MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to Mathematical Mayhem, Department of Mathematics, University of Toronto, 100 St. George St., Toronto, Ontario, Canada. M5S 3G3. The electronic address is still

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The Assistant Mayhem Editor is Cyrus Hsia (University of Western Ontario). The rest of the staff consists of Adrian Chan (Upper Canada College), Jimmy Chui (University of Toronto), David Savitt (Harvard University) and Wai Ling Yee (University of Waterloo).

Shreds and Slices

A Combinatorial Proof of a Trigonometric Identity

Douglass Grant

A friend who attended university in Germany once told me that his course in first year calculus was made memorable by the fact that the basic definition of the sine and cosine functions used by his professor were the Maclaurin series for the two functions. At the time, I made the flippant remark that such an approach would make it a challenge even to prove that $\sin^2 x + \cos^2 x = 1$. The details of that proof, in fact, involve some identities more commonly encountered in discrete mathematics or combinatorics than in calculus.

Since

$$\sin x \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1}\,,$$

squaring the series and shifting the index by one on the inner series yields

$$\begin{aligned}\sin^2 x &= \sum_{k=0}^{\infty}\left(\sum_{\substack{n+m=k\\ n,m\geq 0}} \frac{(-1)^n(-1)^m}{(2n+1)!(2m+1)!}\right) x^{2k+2}\\ &= \sum_{k=1}^{\infty}\left(\sum_{\substack{n+m=k-1\\ n,m\geq 0}} \frac{(-1)^{k-1}}{(2n+1)!(2m+1)!}\right) x^{2k}\\ &= \sum_{k=1}^{\infty}\left(\sum_{n=0}^{k-1} \frac{(-1)^{k-1}}{(2n+1)!(2k-2n-1)!}\right) x^{2k}\,.\end{aligned}$$

Similarly,

$$\cos^2 x \;=\; \sum_{k=0}^{\infty}\left(\sum_{n=0}^{k} \frac{(-1)^k}{(2n)!(2k-2n)!}\right) x^{2k}\,.$$

Since the constant term in the series for $\cos^2 x$ is clearly unity, it suffices to show that the sum of the coefficients of x^{2k} for $\sin^2 x$ and $\cos^2 x$ is zero for $k \geq 1$. Note that the integers whose factorials appear in the denominators of both inner summations sum to $2k$.

For $k \geq 1$, let

$$S_k \;=\; \sum_{n=0}^{k-1} \frac{(-1)^{k-1}}{(2n+1)!(2k-2n-1)!} + \sum_{n=0}^{k} \frac{(-1)^k}{(2n)!(2k-2n)!}\,.$$

Then

$$(2k)!S_k \;=\; \sum_{n=0}^{2k} \frac{(-1)^n(2k)!}{(2n)!(2k-2n)!} = \sum_{n=0}^{2k}(-1)^n\binom{2k}{n}\,.$$

But by the Binomial Theorem,

$$(a-b)^{2k} \;=\; \sum_{n=0}^{2k}(-1)^n\binom{2k}{n}a^{2k-n}b^n\,,$$

so letting $a = b = 1$, we obtain

$$0 \;=\; \sum_{n=0}^{2k}(-1)^n\binom{2k}{n} \;=\; (2k)!S_k\,,$$

whence $S_k = 0$ for $k \geq 1$, as required.

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Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from this issue be submitted in time for issue 8 of 2000.

High School Problems

Editor: Adrian Chan, 229 Old Yonge Street, Toronto, Ontario, Canada. M2P 1R5 <all238@sprint.com>

We correct problem H253 again, which we tried to correct in Issue 5.

H253. Find all real solutions to the equation

$$\sqrt{3x^2 - 18x + 52} + \sqrt{2x^2 - 12x + 162} \;=\; \sqrt{-x^2 + 6x + 280}\,.$$

H261. Solve for x:

$$\left(\sqrt{7 - \sqrt{48}}\right)^x + \left(\sqrt{7 + \sqrt{48}}\right)^x \;=\; 14\,.$$

H262. *Proposed by Mohammed Aassila, CRM, Montréal, Québec.*
Solve the equation

$$x - \frac{x}{\sqrt{x^2 - 1}} \;=\; \frac{91}{60}\,.$$

H263. Let ABC be an acute-angled triangle such that $a = 14$, $\sin B = 12/13$, and c, a, b form an arithmetic sequence (in that order). Find $\tan A + \tan B + \tan C$.

H264. Find all values of a such that $x^3 - 6x^2 + 11x + a - 6 = 0$ has exactly three integer solutions.

Advanced Problems

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A237. Show that for any sequence of decimal digits that does not begin with 0, there is a Fibonacci number whose decimal representation begins with this sequence. (The Fibonacci sequence is the sequence F_n generated by the initial conditions $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.)

A238. Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at P and Q. A line through P intersects $\mathcal{C}_1$ and $\mathcal{C}_2$ again at A and B, respectively, and X is the mid-point of AB. The line through Q and X intersects $\mathcal{C}_1$ and $\mathcal{C}_2$ again at Y and Z, respectively. Prove that X is the mid-point of YZ.

(1997 Baltic Way)

A239. *Proposed by Mohammed Aassila, CRM, Montréal, Québec.*

Let a_1, a_2, ... , a_n be n distinct numbers, $n \geq 3$. Prove that

$$\sum_{i=1}^{n} \left(a_i \cdot \prod_{j \neq i} \frac{1}{a_i - a_j} \right) = 0 .$$

A240. *Proposed by Mohammed Aassila, CRM, Montréal, Québec.*

Let a, b, and c be integers, not all equal to 0. Show that

$$\frac{1}{4a^2 + 3b^2 + 2c^2} \leq \left| \sqrt[3]{4}a + \sqrt[3]{2}b + c \right| .$$

Challenge Board Problems

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C89. *Proposed by Tal Kubo, Brown University.*

Show that the formal power series (in x and y) $\sum_{n=0}^{\infty} (xy)^n$ cannot be expressed as a finite sum $\sum_{i=1}^{m} f_i(x) g_i(y)$, where $f_i(x)$ and $g_i(y)$ are formal power series in x and y, respectively, $1 \leq i \leq m$.

C90. *Proposed by Noam Elkies, Harvard University.*

Let S_1, S_2, and S_3 be three spheres in $\mathbb{R}^3$ whose centres are not collinear. Let $k \le 8$ be the number of planes which are tangent to all three spheres. Let A_i, B_i, and C_i be the point of tangency between the i^{th} such tangent plane, $1 \le i \le k$, and S_1, S_2, and S_3, respectively, and let O_i be the circumcentre of triangle $A_iB_iC_i$. Prove that all the O_i are collinear. (If $k = 0$, then this statement is vacuously true.)

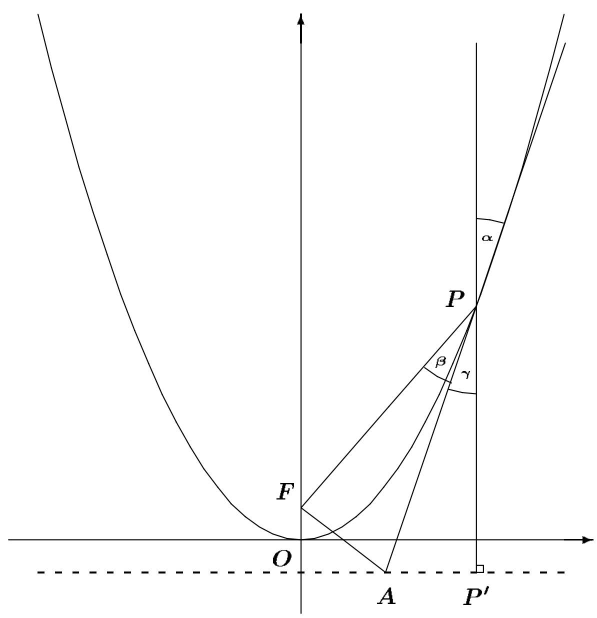
Problem of the Month

Jimmy Chui, student, University of Toronto

Problem. The point $P(a, b)$ is on the parabola $x^2 = 4y$. The tangent at P meets the line $y = -1$ at the point A. For the point $F(0, 1)$, prove that $\angle AFP = 90°$ for all positions of P, except $(0, 0)$.

(Descartes 1998, C3)

Solution. This question can be done using calculus; however, here we show a clever method that makes the problem fall apart quite easily.



First we note that the point F is the focus and the line $y = -1$ is the directrix of the given parabola.

Drop the perpendicular from P to the line $y = -1$, and call that point P'. Now, let α, β, and γ be the angles as in the diagram; that is, let $\beta = \angle APF$, let $\gamma = APP'$, and let α represent the angle opposite $\angle APP'$.

Observe that $PF = PP'$, because any point on a parabola is equidistant to both the focus and directrix.

From the opposite angle theorem we get $\alpha = \gamma$.

Now, it is known that a ray, parallel to the axis of symmetry in a parabola, will pass through the focus when it reflects off the interior of the parabola. The line $P'P$ extended is parallel to the line $x = 0$ (the axis of symmetry) and can be taken as the ray of incidence. The ray PF is then the ray of reflection. The line AP extended is the tangent at the point P of the parabola, and the angle of incidence equals the angle of reflection, or $\alpha = \beta$. So we have $\beta = \gamma$.

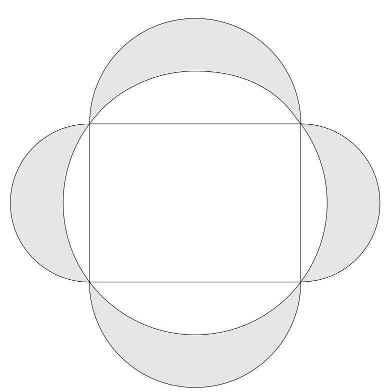
Hence, from SAS congruency, we have that $\triangle APF \equiv \triangle APP'$. This implies that $\angle AFP = \angle AP'P = 90°$, QED.

J.I.R. McKnight Problems Contest 1991

1. (a) The vertices of a right-angled triangle ABC are $A(4, 6)$, $B(3, -3)$ and $C(8, y)$. Find all possible values of y.

   (b) Simplify:
$$\frac{5^{3x+1} - 5^{3x-1} + 24}{(2.4)5^{3x} + 12}.$$

2. (a) A rectangle has been inscribed in a circle and semi-circles have been drawn on its sides (as shown in the diagram). Determine the ratio of the sum of the area of the four lunes (shaded regions) to the area of the rectangle.



(b) Find the value of $(2^{2^0}+1)(2^{2^1}+1)(2^{2^2}+1)\cdots(2^{2^{50}}+1)$.

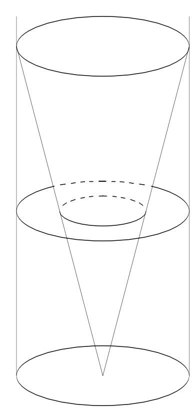
3. (a) Find the sum of the first one thousand terms for the series: $1\cdot 4+2\cdot 7+3\cdot 10+\cdots$.

(b) Solve for x, $y\in\mathbb{R}$:

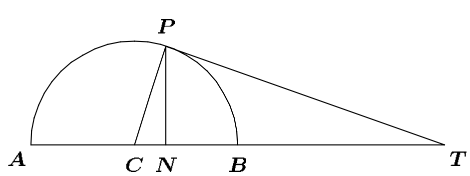
$$\begin{aligned} x+y+\sqrt{xy} &= 19 \\ x^2+y^2+xy &= 133 \end{aligned}$$

4. The two equations $a^2x^2+192bx+1991a=0$ and $bx^2+192a^2x+1991b=0$ have a common root. If $a^2\neq b$, determine all possible values of the common root.

5. A metal cone of height 12 cm and radius 4 cm just fits into a cylinder of radius 4 cm and height 100 cm which is filled with water to a depth of 50 cm. The cone is lowered at a constant rate of 4 cm per second (with respect to the walls of the cylinder). At what rate is the water in the cylinder rising when the vertex of the cone is immersed to a depth of 6 cm?



6. A semi-circle has centre C and diameter AB. The point N is on CB and AB is produced to T so that $AT:AC=AN:CN$. The tangent from T meets the semi-circle at P. Prove that $\angle CNP=90°$.



7. (a) Prove that the product of 4 consecutive positive integers cannot be a perfect square.

(b) What must be added to the product of 4 consecutive terms of any arithmetic sequence to produce a perfect square?

(c) What must be added to the product of 4 consecutive terms of any geometric sequence to produce a perfect square?

8. Triangle ABC has sides $AB = 6$, $AC = 4$ and $BC = 5$. Point P is on AB and point Q is on AC such that PQ bisects the area of triangle ABC. Prove that the minimum length of PQ is $\sqrt{42}/2$.
9. Given $b > 0$, $b \neq 1$, find the set of values of k for which the equation $\log_{b^2}(x^2 - b^2) = \log_b(x - bk)$ has real solutions.
10. Six points are chosen in space such that no three are collinear and no four are coplanar. The 15 line segments joining the points in pairs are drawn then painted, some red and some blue. Prove that some triangle has all its sides the same colour.

IMO Report

Jimmy Chui

student, University of Toronto

The 1999 Canadian IMO team members commenced their summer with one and a half weeks of training at the University of Waterloo. During this time, they managed to hike twice, and it was a shame that no one was lost this year. On the 13$^{\text{th}}$ of July, after an exhausting full day of travel, the team found itself in Bucharest, Romania, ready for the 40$^{\text{th}}$ International Mathematical Olympiad.

The members of this year's team were David "Pippy" Arthur, Jimmy "The Squeeze" Chui, James "Roadkill" Lee, Jessie "Pyromaniac" Lei, David "Monkey Matrix" Nicholson, and David "23 Across" Pritchard. Team leader Dr. Ed "81" Barbeau gave incessant lectures on continuity while deputy leader Dr. Arthur "Put down that math and deal!" Baragar could be found playing Tetris on some particular portable game machine. Meanwhile, the deputy leader observer, Dr. Dorette "Maybe this works..." Pronk, dutifully took pictures of the other team members while they were not looking. The team is also grateful to Dr. Ed Wang and Richard Hoshino for their wise words in combinatorics and inequalities, and to Dr. Christopher Small for sharing his functional equations knowledge, as well as for his outstanding hospitality in Elora.

This year's contest was immensely challenging, and it continued the low medal cut-off scores the last few IMOs have seen. Considering the difficulty of the questions, Canada performed respectably and brought home 3 bronzes. The scores were as follows:

CAN 1	David Arthur	18	Bronze Medal
CAN 2	Jimmy Chui	16	Bronze Medal
CAN 3	James Lee	6	
CAN 4	Jessie Lei	9	
CAN 5	David Nicholson	8	
CAN 6	David Pritchard	17	Bronze Medal

Unofficially, Canada's total score of 74 was enough for 32nd place out of the 83 competing countries. Best of luck to CAN 3 and 5 as they pursue their university studies at the University of Waterloo, and to CAN 2 and 4 as they move on to the University of Toronto. The remaining two members are still eligible for next year's team. Hopefully there will be shouts of "We like to party!" after the competition next year!

Special thanks must also go to Dr. Graham Wright of the Canadian Mathematical Society for once again supplying the funds for the team, and again to team leader Dr. Ed Barbeau for his continual efforts training IMO potentials through the CMS's correspondence program.

It was the first flight to Europe for many of us, and quite an experience it turned out to be. We were surprised at the endless supply of cheese that the cafeteria managed to put on our plates. We found it a great object to ward off stray dogs. Visits to several museums, including the infamous Transylvania Castle, were a real treat to the competitors, but it was a shame that Dracula was nowhere to be found. However, the cheese did find its way along with us. With all that said and done, the IMO was once again a success. We wish the best of luck to all hopefuls for the 2000 Canadian IMO team, bound for the Republic of Korea for the 41st IMO.

Stan Wagon's e–mail problem of the week

For the benefit of those readers of ***CRUX with MAYHEM*** who make use of Stan Wagon's *e-mail Problem of the Week*, and for the information of those who do not know about it, the Math Forum at Swarthmore has just taken over the handling of the e-list. The instructions for subscribing are:

> to subscribe to the Problem of the Week send a message to `<<majordomo@forum.swarthmore.edu>>`. Body of message should read simply SUBSCRIBE MACPOW. Macalester students should NOT subscribe to the e–list, but get printed postings instead.

Here is a sample of the type of problem that you can expect:

Problem 887 Square Division

Show how to divide a unit square into two rectangles so that the smaller rectangle can be placed on the larger with every vertex of the smaller on exactly one of the edges of the larger.

Source: 1994 Dutch Mathematical Olympiad; as reported in Crux Mathematicorum, Sept 1998, Vol. 24, No. 5, p. 264.

An Identity of a Tetrahedron

Murat Aygen

Problem. Let $ABCD$ be a tetrahedron with sides $a = BC$, $b = AC$, $c = AB$, $a_1 = AD$, $b_1 = BD$, and $c_1 = CD$ (see Figure 1(a)). Let V and R denote the volume and circumradius of the tetrahedron, respectively. Show that $6VR$ equals the area of the triangle with sides aa_1, bb_1, and cc_1.

Solution. Consider the plane of triangle ABC and the parallel plane through D. These intersect the circumsphere of $ABCD$ in two circles; let their radii be r_1 and r_2, respectively (see Figure 1(b)). Let h be the distance between the two planes.

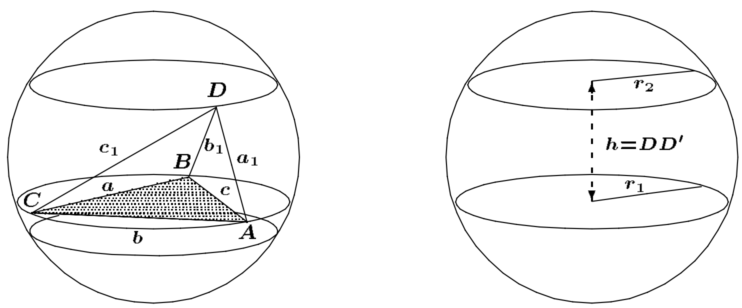


Figure 1(a). Figure 1(b).

Let O be the circumcentre of triangle ABC, let D' be the projection of D onto the plane of triangle ABC, and let $2\theta = \angle D'OC$ (see Figure 2).

By the Cosine Law on triangle $D'OC$,

$$\begin{aligned}(CD')^2 &= r_1^2 + r_2^2 - 2r_1r_2\cos 2\theta \\ &= r_1^2 + r_2^2 - 2r_1r_2(1 - 2\sin^2\theta) \\ &= (r_1 - r_2)^2 + 4r_1r_2\sin^2\theta\,.\end{aligned}$$

Since $DD' = h$, we obtain similarly that

$$\begin{aligned}a_1^2 &= AD^2 = (DD')^2 + (AD')^2 \\ &= h^2 + (r_1 - r_2)^2 + 4r_1r_2\sin^2(B - \theta)\,, \\ b_1^2 &= BD^2 = (DD')^2 + (BD')^2 \\ &= h^2 + (r_1 - r_2)^2 + 4r_1r_2\sin^2(A + \theta)\,, \\ c_1^2 &= CD^2 = (DD')^2 + (CD')^2 \\ &= h^2 + (r_1 - r_2)^2 + 4r_1r_2\sin^2\theta\,.\end{aligned}$$

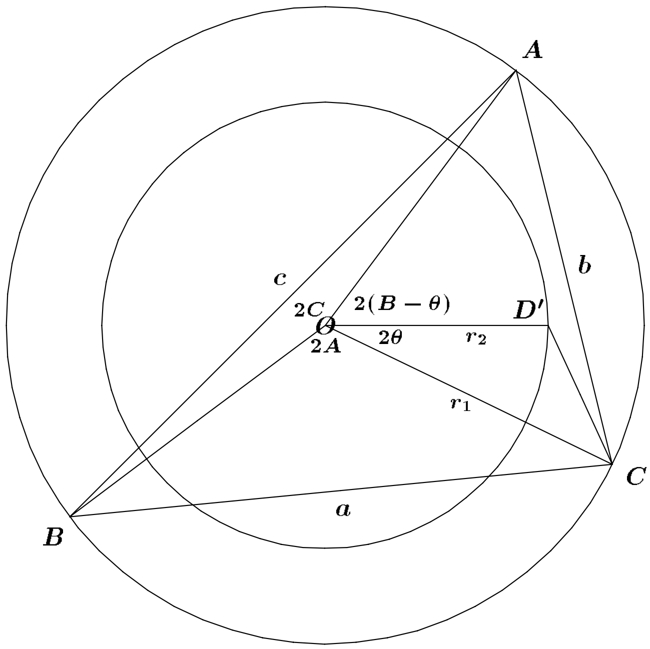


Figure 2.

We now present a geometrical derivation of these lengths. Let K and K_1 denote the area of the triangle with sides a, b, and c, and the triangle with sides aa_1, bb_1, and cc_1, respectively. We will derive a relationship between K and K_1.

Let P_0 be the plane containing triangle ABC. Consider another plane P_1, passing through A, and meeting P_0 at an angle of ϕ. Let ψ be the angle between AB and the intersection of P_0 and P_1 (see Figure 3), where ϕ and ψ are angles that will be specified later.

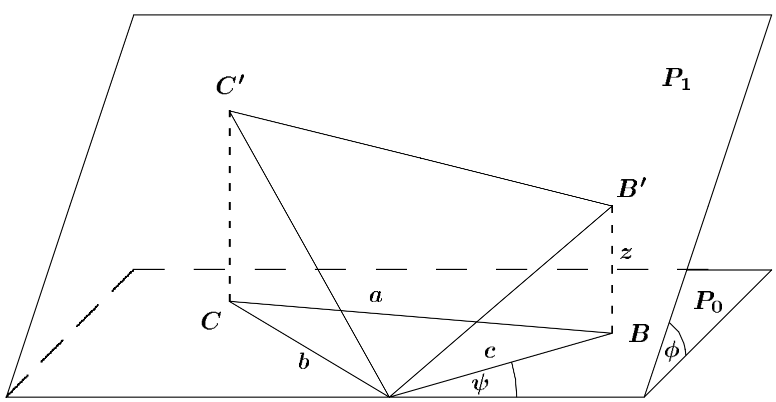


Figure 3.

Let B' and C' be the points in P_1 that project to B and C in P_0, respectively. Then by some elementary trigonometry,

$$\begin{aligned}(B'C')^2 &= a^2 + a^2\tan^2\phi\,\sin^2(B-\psi)\,,\\ (AC')^2 &= b^2 + b^2\tan^2\phi\,\sin^2(A+\psi)\,,\\ (AB')^2 &= c^2 + c^2\tan^2\phi\,\sin^2\psi\,.\end{aligned}$$

Now, assign ψ and ϕ such that

$$\psi = \theta \quad\text{and}\quad \tan^2\phi = \frac{4r_1r_2}{h^2+(r_1-r_2)^2}\,.$$

Then the equations above become

$$\begin{aligned}(B'C')^2 &= a^2(1+\tan^2\phi\,\sin^2\theta)\\ &= \frac{a^2[h^2+(r_1-r_2)^2+4r_1r_2\sin^2\theta]}{h^2+(r_1-r_2)^2}\\ &= \frac{a^2a_1^2}{h^2+(r_1-r_2)^2}\,,\\ (AC')^2 &= b^2[1+\tan^2\phi\,\sin^2(B-\theta)]\\ &= \frac{b^2b_1^2}{h^2+(r_1-r_2)^2}\,,\\ (AB')^2 &= c^2[1+\tan^2\phi\,\sin^2(A+\theta)]\\ &= \frac{c^2c_1^2}{h^2+(r_1-r_2)^2}\,.\end{aligned}$$

Hence, $AB'C'$ is proportional to the triangle with sides aa_1, bb_1, and cc_1, with ratio of areas

$$\frac{1}{h^2+(r_1-r_2)^2}\,.$$

Projecting from P_1 to P_0 scales the area by a further factor of $1/\cos\phi$. Hence,

$$K = \frac{K_1\cos\phi}{h^2+(r_1-r_2)^2}\,. \tag{1}$$

We also know that

$$V = \frac{1}{3}Kh\,, \tag{2}$$

$$h = \sqrt{R^2-r_1^2}+\sqrt{R^2-r_2^2}\,. \tag{3}$$

Squaring both sides of (3), we obtain

$$h^2+r_1^2+r_2^2-2R^2 = 2\sqrt{(R^2-r_1^2)(R^2-r_2^2)}\,.$$

Therefore,

$$\begin{aligned}(h^2 + r_1^2 + r_2^2)^2 - 4R^2h^2 - 4R^2r_1^2 - 4R^2r_2^2 + 4R^4 \\ = \; 4R^4 - 4R^2r_1^2 - 4R^2r_2^2 + 4r_1^2r_2^2 \,,\end{aligned}$$

yielding

$$\begin{aligned} 4R^2h^2 &= h^4 + r_1^4 + r_2^4 + 2h^2r_1^2 + 2h^2r_2^2 - 2r_1^2r_2^2 \\ &= h^4 + r_1^4 + r_2^4 + 2h^2r_1^2 + 2h^2r_2^2 + 2r_1^2r_2^2 - 4r_1^2r_2^2 \\ &= (h^2 + r_1^2 + r_2^2)^2 - (2r_1r_2)^2 \\ &= [h^2 + (r_1 + r_2)^2][h^2 + (r_1 - r_2)^2] \,. \qquad (4) \end{aligned}$$

Finally, note that

$$\cos^2\phi \;=\; \frac{1}{1 + \tan^2\phi} \;=\; \frac{h^2 + (r_1 - r_2)^2}{h^2 + (r_1 + r_2)^2} \,. \qquad (5)$$

Therefore,

$$\begin{aligned} (6VR)^2 &= 4K^2R^2h^2 && \text{(from (2))} \\ &= \frac{4K_1^2R^2h^2\cos^2\phi}{[h^2 + (r_1 - r_2)^2]^2} && \text{(from (1))} \\ &= \frac{K_1^2[h^2 + (r_1 + r_2)^2][h^2 + (r_1 - r_2)^2]^2}{[h^2 + (r_1 + r_2)^2][h^2 + (r_1 - r_2)^2]^2} && \text{(from (4) and (5))} \\ &= K_1^2 \,, \end{aligned}$$

so that

$$6VR \;=\; K_1 \,.$$

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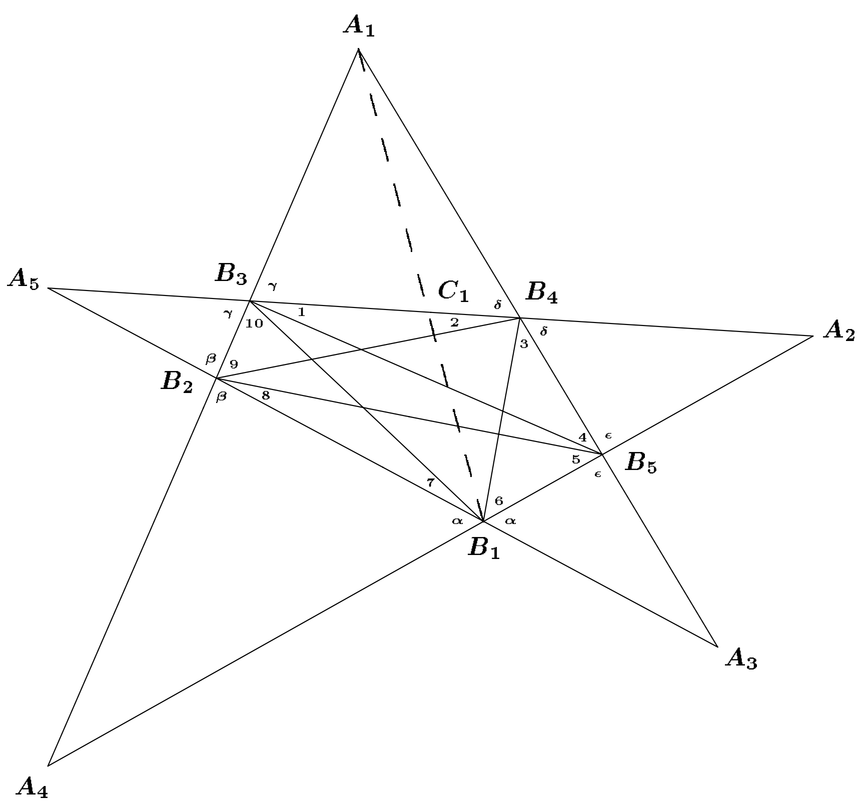
A Simple Proof of a Pentagram Theorem

Geoffrey A. Kandall

We will give a short, transparent proof of Eiji Konishi's pentagram theorem, which was communicated by Hiroshi Kotera [1]. The proof is really just an exercise in the Law of Sines.

Theorem. Let $A_1A_2A_3A_4A_5$ be a pentagram with pentagon $B_1B_2B_3B_4B_5$ as shown in the figure. Let C_k be the intersection of line segment A_kB_k and side $B_{k+2}B_{k+3}$, for $k = 1, 2, \ldots, 5$. Then

$$\frac{B_3C_1}{C_1B_4} \cdot \frac{B_4C_2}{C_2B_5} \cdot \frac{B_5C_3}{C_3B_1} \cdot \frac{B_1C_4}{C_4B_2} \cdot \frac{B_2C_5}{C_5B_3} = 1 .$$



Proof. Let $[P]$ denote the area of polygon P. Then we have that

$$\begin{aligned}\frac{B_3C_1}{C_1B_4} &= \frac{[A_1B_3B_1]}{[A_1B_4B_1]}\\ &= \frac{A_1B_3\cdot B_3B_1\cdot\sin\angle A_1B_3B_1}{A_1B_4\cdot B_4B_1\cdot\sin\angle A_1B_4B_1}\\ &= \frac{\sin\delta\cdot B_3B_1\cdot\sin\theta_{10}}{\sin\gamma\cdot B_4B_1\cdot\sin\theta_3}\,.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{B_4C_2}{C_2B_5} &= \frac{\sin\epsilon\cdot B_4B_2\cdot\sin\theta_2}{\sin\delta\cdot B_5B_2\cdot\sin\theta_5}\,,\\ \frac{B_5C_3}{C_3B_1} &= \frac{\sin\alpha\cdot B_5B_3\cdot\sin\theta_4}{\sin\epsilon\cdot B_1B_3\cdot\sin\theta_7}\,,\\ \frac{B_1C_4}{C_4B_2} &= \frac{\sin\beta\cdot B_1B_4\cdot\sin\theta_6}{\sin\alpha\cdot B_2B_4\cdot\sin\theta_9}\,,\\ \frac{B_2C_5}{C_5B_3} &= \frac{\sin\gamma\cdot B_2B_5\cdot\sin\theta_8}{\sin\beta\cdot B_3B_5\cdot\sin\theta_1}\,.\end{aligned}$$

Multiplying these five equations together, we obtain

$$\begin{aligned}&\frac{B_3C_1}{C_1B_4}\cdot\frac{B_4C_2}{C_2B_5}\cdot\frac{B_5C_3}{C_3B_1}\cdot\frac{B_1C_4}{C_4B_2}\cdot\frac{B_2C_5}{C_5B_3}\\ &\quad= \frac{\sin\theta_{10}\cdot\sin\theta_2\cdot\sin\theta_4\cdot\sin\theta_6\cdot\sin\theta_8}{\sin\theta_7\cdot\sin\theta_9\cdot\sin\theta_1\cdot\sin\theta_3\cdot\sin\theta_5}\\ &\quad= \frac{B_1B_2}{B_2B_3}\cdot\frac{B_2B_3}{B_3B_4}\cdot\frac{B_3B_4}{B_4B_5}\cdot\frac{B_4B_5}{B_5B_1}\cdot\frac{B_5B_1}{B_1B_2}\\ &\quad= 1\,.\end{aligned}$$

Reference

[1] H. Kotera, *The Pentagram Theorem*, ***CRUX with MAYHEM*** 24:5 (1998), 291–295.

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