PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}$"×11" or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 April 2000**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

Last month's problem section asked, in error, for solutions by 1 January 2000. That was a Y2K bug! It should have read **1 March 2000**.

2452. Correction. *Proposed by Antal E. Fekete, Memorial University of Newfoundland, St. John's, Newfoundland.*

Establish the following equalities:

(a) $$\sum_{n=0}^{\infty} \frac{(2n+1)^2}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(2n+2)^2}{(2n+2)!}.$$

(b) and (c) are correct as originally printed.

2453. Correction. *Proposed by Antal E. Fekete, Memorial University of Newfoundland, St. John's, Newfoundland.*

Establish the following equalities:

(a) $$\sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)^3}{(2n+1)!} = -3 \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!}.$$

(b) $$\sum_{n=0}^{\infty} (-1)^n \frac{(2n)^3}{(2n)!} = -3 \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!}.$$

(c) $$\left(\sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)^2}{(2n+1)!}\right)^2 + \left(\sum_{n=0}^{\infty} (-1)^n \frac{(2n)^2}{(2n)!}\right)^2 = 2.$$

2476. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

Let n be a positive integer and consider the set $\{1, 2, 3, \ldots, 2n\}$. Give a **combinatorial** proof that the number of subsets A such that

1. A has exactly n elements, and
2. the sum of all elements in A is divisible by n,

is equal to

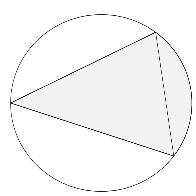
$$\frac{1}{n}\sum_{d|n}(-1)^{n+d}\phi\left(\frac{n}{d}\right)\binom{d}{2d},$$

where ϕ is the Euler function.

Note: When n is prime, proving the formula is problem 6 of the 1995 IMO. A non-combinatorial proof of the formula is due to Roberto Dvornicich and Nikolay Nikolov.

2477. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Given a non-degenerate $\triangle ABC$ with circumcircle Γ, let r_A be the inradius of the region bounded by BA, AC and $\mathrm{arc}(CB)$ (so that the region includes the triangle).



Similarly, define r_B and r_C. As usual, r and R are the inradius and circumradius of $\triangle ABC$.

Prove that

(a) $\frac{64}{27}r^3 \;\le\; r_A r_B r_C \;\le\; \frac{32}{27}Rr^2$;

(b) $\frac{16}{3}r^2 \;\le\; r_B r_C + r_C r_A + r_A r_B \;\le\; \frac{8}{3}Rr$;

(c) $4r \;\le\; r_A + r_B + r_C \;\le\; \frac{4}{3}(R+r)$,

with equality occurring in all cases if and only if $\triangle ABC$ is equilateral.

2478. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

For $n \in \mathbb{N}$, evaluate $\displaystyle\sum_{k=0}^{n}\frac{n-k}{k+1}\binom{2k}{k}\binom{2n-2k}{n-k}$.

2479. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

Writing $\tau(n)$ for the number of divisors of n, and $\omega(n)$ for the number of distinct prime factors of n, prove that

$$\sum_{k=1}^{n} (\tau(k))^2 \;=\; \sum_{k=1}^{n} 2^{\omega(k)} \sum_{j=1}^{\lfloor n/k \rfloor} \left\lfloor \frac{\lfloor n/k \rfloor}{j} \right\rfloor .$$

2480. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

Writing $\phi(n)$ for Euler's totient function, evaluate

$$\sum_{d|n} d \sum_{k|d} \frac{\phi(k)\phi(d/k)}{k} .$$

2481. *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that A, B, C are 2×2 commutative matrices. Prove that

$$\det\left((A+B+C)(A^3+B^3+C^3-3ABC)\right) \;\geq\; 0 .$$

2482. *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that p, q, r are complex numbers. Prove that

$$|p+q| + |q+r| + |r+p| \;\leq\; |p| + |q| + |r| + |p+q+r| .$$

2483. *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Suppose that $0 \leq A$, B, C and $A+B+C \leq \pi$. Show that

$$0 \;\leq\; A - \sin A - \sin B - \sin C + \sin(A+B) + \sin(A+C) \;\leq \pi .$$

There are, of course, similar inequalities with the angles permuted cyclically.

[The proposer notes that this came up during an attempt to generalise problem 2383.]

2484. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given a square $ABCD$, suppose that E is a point on AB produced beyond B, that F is a point on AD produced beyond D, and that $EF = 2AB$. Let P and Q be the intersections of EF with BC and CD, respectively. Prove that

(a) $\triangle APQ$ is acute-angled;

(b) $\angle PAQ \geq 45^\circ$.

2485. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a convex quadrilateral with $AB = BC = CD$. Let P be the intersection of the diagonals AC and BD. Suppose that $AP : BD = DP : AC$.

Prove that either $BC \parallel AD$ or $AB \perp CD$.

2486. *Proposed by Joe Howard, New Mexico Highlands University, Las Vegas, NM, USA.*

It is well-known that $\cos(20^\circ)\cos(40^\circ)\cos(80^\circ) = \frac{1}{8}$.

Show that $\sin(20^\circ)\sin(40^\circ)\sin(80^\circ) = \frac{\sqrt{3}}{8}$.

2487. *Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain.*

If a, b, c, d are distinct real numbers, prove that

$$\frac{a^4+1}{(a-b)(a-c)(a-d)} + \frac{b^4+1}{(b-a)(b-c)(b-d)}$$

$$+ \frac{c^4+1}{(c-a)(c-b)(c-d)} + \frac{d^4+1}{(d-a)(d-b)(d-c)} \;=\; a+b+c+d\,.$$

2488. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $S_n = \mathrm{A}_1\mathrm{A}_2\ldots\mathrm{A}_{n+1}$ be a simplex in $\mathbb{E}^n$, and M a point in S_n. It is known that there are real positive numbers λ_1, λ_2, ..., λ_{n+1} such that $\sum\limits_{j=1}^{n+1}\lambda_j = 1$ and $M = \sum\limits_{j=1}^{n+1}\lambda_j\mathrm{A}_j$ (here, by a point $\mathbf{P}$, we mean the position vector $\overrightarrow{\mathrm{OP}}$). Suppose also that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and let $\mathrm{B}_k = \dfrac{1}{k}\sum\limits_{j=1}^{k}\mathrm{A}_j$.

Prove that

$$M \;\in\; \textbf{convex cover of } \{\mathrm{B}_1, \mathrm{B}_2, \ldots, \mathrm{B}_{n+1}\}\,;$$

that is, there are real positive numbers μ_1, μ_2, ..., μ_{n+1} such that

$$M \;=\; \sum_{k=1}^{n+1}\mu_k\mathrm{B}_k\,.$$