C$ for some constant C. Since $f(-\beta)=\sin^2(\alpha-\beta)$, we conclude that $f(x)=\sin^2(\alpha-\beta)$.

4. In Scottish Dancing, there are three types of dances, two of which are fast rhythms, Jigs and Reels, and one is a slow rhythm, Strathspey.

   A Scottish Dance program always starts with a Jig. The following dances are selected (by type) according to the following rules:

   (i) the next dance is always of a different type from the previous one,

   (ii) no more than two fast dances can be consecutive.

   Find how many different arrangements of Jigs, Reels and Strathspeys are possible in a Scottish Dance list which has (a) seven dances, (b) fifteen dances.

Solution to (a) by Michael Parmenter, Memorial University of Newfoundland, St. John's, Newfoundland.

We list the cases according to which of the seven dances are Strathspeys, and count the number of arrangements in each case:

Strathspey positions	Number
3, 5	$2 \times 2 = 4$
3, 6	$2 \times 2 = 4$
2, 5	$2 \times 2 = 4$
3, 5, 7	$2 \times 2 = 4$
2, 5, 7	$2 \times 2 = 4$
2, 4, 7	$2 \times 2 = 4$
2, 4, 6	$2 \times 2 \times 2 = 8$
Total	32

Parmenter also solved (b).

Solution and generalisation by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We consider the more general problem when there are n dances. Let $f(n)$, $g(n)$, respectively, denote the number of different arrangements satisfying conditions (i) and (ii), starting with a Jig, Reel, respectively. It is easy to see that $f(1) = 1$, $f(2) = 2$, and $f(3) = 3$.

For $n > 3$, we claim that $f(n)$ satisfies the 3^{rd}–order recurrence relation

$$f(n) \;=\; 2\big(f(n-2) + f(n-3)\big)\,.$$

For convenience of notation, we denote Jig, Reel and Strathspey by J, R, and S respectively. Since the 1^{st} dance is a J, the 2^{nd} dance must be an S or an R. Among all those arrangements starting with J, S, there are $f(n-2)$ of them which start with J, S, J, and $g(n-2)$ of them which start with J, S, R. On the other hand, if the 2^{nd} dance is an R, then the 3^{rd} dance must be an S. Among all those arrangements starting with J, R, S, there are $f(n-3)$ of them which start with J, R, S, J, and $g(n-3)$ of them which start with J, R, S, R. Hence we get

$$f(n) \;=\; f(n-2) + g(n-2) + f(n-3) + g(n-3)\,.$$

Note that for any admissible arrangement, if we replace any J by an R and *vice versa*, we obtain an arrangement for which conditions (i) and (ii) still hold. This one-to-one correspondence shows that $f(n) = g(n)$ for all n. Therefore, we obtain the recurrence relation $f(n) = 2f(n-2) + 2f(n-3)$.

Using the initial values of $f(1)$, $f(2)$, and $f(3)$, and iterating the recurrence relation, we easily find that $f(4) = 6$, $f(5) = 10$, $f(6) = 18$, $f(7) = 32$, $f(8) = 56$, $f(9) = 100$, $f(10) = 176$, $f(11) = 312$, $f(12) = 552$, $f(13) = 976$, $f(14) = 1728$, and $f(15) = 3056$.

Comments:

(a) This is indeed a very interesting problem.

(b) To obtain an explicit formula for the value of $f(n)$ for arbitrary n seems to be difficult, since the characteristic equation of the recurrence relation is the cubic equation $x^3 - 2x - 2 = 0$. Standard theory tells us that if the three roots of this equation are denoted by r_1, r_2 and r_3, then the general solution is given by $c_1 r_1^n + c_2 r_2^n + c_3 r_3^n$ for some constants c_1, c_2 and c_3.

5. Find all differentiable functions $f(x)$ which satisfy the integral equation

$$\big(f(x)\big)^{2000} = \int_1^x \big(f(t)\big)^{1999}\,dt\,.$$

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The only such functions are $f(x) \equiv 0$ and $f(x) = \frac{x-1}{2000}$.

Clearly, the function $f(x) \equiv 0$ satisfies the given equation. Suppose then that $f(x) \not\equiv 0$. Differentiating both sides of the given equation yields $2000\big(f(x)\big)^{1999} f'(x) = \big(f(x)\big)^{1999}$, and so $f'(x) = \frac{1}{2000}$. Hence, $f(x) = \frac{x}{2000} + C$ for some constant C. Setting $x = 1$ in the given equation yields $f(1) = 0$, giving $C = -\frac{1}{2000}$, and so $f(x) = \frac{x-1}{2000}$.

Also solved by Richard Tod, The Royal Forest of Dean, Gloucestershire, England.

7. Pat has a method for solving quadratic equations. For example, Pat solves $6x^2 + x - 2 = 0$ as follows:

Step 1. Pat multiplies the leading coefficient by the constant, and solves the simpler equation $x^2 + x - 12 = 0$ to get $(x+4)(x-3) = 0$.

Step 2. Pat then replaces each x by $6x$ (x times the leading coefficient) to get $(6x+4)(6x-3) = 0$.

Step 3. Pat then simplifies this equation to get $(3x+2)(2x-1) = 0$, which solves the original equation.

Prove or disprove that Pat's method always works.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Pat's method always works. Let $ax^2 + bx + c = 0$ be an arbitrary quadratic equation with $a \neq 0$. Then step 1 yields $x^2 + bx + ac = 0$.

Suppose that $x^2 + bx + ac = (x-\alpha)(x-\beta)$. Then $\alpha + \beta = -b$ and $\alpha\beta = ac$.

Step 2 then gives $(ax-\alpha)(ax-\beta) = 0$, or $a^2x^2 - a(\alpha+\beta)x + \alpha\beta = 0$.

Hence, $a^2x^2 + abx + ac = 0$. Since $a \neq 0$, step 3 finally yields $ax^2 + bx + c = 0$.