SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2370. [1998: 364] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Determine the exact values of the roots of the polynomial equation

$$x^5 - 55x^4 + 330x^3 - 462x^2 + 165x - 11 \;=\; 0\,.$$

Solution by Joe Howard, New Mexico Highlands University, Las Vegas, NM, USA.

Let $c = \cos\theta$, $s = \sin\theta$, where $\theta = \frac{k\pi}{11}$, $k = 1, 3, 5, 7, 9$. Then, by DeMoivre's formula, we have

$$(c+is)^{11} \;=\; \left(e^{i\theta}\right)^{11} \;=\; e^{11i\theta} \;=\; e^{k\pi i} \;=\; -1\,.$$

Equating the imaginary parts, we get

$$\binom{11}{1}c^{10}s - \binom{11}{3}c^8s^3 + \binom{11}{5}c^6s^5 - \binom{11}{7}c^4s^7 + \binom{11}{9}c^2s^9 - \binom{11}{11}s^{11} \;=\; 0\,.$$

Since $s \neq 0$, we obtain

$$s^{10} - 55s^8c^2 + 330s^6c^4 - 462s^4c^6 + 165s^2c^8 - 11c^{10} \;=\; 0\,.$$

Since $c \neq 0$, dividing the last equation by c^{10} yields

$$\tan^{10}\theta - 55\tan^8\theta + 330\tan^6\theta - 462\tan^4\theta + 165\tan^2\theta - 11 \;=\; 0\,.$$

Therefore, the roots of the given polynomial equation are given by $x = \tan^2\left(\frac{k\pi}{11}\right)$, $k = 1, 3, 5, 7, 9$.

Also solved by MANUEL BENITO MUÑOZ and EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; DOUGLASS L. GRANT, University College of Cape Breton, Sydney, Nova Scotia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece (2 solutions); GERRY LEVERSHA, St. Paul's School, London, England; KEE-WAI LAU, Hong Kong; VEDULA N. MURTY, Visakhapatnam, India; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ARAM TANGBOONDOUANGJIT, Carnegie Mellon University, Pittsburgh, PA, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

All the submitted solutions are more or less equivalent to the one given above. Almost all the solvers gave the answers $x = \tan^2\left(\frac{k\pi}{11}\right)$, $k = 1, 2, 3, 4, 5$. *Benito and Fernández gave the answer* $x = \cot^2\left(\frac{k\pi}{22}\right)$, $k = 1, 3, 5, 7, 9$. *It is easy to see that all these expressions are the same as the one obtained by Howard.*

2371. [1998: 364] *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

For n an integer greater than 4, let $f(n)$ be the number of five-element subsets, S, of $\{1, 2, \ldots, n\}$ which have *no isolated points*, that is, such that if $s \in S$, then either $s-1$ or $s+1$ (**not** taken $\mathrm{modulo}\ n$) is in S.

Find a "nice" formula for $f(n)$.

I. *Solution by Manuel Benito and Emilio Fernandez, I.B. Praxedes Mateo Sagasta, Logroño, Spain.*

Let us represent any such subset S as $[a_1, a_2, a_3, a_4, a_5]$, where $a_1 < a_2 < a_3 < a_4 < a_5$. It is clear that the number a_3 can take on each of the $n-4$ values from 3 to $n-2$, but let us ask: how many times can it take on each one of these values?

Well, a_1 is not isolated, so that $a_2 = a_1+1$, and a_5 is not isolated either so $a_4 = a_5 - 1$. Then, for each fixed k from 3 to $n-2$ we can enumerate the subsets S_k for which $a_3 = k$, in the following manner:

- type $[k-2, k-1, k, k+1, k+2]$ .... 1 subset;
- from $a_1 = 1$ to $a_1 = k-3$, type $[a_1, a_2, k, k+1, k+2]$ .... $k-3$ subsets;
- from $a_5 = k+3$ to $a_5 = n$, type $[k-2, k-1, k, a_5-1, a_5]$ .... $n-(k+3)+1$ subsets;

which gives a total of $n-4$ subsets S_k for each k. Since there are $n-4$ values of k, this gives

$$f(n) \;=\; (n-4)^2\,.$$

II. *Solution by Kathleen E. Lewis, SUNY Oswego, Oswego, NY, USA.*

The characteristic function for such a subset will contain $n-5$ zeros and five ones. To be non-isolated, the ones must either consist of a single block or be broken into a block of size two and a block of size three (in either order). Looking at the case of two blocks, we can think of this as a problem of placing two distinct items plus $n-5$ indistinguishable items (the zeros) in a sequence, which can be done in $(n-3)(n-4)$ ways. But if the two blocks of ones end up adjacent to each other, we get the case of all five ones together. Since it does not matter in this case which block of ones is first, we have double-counted these arrangements. There are $n-4$ ways to put all the ones together among the $n-5$ zeros. Therefore, when we compensate for double-counting this case earlier, we get a final answer of

$$(n-3)(n-4) - (n-4) \;=\; (n-4)^2\,.$$

III. *Solution by Jeremy Young, student, Nottingham High School, Nottingham, UK.*

Given $f(n)$, consider $f(n+1)$ and every corresponding subset S. There are by definition $f(n)$ subsets S with $n+1 \notin S$. If $n+1 \in S$, $n \in S$ too.

Consider subsets of this form. Now if $n-1 \in S$, the remaining two elements must be adjacent; that is, k and $k+1$ with $1 \leq k \leq n-3$, and there are $n-3$ such subsets. If $n-1 \notin S$, the three remaining elements must be adjacent; that is, $k, k+1, k+2$ with $1 \leq k \leq n-4$, and there are $n-4$ such subsets.

We have thus established the recurrence

$$f(n+1) \;=\; f(n)+(n-3)+(n-4) \;=\; f(n)+2n-7\,,$$

with $f(5)=1$ (obviously) giving $f(6)=4$, $f(7)=9$, $f(8)=16$, ... Note that $f(k)=(k-4)^2$ is true for $k=5$ and gives

$$f(k+1) \;=\; (k-4)^2+2k-7 \;=\; k^2-6k+9 \;=\; (k-3)^2\,.$$

Therefore $f(n)=(n-4)^2$ by induction, for all integers $n \geq 5$.

Also solved by SAM BAETHGE, Nordheim, TX, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong, China; GERRY LEVERSHA, St. Paul's School, London, England; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer. One incorrect solution was received.

Bradley's solution was the same as Lewis's, and Lau, Leversha and Perz all had solutions similar to Young's.

2372. [1998: 365] *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

For n and k positive integers, let $f(n,k)$ be the number of k–element subsets S of $\{1, 2, \ldots, n\}$ satisfying:

(i) $1 \in S$ and $n \in S$; and

(ii) whenever $s \in S$ with $s < n-1$, then either $s+2 \in S$ or $s+3 \in S$.

Prove that $f(n,k) = f(4k-2-n, k)$ for all n and k; that is, the sequence

$$f(k,k)\,,\; f(k+1,k)\,,\; f(k+2,k)\,,\; \ldots\,,\; f(3k-2,k)$$

of non-zero values of $\{f(n,k)\}_{n=1}^{\infty}$ is a palindrome for every k.

Solution by Gerry Leversha, St. Paul's School, London, England.

I shall call sets which satisfy the condition *permissible*.

In general, it is clear that $f(k,k)=1$. Also $f(3k-2,k)=1$, since the only possible set is $\{1, 4, 7, \ldots, 3k-5, 3k-2\}$, an arithmetic progression with common difference 3 and k terms. Since this is the "sparsest" possible k–element permissible set, it is clear that $f(n,k)=0$ for $n \geq 3k-1$. Equally obviously $f(n,k)=0$ for $n < k$. So we need only consider n such that $k < n < 3k-2$. For $n = 2k-1$, we have $4k-2-n = 2k-1$, so the palindromic condition is trivial here. It remains to show that for $k < n < 2k - 1$, $f(n,k) = f(4k - 2 - n, k)$.

Consider a permissible set $S \subset \{1, 2, 3, \ldots, n\}$ which has k elements. Let T be the sequence of *differences* between successive elements of S; then

- T is an ordered $(k-1)$–tuple each of whose elements is either 1, 2 or 3;
- The sum of the elements of T is $n - 1$;
- T cannot contain the elements 1, 3 in that order.

Any such T will yield a permissible set S. The restriction on subsequences of the form 1, 3 ensures that we cannot have a, $a + 1$, $a + 4$ appearing in S, which would contravene the condition since neither $a + 2$ nor $a + 3$ would be in S. Hence

$f(n,k)$ is equal to the number of such sequences T.

Now from any such sequence T construct a sequence T^* as follows:

- Replace every element x of T by $4 - x$;
- Reverse the sequence so formed.

Then the new sequence T^* has the following properties:

- It is an ordered $(k-1)$–tuple formed from the elements 1, 2 and 3;
- The sum of the elements is $4(k - 1) - (n - 1) = 4k - n - 3$;
- It cannot contain a subsequence 1, 3 (since there would have to have been such a subsequence in T in the first place).

Thus there is a one-to-one correspondence between sequences of type T and those of type T^*. But sequences of type T^* correspond in turn to permissible k–element sets $S \subset \{1, 2, \ldots, 4k - n - 2\}$. Hence we have shown that

$$f(n,k) \;=\; f(4k - n - 2, k)\,.$$

I shall illustrate this process in the case of $f(7,5)$ and $f(11,5)$. The four columns below show, respectively,

- a permissible 5–element subset of $\{1, 2, 3, 4, 5, 6, 7\}$,
- the corresponding T sequence,
- the corresponding T^* sequence,
- the permissible 5–element subset of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$.

$\{1,2,3,5,7\}$	$(1,1,2,2)$	$(2,2,3,3)$	$\{1,3,5,8,11\}$
$\{1,2,4,5,7\}$	$(1,2,1,2)$	$(2,3,2,3)$	$\{1,3,6,8,11\}$
$\{1,2,4,6,7\}$	$(1,2,2,1)$	$(3,2,2,3)$	$\{1,4,6,8,11\}$
$\{1,3,4,5,7\}$	$(2,1,1,2)$	$(2,3,3,2)$	$\{1,3,6,9,11\}$
$\{1,3,4,6,7\}$	$(2,1,2,1)$	$(3,2,3,2)$	$\{1,4,6,9,11\}$
$\{1,3,5,6,7\}$	$(2,2,1,1)$	$(3,3,2,2)$	$\{1,4,7,9,11\}$
$\{1,4,5,6,7\}$	$(3,1,1,1)$	$(3,3,3,1)$	$\{1,4,7,10,11\}$

Also solved by MANUEL BENITO and EMILIO FERNANDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

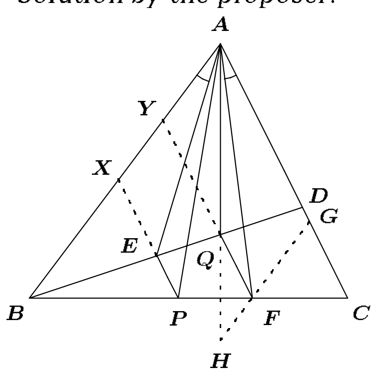
Lambrou, Lewis and Young gave "combinatorial" solutions similar to Leversha's.

2375. [1998: 365] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let D be a point on side AC of triangle ABC. Let E and F be points on the segments BD and BC, respectively, such that $\angle BAE = \angle CAF$. Let P and Q be points on BC and BD, respectively, such that $EP \parallel DC$ and $FQ \parallel CD$.

Prove that $\angle BAP = \angle CAQ$.

Solution by the proposer.



Let PE and FQ meet AB at X and Y, respectively.

Since $PX \parallel FY$ and FP, QE and YX concur at B, we have

$$\frac{XE}{XP} = \frac{YQ}{YF}. \qquad (1)$$

Let the line through F parallel to AB meet AC and AQ at G and H, respectively.

Since $AY \parallel FH$ and $QF \parallel AG$, we get

$$\frac{YQ}{YF} = \frac{AQ}{AH} = \frac{GF}{GH}. \qquad (2)$$

From (1) and (2), we now have

$$\frac{XE}{XP} = \frac{GF}{GH}. \qquad (3)$$

Since $XE \parallel AG$ and $AX \parallel GF$, we have

$$\angle AXP = 180^\circ - \angle XAG = \angle AGF. \qquad (4)$$

Since $\angle XAE = \angle BAE = \angle CAF = \angle GAF$, we get from (4) that

$$\triangle AXE \sim \triangle AGF\,. \qquad (5)$$

From (3) and (5), we have that

$$\triangle AXP \sim \triangle AGH\,.$$

Therefore, we obtain that $\angle XAP = \angle GAH$, and this implies that $\angle BAP = \angle CAQ$.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; MANUEL BENITO MUÑOZ and EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; C. FESTRAETS-HAMOIR, Brussels, Belgium; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; and D.J. SMEENK, Zaltbommel, the Netherlands.

No-one other than the proposer used pure geometric methods. Lambrou made use of vectors, and all the other solvers made use of trigonometry.

2376. [1998: 424] *Proposed by Albert White, St. Bonaventure University, St. Bonaventure, NY, USA.*

Suppose that ABC is a right-angled triangle with the right angle at C. Let D be a point on hypotenuse AB, and let M be the mid-point of CD. Suppose that $\angle AMD = \angle BMD$. Prove that

1. $\overline{AC}^2\,\overline{MC}^2 + 4[ABC]\,[BCD] = \overline{AC}^2\,\overline{MB}^2$;
2. $4\overline{AC}^2\,\overline{MC}^2 - \overline{AC}^2\overline{BD}^2 = 4[ACD]^2 - 4[BCD]^2$,

where $[XYZ]$ denotes the area of $\triangle XYZ$.

(This is a continuation of problem **1812**, [1993: 48].)

Solution by Gerry Leversha, St. Paul's School, London, England.

We do not make use of the assumption that $\angle AMD = \angle BMD$. Let $AB = c, BC = a, CA = b, BD = \lambda c, \angle BCD = \theta$, and let X be the foot of the perpendicular from D to BC.

$$\begin{aligned} MB^2 - MC^2 &= BC^2 - 2MC \cdot BC\cos\theta \\ &= a^2 - CD \cdot a\cos\theta \\ &= a^2 - a \cdot CX \;=\; a \cdot BX\,. \end{aligned}$$

But by similar triangles $BX = \lambda a$ and so $MB^2 - MC^2 = \lambda a^2$. Now $[ABC] = \frac{1}{2}ab$ and $[BCD] = \frac{1}{2}\lambda ab$ and so

$$4[ABC][BCD] = \lambda a^2 b^2 = AC^2(MB^2 - MC^2)\,,$$

which is part 1 of the problem. In a similar vein

$$\begin{aligned} AC^2(4MC^2 - BD^2) &= AC^2(CD^2 - BD^2) \\ &= b^2(BC^2 - 2BC \cdot BD \cos B) \\ &= b^2(a^2 - 2a \cdot \lambda c \cos B) \;=\; a^2b^2(1-2\lambda) \\ &= 4((1-\lambda)^2 - \lambda^2)[ABC]^2 \\ &= 4[ACD]^2 - 4[BCD]^2\,, \end{aligned}$$

as required for part 2.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; D.J. SMEENK, Zaltbommel, the Netherlands; PARAGIOU THEOKLITOS, Limassol, Cyprus; and the proposer.

Smeenk's solution is virtually identical to our featured solution, while that of Lambrou is quite similar although he avoids the cosine law by using Cartesian coordinates. All submitted solutions avoided the unnecessary condition involving the pair of angles at M; the proposer evidently arrived at his problem while investigating problem **1812** [1993: 48, 1994: 20-22], *and he failed to notice that his result is valid for any choice of D on AB.*

2377. [1998: 425] *Proposed by Nikolaos Dergiades, Thessaloniki, Greece.*

Let ABC be a triangle and P a point inside it. Let $BC = a$, $CA = b$, $AB = c$, $PA = x$, $PB = y$, $PC = z$, $\angle BPC = \alpha$, $\angle CPA = \beta$ and $\angle APB = \gamma$.

Prove that $ax = by = cz$ if and only if $\alpha - A = \beta - B = \gamma - C = \frac{\pi}{3}$.

Solution by Jun-hua Huang, the Middle School Attached To Hunan Normal University, Changsha, China.

Let M, N, L be the feet of the perpendiculars from P to CA, AB, BC, respectively.

Thus $ax = 2R\sin A \cdot x = 2R \cdot MN$, where R is the circumradius of $\triangle ABC$. Similarly $by = 2R \cdot NL$ and $cz = 2R \cdot ML$. So $ax = by = cz \iff MN = NL = ML$. Now $\alpha - A = \angle ABP + \angle ACP = \angle NLP + \angle MLP = \angle NLM$ and $\beta - B = \angle LMN$. $\gamma - C = \angle MNL$. So

$$MN = NL = ML \iff \alpha - A = \beta - B = \gamma - C = \frac{\pi}{3}\,,$$

which completes the proof.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; G. TSINTSIFAS, Thessaloniki, Greece; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

2380. [1998: 425] *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

When the price of a certain book in a store is reduced by $1/3$ and rounded to the nearest cent, the cents and dollars are switched. For example, if the original price was \$43.21, the new price would be \$21.43 (this does not satisfy the "reduced by $1/3$" condition, of course). What was the original price of the book? [For the benefit of readers unfamiliar with North American currency, there are 100 cents in one dollar.]

Solution by Gerry Leversha, St. Paul's School, London, England.

Let the price be a dollars and b cents. Then

$$\frac{2}{3}(100a+b) \;=\; 100b+a+\frac{x}{3}\,,$$

where $x \in \{-1,0,1\}$ is included to deal with any possible rounding. This simplifies to

$$197a \;=\; 298b + x\,.$$

Now 197 and 298 are coprime, so the smallest solution in the case $x = 0$ is $a = 298$, $b = 197$, which is impossible since $0 \le b \le 99$. The usual Euclidean algorithm procedure yields

$$59\times 197 \;-\; 39\times 298 \;=\; 1\,,$$

and this shows that we should take $a = 59$, $b = 39$ and $x = 1$. Hence the price of the book was **\$59.39**.

Also solved by CHARLES ASHBACHER, Cedar Rapids, IA, USA; MIGUEL CARRIÓN ÁLVAREZ, Universidad Complutense de Madrid, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; SHAWN GODIN, Cairine Wilson S.S., Orleans, Ontario; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; J.A. McCALLUM, Medicine Hat, Alberta; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; D.J. SMEENK, Zaltbommel, the Netherlands; DIGBY SMITH, Mount Royal College, Calgary, Alberta; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer. Two incorrect solutions were received, at least one of which was due to misunderstanding the problem.

Many solvers noted that the answer is unique, as can be seen from the above solution.

Diminnie opines that "in the U.S. there would be no nontrivial solution to this problem, since U.S. stores seem to insist on rounding up in all circumstances!".

2381. [1998: 425] *Proposed by Angel Dorito, Geld, Ontario.*

Solve the equation $\log_2 x \;=\; \log_4(x+1)$.

Solution by Chris Cappadocia, student, St. Joseph Scollard Hall SS, North Bay, Ontario.

Let both of them equal y. Then

$$2^y = x \qquad \text{and} \qquad 4^y = 2^{2y} = x+1\,.$$

And so

$$2^y = \frac{2^{2y}}{2^y} = \frac{x+1}{x}.$$

Comparing, we get $x = (x+1)/x$, and solving for x and throwing away the negative value, we get a final answer of

$$x = \frac{1+\sqrt{5}}{2}.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts, USA; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; BOOKERY PROBLEM GROUP, Walla Walla, Washington, USA; MIGUEL CARRIÓN ÁLVAREZ, Universidad Complutense de Madrid, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; RUSSELL EULER and JAWAD SADEK, NW Missouri State University, Maryville, MO, USA; C. FESTRAETS-HAMOIR, Brussels, Belgium; SHAWN GODIN, Cairine Wilson S. S., Orleans, Ontario; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; JOE HOWARD, New Mexico Highlands University, Las Vegas, NM, USA; JUN-HUA HUANG, the Middle School Attached To Hunan Normal University, Changsha, China; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ANGEL JOVAL ROQUET, Instituto Español de Andorra, Andorra; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; J.A. McCALLUM, Medicine Hat, Alberta; JOHN GRANT McLOUGHLIN, Faculty of Education, Memorial University, St. John's, Newfoundland; HENRY J. RICARDO, Medger Evers College (CUNY), Brooklyn, New York, USA; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; MAX SHVARAYEV, Tucson, Arizona, USA; SKIDMORE COLLEGE PROBLEM GROUP, Saratoga Springs, New York, USA; D.J. SMEENK, Zaltbommel, the Netherlands; DIGBY SMITH, Mount Royal College, Calgary, Alberta; ARAM TANGBOONDOUANGJIT, Carnegie Mellon University, Pittsburgh, PA, USA; PARAGIOU THEOKLITOS, Limassol, Cyprus; PANOS E. TSAOUSSOGLOU, Athens, Greece; UNIVERSITY OF ARIZONA PROBLEM SOLVING GROUP, Tucson, Arizona, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

Joval's solution must be the first one ever received by this journal from Andorra! Welcome — and can we now hear from Liechtenstein and Monaco?

2382. [1998: 425] *Proposed by Mohammed Aassila, Université Louis Pasteur, Strasbourg, France.*

If $\triangle ABC$ has inradius r and circumradius R, show that

$$\cos^2\left(\frac{B-C}{2}\right) \geq \frac{2r}{R}.$$

Solution by Vedula N. Murty, Dover, PA, USA.

We have

$$\left(\cos\frac{B-C}{2} - 2\sin\frac{A}{2}\right)^2 \geq 0,$$

so

$$\begin{aligned}\cos^2\left(\frac{B-C}{2}\right) &\geq 4\cos\frac{B-C}{2}\sin\frac{A}{2} - 4\sin^2\frac{A}{2}\\ &= 4\sin\frac{A}{2}\left[\cos\frac{B-C}{2} - \cos\frac{B+C}{2}\right]\\ &= 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{2r}{R}\,,\end{aligned}$$

since $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ [see for example, Roger A. Johnson, *Modern Geometry* (1929) **298a**].

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; JUN-HUA HUANG, the Middle School Attached To Hunan Normal University, Changsha, China; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; ISAO NAOI and HIDETOSHI FUKAGAWA, Gifu, Japan; ISTVÁN REIMAN, Budapest, Hungary; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; ARAM TANGBOONDOUANGJIT, Carnegie Mellon University, Pittsburgh, PA, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; G. TSINTSIFAS, Thessaloniki, Greece; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; and the proposer.

Arslanagić, Bellot Rosado, Janous, Naoi and Fukagawa, and Seiffert all note that we have equality precisely when $2a = b + c$. Konečný, Smeenk and Yiu provide the equivalent condition, $\tan\frac{B}{2}\tan\frac{C}{2} = \frac{1}{3}$.

*Iftimie Simion (Math Teacher, Stuyvesant HS, New York, NY, USA) points out that this problem appears in a 10th grade textbook [*Matematică: Geometrie si trigonometrie*, A. Cota et al., p. 106] used in Romania.*

Bellot Rosado refers us to two notes on this inequality by Dan Plaesu (Iasi) and Gheorge Marchidan (Suceava) in the Romanian journal Gazeta matematica *(1991) nos. 6 and 7. In the second note the authors prove the related inequality*

$$\cos^2\frac{B-C}{2} \geq \frac{a^2bc}{R^2(b+c)^2}\,.$$

Bellot Rosado also shows that

$$\frac{a^2bc}{R^2(b+c)^2} > \frac{2r}{R}$$

precisely when $\frac{a}{s} \in (3-\sqrt{5}, 1)$.

2383. [1998: 425] *Proposed by Mohammed Aassila, Université Louis Pasteur, Strasbourg, France.*

Suppose that three circles, each of radius 1, pass through the same point in the plane. Let A be the set of points which lie inside at least two of the circles. What is the least area that A can have?

Solution by Kathleen E. Lewis, SUNY Oswego, Oswego, NY, USA.

Let the point of intersection of the circles be labelled O and the centres be C_1, C_2 and C_3. Consider first the area of intersection of the first two circles as a function of the angle C_1OC_2, which we will call θ. If D is the other point of intersection of these circles, then the area of intersection is twice the difference between the area of the sector created by the angle OC_1D and the area of the triangle OC_1D. The angle OC_1D has measure $\pi-\theta$, so the sector has area $(\pi-\theta)/2$ and the triangle has area $\sin(\theta/2)\cos(\theta/2) = (1/2)\sin\theta$. Therefore the area of overlap of the two circles is $\pi-\theta-\sin\theta$, which we call $f(\theta)$. Since $f'(\theta) = -1-\cos\theta$, which is never positive, the area decreases as θ increases from 0 to π.

Suppose that the circles with centres at C_1 and C_2 are in fixed position, with an angle of $\theta \le \pi$ between them, and we wish to place the third circle in a way that minimizes the total area of overlap. Since we want large angles between the centres of the circles, we want to put the third circle so that the new angles formed at O divide the angle $2\pi-\theta$ rather than θ. If we call these angles ϕ and ρ, then the total area of overlap is

$$(\pi-\theta-\sin\theta)+(\pi-\phi-\sin\phi)+(\pi-\rho-\sin\rho)\,.$$

Since θ is fixed and π is constant, we need to minimize

$$\begin{aligned}-\phi-\sin\phi-\rho-\sin\rho &= -\phi-\sin\phi-(2\pi-\theta-\phi)-\sin(2\pi-\theta-\phi)\\ &= \theta-2\pi-\sin\phi-\sin(2\pi-\theta-\phi)\end{aligned}$$

as a function of ϕ. Taking the derivative, we get $-\cos\phi+\cos(2\pi-\theta-\phi)$, which is zero if and only if $\cos\phi = \cos(2\pi-\theta-\phi)$. Since the sum of these two angles is strictly less than 2π, the cosines can only be equal if the angles are. Thus, the minimum area occurs when $\phi=\rho$. This argument could equally well be used to argue that if any two of the angles are unequal, then the area could be reduced by moving the circle between those angles to equalize them (leaving the other two circles fixed). The minimum area therefore occurs when all three angles are equal to $2\pi/3$, giving an overlap area of $3[\pi/3-\sin(2\pi/3)] = \pi-3\sqrt{3}/2$.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; MAX SHKARAYEV, Tuscon, AZ, USA; THE UNIVERSITY OF ARIZONA PROBLEM SOLVING GROUP, Tuscon, AZ, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer. There were one incomplete and three incorrect solutions.

2384. [1998: 425] *Proposed by Paul Bracken, CRM, Université de Montréal, Québec.*

Prove that $2(3n-1)^n \ge (3n+1)^n$ for all $n \in \mathbb{N}$.

Solution by Michel Bataille, Rouen, France.

The inequality is obvious for $n=0$, so we may assume $n \ge 1$. We have to prove

$$\left(\frac{3n-1}{3n+1}\right)^n \ge \frac{1}{2}\,,$$

or, equivalently,

$$n \ln \frac{3n-1}{3n+1} \;\geq\; \ln \frac{1}{2} \,. \tag{1}$$

To this aim, we introduce the function

$$f(x) \;=\; x \ln \frac{3x-1}{3x+1}$$

defined on $[1, \infty)$. We compute

$$f'(x) \;=\; \ln \frac{3x-1}{3x+1} + \frac{6x}{9x^2-1} \qquad \text{and} \qquad f''(x) \;=\; \frac{-12}{(9x^2-1)^2} \,.$$

Since $f''(x) < 0$, $f'(x)$ is strictly decreasing on $[1, \infty)$. Moreover,

$$\lim_{x \to \infty} f'(x) = 0 \,,$$

so $f'(x) > 0$ for all $x \in [1, \infty)$. Hence f is increasing on $[1, \infty)$ and, since $f(1) = \ln \frac{1}{2}$, the inequality (1) follows.

Also solved by MIGUEL CARRIÓN ÁLVAREZ, Universidad Complutense de Madrid, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; KEITH EKBLAW, Walla Walla, WA, USA; RUSSELL EULER and JAWAD SADEK, NW Missouri State University, Maryville, MO, USA; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JUN-HUA HUANG, the Middle School Attached To Hunan Normal University, Changsha, China; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece (three solutions); NORVALD MIDTTUN, Royal Norwegian Navy Academy, Norway; VEDULA N. MURTY, Visakhapatnam, India; VICTOR OXMAN, University of Haifa, Haifa, Israel; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; MAX SHKARAYEV, Tucson, AZ, USA; DIGBY SMITH, Mount Royal College, Calgary, Alberta; PANOS E. TSAOUSSOGLOU, Athens, Greece; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer. There were also three incorrect solutions submitted.

Janous and Seiffert have proved the more general inequality

$$\frac{a+b}{a-b}(ax-b)^x \geq (ax+b)^x$$

for all real $x \in [1, \infty)$ and real a and b such that $a > b > 0$.

2385. [1998: 426] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

A die is thrown $n \geq 3$ consecutive times. Find the probability that the sum of its n outcomes is greater than or equal to $n+6$ and less than or equal to $6n-6$.

Solution by Jeremy Young, student, Nottingham High School, Nottingham, UK, modified by the editor.

Consider the number of ways of scoring a sum of $n+k$ for some k such that $0 \le k \le 5$. This is the number of ways of expressing $n+k$ as a sum of n positive integers, where the order of the summands matters. This number is well-known to be $\binom{n+k-1}{n-1}$. [*Ed.*: see Theorem 1.5.3 on p. 142 of [1].] Furthermore, the number of ways of scoring a sum of $6n-k$ for some k such that $0 \le k \le 5$ is the same as that of scoring a sum of $n+k$. This can be seen by replacing each outcome λ_j by $7-\lambda_j$ for $j = 1, 2, \ldots, n$, since $n \le \sum_{j=1}^{n} \lambda_j \le n+5$ if and only if $6n-5 \le \sum_{j=1}^{n}(7-\lambda_j) \le 6n$.

The total number of possible scoring sequences is 6^n. Hence, the required probability is

$$\begin{aligned} p_n &= \frac{1}{6^n}\left(6^n - 2\sum_{k=0}^{5}\binom{n+k-1}{n-1}\right) \\ &= 1 - \frac{2}{6^n}\sum_{k=0}^{5}\binom{n+k-1}{n-1} = 1 - \frac{2}{6^n}\binom{n+5}{n} \end{aligned}$$

by the well-known combinatorial identity $\sum_{m=r}^{n}\binom{m}{r} = \binom{n+1}{r+1}$, where $0 \le r \le n$. [*Ed.*: See Theorem 1.6.4 on p. 156 of [1].]

Reference:

[1]. H. Joseph Straight, *Combinatorics, An Invitation*, Brooks/Cole, 1993.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; NORVALD MIDTTUN, Royal Norwegian Navy Academy, Norway; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. There were two partially incorrect solutions.

Janous, Leversha and Midttun all obtained the answer $1 - \frac{2}{6^n}\sum_{k=0}^{5}\binom{5}{5-k}\binom{n}{k}$ *first, and then stated or showed that* $\sum_{k=0}^{5}\binom{5}{5-k}\binom{n}{k} = \binom{n+5}{5}$*, which, of course, is just a special case of the well-known and easy-to-prove Vandermonde's Identity:*

$$\sum_{k=0}^{l}\binom{m}{k}\binom{n}{l-k} = \binom{m+n}{l} \qquad \text{for integers } l,\ m,\ n \text{ such that } 0 \le l \le m,\ n.$$

[Ed.: *See Ex. 26 on p. 167 of* [1].]

Janous remarked that, in general, for $t \in \{0, 1, 2, \ldots, \lfloor 5n/2 \rfloor\}$*, we have*

$$p(n+t \le X \le 6n-t) = 1 - \frac{2}{6^n}\binom{n+t+1}{t-1},$$

where X *is the random variable denoting the sum of the* n *outcomes.*

Leversha based his solution on the fact that the probability generating function for X *is* $G(t) = \frac{1}{6^n}\left(t+t^2+t^3+t^4+t^5+t^6\right)^n$.

2388 [1998, 503; Correction 1999, 171]. *Proposed by Daniel Kupper, Büllingen, Belgium.*

Suppose that $n \geq 1 \in \mathbb{N}$ is given and that, for each integer $k \in \{0, 1, \ldots, n-1\}$, the numbers a_k, b_k, $z_k \in \mathbb{C}$ are given, with the z_k^2 distinct. Suppose that the polynomials

$$A_n(z) = z^n + \sum_{k=0}^{n-1} a_k z^k \quad \text{and} \quad B_n(z) = z^n + \sum_{k=0}^{n-1} b_k z^k$$

satisfy $A_n(z_j) = B_n(z_j^2) = 0$ for all $j \in \{0, 1, \ldots, n-1\}$.

Find an expression for $b_0, b_1, \ldots, b_{n-1}$ in terms of $a_0, a_1, \ldots, a_{n-1}$.

Solution by Kee-Wai Lau, Hong Kong.

We have $A_n(z) \equiv \prod_{k=1}^{n} (z - z_k)$ and $B_n(z) = \prod_{k=1}^{n} (z - z_k^2)$.

Hence, $B_n\left(z^2\right) = \prod_{k=1}^{n} (z - z_k)(z + z_k) \equiv (-1)^n A_n(z) A_n(-z)$.

Thus, $\sum_{k=0}^{n} b_k z^{2k} \equiv (-1)^n \left(\sum_{k=0}^{n} a_k z^k\right)\left(\sum_{k=0}^{n} (-1)^k a_k z^k\right)$.

It follows that, for $k = 0, 1, 2, \ldots, n$,

$$b_k = \sum_{j=\max\{0,2k-n\}}^{\min\{n,2k\}} (-1)^{n-j} a_j a_{2k-j} \,.$$

Also solved by MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; JOSÉ LUIS DIAZ, Universidad Politécnica de Cataluña, Terrassa, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. There was one incorrect solution.

We thank Michael Lambrou for the correction to the statement of this problem.

The answer obtained by Seiffert is the same as the one given above, while those given by Lambrou, Leversha and the proposer are minor variations thereof. On the other hand, the answer obtained by Luis has a different "appearance":

$$b_k = (-1)^{n-k}\left(a_k^2 + 2\sum_{j=1}^{n-k+1} (-1)^j a_{k-j} a_{k+j}\right), \qquad k = 0, 1, \ldots, n$$

with $a_j = 0$ if $j < 0$ or $j > n$.

2392. [1998: 504] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*
Suppose that x_i, y_i, $(1 \le i \le n)$ are positive real numbers. Let

$$A_n = \sum_{i=1}^{n} \frac{x_i y_i}{x_i + y_i}, \qquad B_n = \frac{\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{\sum_{i=1}^{n}(x_i + y_i)},$$

$$C_n = \frac{\left(\sum_{i=1}^{n} x_i\right)^2 + \left(\sum_{i=1}^{n} y_i\right)^2}{\sum_{i=1}^{n}(x_i + y_i)}, \qquad D_n = \sum_{i=1}^{n} \frac{x_i^2 + y_i^2}{x_i + y_i}.$$

Prove that

1. $A_n \le C_n$,
2. $B_n \le D_n$,
3. $2A_n \le 2B_n \le C_n \le D_n$.'

Solution by Heinz-Jürgen Seiffert, Berlin, Germany.
It is sufficient to prove the inequalities in the third part.

Let

$$X_n = \sum_{j=1}^{n} x_j, \qquad Y_n = \sum_{j=1}^{n} y_j, \quad \text{and} \quad Z_n = \sum_{j=1}^{n} \frac{(x_j - y_j)^2}{x_j + y_j}.$$

The Cauchy-Schwarz Inequality gives

$$\left(\sum_{j=1}^{n}(x_j - y_j)\right)^2 = \left(\sum_{j=1}^{n} \sqrt{x_j + y_j}\, \frac{x_j - y_j}{\sqrt{x_j + y_j}}\right)^2 \le \left(\sum_{j=1}^{n}(x_j + y_j)\right)\left(\sum_{j=1}^{n} \frac{(x_j - y_j)^2}{x_j + y_j}\right),$$

or

$$(X_n - Y_n)^2 \le (X_n + Y_n)\, Z_n. \tag{1}$$

Using the easily verified identities

$$4A_n = X_n + Y_n - Z_n, \qquad B_n = \frac{X_n Y_n}{X_n + Y_n},$$

$$C_n = \frac{X_n^2 + Y_n^2}{X_n + Y_n}, \quad \text{and} \quad 2D_n = X_n + Y_n + Z_n,$$

we see that the inequalities $A_n \le B_n$ and $C_n \le D_n$ both follow from (1), while the inequality $2B_n \le C_n$ is an immediate consequence of the AM–GM Inequality.

Also solved by MICHEL BATAILLE, Rouen, France; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; PANOS E. TSAOUSSOGLOU, Athens, Greece; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

All the solvers, except Tsaoussoglou and the proposer, had solutions similar to the one given above. Tsaoussoglou proved stronger inequalities than the proposed 1. *and* 2.*, to obtain his solution.*

2393. [1998: 504] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*
Suppose that a, b, c and d are positive real numbers. Prove that

1. $$\Big((a+b)(b+c)(c+d)(d+a)\Big)^{3/2} \;\geq\; 4abcd(a+b+c+d)^2,$$

2. $$\Big((a+b)(b+c)(c+d)(d+a)\Big)^{3} \;\geq\; 16(abcd)^2 \prod_{\substack{a,b,c,d\\ \text{cyclic}}} (2a+b+c).$$

Solution by Jun-hua Huang, the Middle School Attached To Hunan Normal University, Changsha, China.

1. The inequality is equivalent to

$$[(a+b)(b+c)(c+d)(d+a)]^3 \;\geq\; 16(abcd)^2(a+b+c+d)^4\,,$$

or, dividing by $(abcd)^6$,

$$\left[\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{c}+\frac{1}{d}\right)\left(\frac{1}{d}+\frac{1}{a}\right)\right]^3$$
$$\geq\; 16\left(\frac{1}{bcd}+\frac{1}{acd}+\frac{1}{abd}+\frac{1}{abc}\right)^4\,.$$

Let $x=a^{-1}$, $y=b^{-1}$, $z=c^{-1}$, and $w=d^{-1}$. The inequality becomes

$$[(x+y)(y+z)(z+w)(w+x)]^3 \;\geq\; 16(xyz+yzw+zwx+wxy)^4\,.$$

We prove it as follows. Applying the Geometric Mean–Arithmetic Mean Inequality, we obtain

$$\begin{aligned}
&4(xyz+yzw+zwx+wxy)^2\\
&\quad= 4\left[xw(y+z)+yz(x+w)\right]^2\\
&\quad= 4\left[\sqrt{xw}\sqrt{xw}(y+z)+\sqrt{yz}\sqrt{yz}(x+w)\right]^2\\
&\quad\leq \left[\sqrt{xw}(x+w)(y+z)+\sqrt{yz}(y+z)(x+w)\right]^2\\
&\quad= (x+w)^2(y+z)^2(\sqrt{xw}+\sqrt{yz})^2\\
&\quad= (x+w)^2(y+z)^2(xw+yz+2\sqrt{xwyz})\\
&\quad\leq (x+w)^2(y+z)^2(xw+yz+xy+wz)\\
&\quad= (x+w)^2(y+z)^2(x+z)(w+y)\,.
\end{aligned}$$

Hence

$$4(xyz + yzw + zwx + wxy)^2 \;\le\; (x+w)^2(y+z)^2(x+z)(w+y)\,.$$

Similarly,

$$4(xyz + yzw + zwx + wxy)^2 \;\le\; (x+w)(y+z)(x+z)^2(w+y)^2\,.$$

The result follows by multiplying the last two inequalities.

2. The inequality follows easily from the first one, because

$$\prod_{\text{cyclic}}(2a+b+c) \;\le\; \left(\frac{1}{4}\sum_{\text{cyclic}}(2a+b+c)\right)^4 \;=\; (a+b+c+d)^4\,,$$

by the Geometric Mean–Arithmetic Mean Inequality.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.