

THE SKOLIAD CORNER

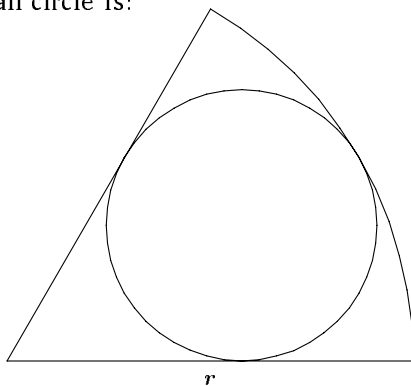
No. 41

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This issue we give the Final Round Parts A and B of the 1998 British Columbia Colleges Senior High School Mathematics Contest. My thanks go to Jim Totten, University College of the Cariboo, one of the organizers, for forwarding the materials for use in the *Corner*.

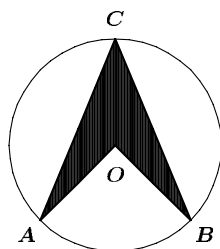
BRITISH COLUMBIA COLLEGES SENIOR HIGH SCHOOL MATHEMATICS CONTEST Final Round 1998 Part A

1. If $(r + \frac{1}{r})^2 = 3$, then $r^3 + \frac{1}{r^3} =$
- (a) 0 (b) 1 (c) 2 (d) $2\sqrt{3}$ (e) $4\sqrt{3}$
2. Kevin has five pairs of socks in his drawer, all of different colours and patterns and, being a typical teenage boy, they are not folded and have been thoroughly mixed up. On the first day of school Kevin reaches into his sock drawer without looking and pulls out three socks. What is the probability that two of the socks match?
- (a) $\frac{3}{10}$ (b) $\frac{3}{5}$ (c) $\frac{1}{3}$ (d) $\frac{1}{24}$ (e) $\frac{1}{15}$
3. A small circle is drawn within a $\frac{1}{6}$ sector of a circle of radius r , as shown. The small circle is tangent to the two radii and the arc of the sector. The radius of the small circle is:



- (a) $\frac{r}{2}$ (b) $\frac{r}{3}$ (c) $\frac{2\sqrt{3}r}{3}$ (d) $\frac{\sqrt{2}r}{2}$ (e) none of these

4. In the accompanying diagram, the circle has radius one, the central angle AOB is a right angle and AC and BC are of equal length. The shaded area is:



- (a) $\frac{\pi}{2}$ (b) $\frac{\sqrt{2}}{2}$ (c) $\frac{\pi-\sqrt{2}}{2}$ (d) $\frac{\sqrt{2}+1}{2}$ (e) $\frac{1}{2}$

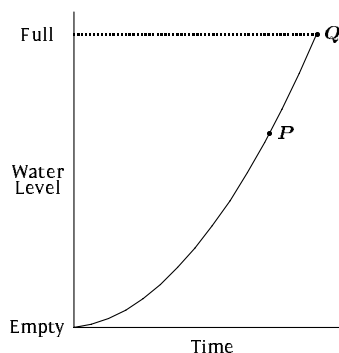
5. The side, front and bottom faces of a rectangular solid have areas $2x$, $\frac{y}{2}$, and xy square centimetres, respectively. The volume of the solid is:

- (a) xy (b) $2xy$ (c) x^2y^2 (d) $4xy$ (e) impossible to determine from the given information

6. The numbers from 1 to 25 are each written on separate slips of paper which are placed in a pile. You draw slips from the pile without replacing any slip you have chosen. You can continue drawing until the *product* of two numbers on any pair of slips you have chosen is a perfect square. The maximum number of slips you can choose before you will be forced to quit is:

- (a) 13 (b) 14 (c) 15 (d) 16 (e) 17

7. A container is completely filled from a tap running at a constant rate. The accompanying graph shows the level of the water in the container at any time while the container is being filled. The segment PQ is a straight line. The shape of the container which corresponds with the graph is:



- (a) (b) (c) (d) (e)

Part B

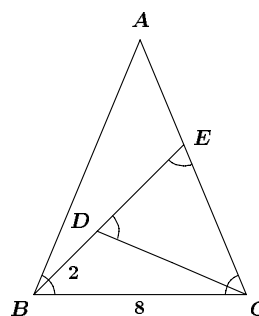
1. A right triangle has an area of 5 and its hypotenuse has length 5. Determine the lengths of the other two sides.

2. Find a set of three consecutive positive integers such that the smallest is a multiple of 5, the second is a multiple of 7 and the largest is a multiple of 9.

3. In the diagram, $BD = 2$, $BC = 8$ and the marked angles are all equal; that is,

$$\angle ABC = \angle BCA = \angle CDE = \angle DEC.$$

Find AB .



4. The ratio of male to female voters in an election was $a : b$. If c fewer men and d fewer women had voted, then the ratio would have been $e : f$. Determine the total number of voters who cast ballots in the election in terms of a , b , c , d , e and f .

5. Three neighbours named Penny, Quincy and Rosa took part in a local recycling drive. Each spent a Saturday afternoon collecting all the aluminum cans and glass bottles he or she could. At the end of the afternoon each person counted up what he or she had gathered, and they discovered that even though Penny had collected three times as many cans as Quincy, and Quincy had collected four times as many bottles as Rosa, each had collected exactly the same number of items, and the three as a group had collected exactly as many cans as bottles. In total, the three collected fewer than 200 items in all. Assuming that each person collected at least one can and one bottle, how many cans and bottles did each person collect?

Last issue we gave the Final Round Parts A and B of the 1998 British Columbia Colleges Junior High School Mathematics Contest. My thanks go to Jim Totten, University College of the Cariboo, one of the organizers, for forwarding the "official solutions" which follow.

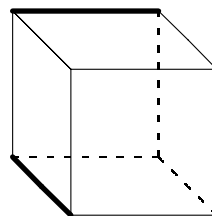
**BRITISH COLUMBIA COLLEGES
JUNIOR HIGH SCHOOL MATHEMATICS CONTEST
Final Round 1998
Part A**

1. Each edge of a cube is coloured either red or black. If every face of the cube has at least one black edge, the smallest possible number of black edges is:

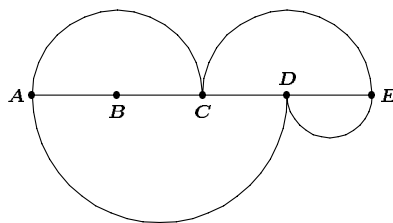
- (a) 6 (b) 5 (c) 4 (d) 3 (e) 2

Answer: The correct answer is (d).

Suppose that every face of a cube has at least one black edge. Since every edge belongs to exactly two faces, and there are six faces, the cube has at least three black edges. On the other hand, three black edges suffice to satisfy the requirement, as we can see on the diagram. The black edges are represented by the thicker lines.



2. Line AE is divided into four equal parts by the points B , C and D . Semicircles are drawn on segments AC , CE , AD and DE creating semicircular regions as shown. The ratio of the area enclosed above the line AE to the area enclosed below the line is:

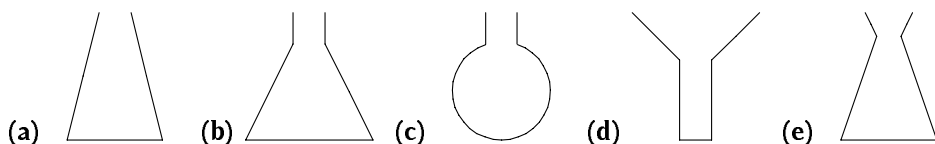
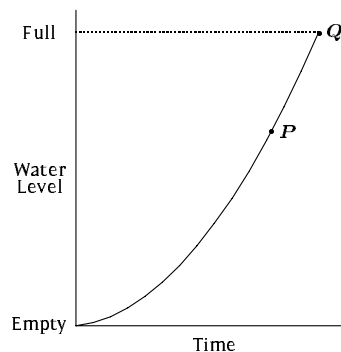


- (a) 4 : 5 (b) 5 : 4 (c) 1 : 1 (d) 8 : 9 (e) 9 : 8

Answer: The correct answer is (a).

Suppose that AB has length 1. Then both semicircles lying above AE have radii of 1, while the semicircles below AE have radii of $1\frac{1}{2}$ and $\frac{1}{2}$. The ratio of the enclosed areas is $[\frac{1}{2}\pi(1)^2 + \frac{1}{2}\pi(1)^2] \div [\frac{1}{2}\pi(1\frac{1}{2})^2 + \frac{1}{2}\pi(\frac{1}{2})^2] = \frac{4}{5}$.

3. A container is completely filled from a tap running at a uniform rate. The accompanying graph shows the level of the water in the container at any time while the container is being filled. The segment PQ is a straight line. The shape of the container which corresponds with the graph is:



Readers may have noticed that this is the same as problem 7 in part A of the British Columbia Colleges **Senior** High School Mathematics Contest Final Round 1998, printed above. For this reason, we delay publishing this official solution until the next issue.

4. The digits 1, 9, 9, and 8 are placed on four cards. Two of the cards are selected at random. The probability that the sum of the numbers on the cards selected is a multiple of 3 is:

- (a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$ (e) $\frac{3}{4}$

Answer: The correct answer is (b).

Let a , b , c , d denote the cards with digits 1, 9, 9, and 8, respectively. There are six possible choices of two cards from the set of four: $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$. For exactly two of these, $\{b, c\}$ and $\{a, d\}$, the corresponding sums, $9 + 9$ and $1 + 8$, are divisible by 3. This gives the probability of $\frac{2}{6} = \frac{1}{3}$.

5. The surface areas of the six faces of a rectangular solid are 4, 4, 8, 8, 18 and 18 square centimetres. The volume of the solid, in cubic centimetres is:

- (a) 24 (b) 48 (c) 60 (d) 324 (e) 576

Answer: The correct answer is (a).

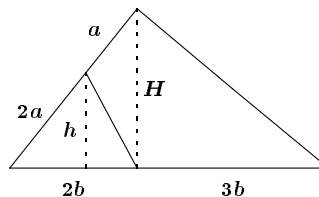
If the edges of a rectangular solid have lengths a , b , and c , then the areas of its nonparallel faces are ab , bc , and ac . Its volume $abc = \sqrt{(ab)(bc)(ac)}$. In our case $abc = \sqrt{4 \cdot 8 \cdot 18} = 24$.

6. The area of the small triangle in the diagram is 8 square units. The area of the large triangle, in square units, is:

- (a) 18 (b) 20 (c) 24 (d) 28 (e) 30

Answer: The correct answer is (e).

The length of the base of the larger triangle is $5b$, while the length of the base of the smaller triangle is $2b$. This gives the ratio of $\frac{5}{2}$. Similarly, the ratio of the corresponding perpendicular heights, $H : h$, is $3a : 2a = \frac{3}{2}$. Hence, the area of the larger triangle is $\frac{5}{2}(\frac{3}{2})(8) = 30$.



7. At 6:15 the hands of the clock form two positive angles with a sum of 360° . The difference of the degree measures of these two angles is:

- (a) 165 (b) 170 (c) 175 (d) 180 (e) 185

Answer: The correct answer is (a).

At 6:15 the minute hand points at 3, while the hour hand is $\frac{1}{4}$ of the way from 6 to 7. The smaller angle between the hands is $[90 + \frac{1}{4}(\frac{360}{12})]^\circ = 97.5^\circ$, while the larger is $(360 - 97.5)^\circ = 262.5^\circ$. This gives the difference of $(262.5 - 97.5)^\circ = 165^\circ$.

8. The last digit of the number 8^{26} is:

- (a) 0 (b) 2 (c) 4 (d) 6 (e) 8

Answer: The correct answer is (c).

By inspecting the last digit of the numbers in the sequence $8^1, 8^2, 8^3, 8^4, \dots$, we discover a repeating pattern of length four: 8, 4, 2, 6. Since $8^{26} = 8^{4(6)+2}$, we conclude that the last digit of 8^{26} is the same as the last digit of 8^2 , that is 4.

9. For the equation $\frac{A}{x+3} + \frac{B}{x-3} = \frac{-x+9}{x^2-9}$ to be true for all values of x for which the expressions in the equation make sense, the value of AB is:

- (a) 2 (b) -1 (c) -2 (d) -3 (e) -6

Answer: The correct answer is (c).

The expression makes sense for all values of x , except ± 3 . By multiplying both sides of the equation by the common denominator $x^2 - 9 = (x - 3)(x + 3)$, we get $A(x - 3) + B(x + 3) = -x + 9$. After multiplying out and collecting the like terms on the left hand side of this equation we get $(A + B)x + 3B - 3A = -x + 9$. Clearly, the polynomials on both sides must be identical; therefore $A + B = -1$ and $3B - 3A = 9$. This system of two equations can be solved in any standard way. For example, we can find $B = 3 + A$ from the second equation and substitute this for B in the first equation. In that way we find $A = -2$ and $B = 1$.

10. A hungry hunter came upon two shepherds, Joe and Frank. Joe had three small loaves of bread and Frank five loaves of the same size. The loaves were divided equally among the three people, and the hunter paid \$8 for his share. If the shepherds divide the money so that each gets an equitable share based on the amount of bread given to the hunter, the amount of money that Joe receives is:

- (a) \$1 (b) \$1.50 (c) \$2 (d) \$2.50 (e) \$3

Answer: The correct answer is (a).

Divide each loaf into 3 parts and distribute equally to each of the three persons. Each person receives 8 parts. The two shepherds start with 9 and 15 parts each, so (after removing their own 8 parts) they contribute 1 and 7 parts, respectively, to the hunter and should receive compensation from the hunter in that ratio. Thus the hunter who originally had 3 loaves should receive \$1.

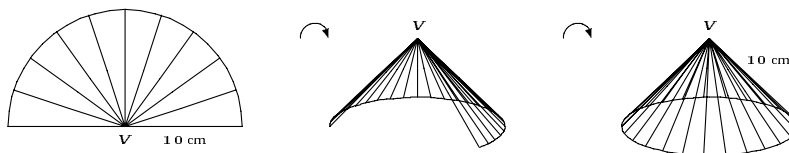
Part B

1. Four positive integers sum to 125. If the first of these numbers is increased by 4, the second is decreased by 4, the third is multiplied by 4 and the fourth is divided by 4, you produce four equal numbers. What are the four original numbers?

Solution. The numbers are 16, 24, 5 and 80.

If x , y , z , and w are the numbers then $x + y + z + w = 125$ and $x + 4 = y - 4 = 4z = \frac{w}{4}$. Hence, $y = x + 8$, $z = \frac{x+4}{4}$, $w = 4(x + 4)$. By substituting these expressions to the first equation, we get $x + (x + 8) + \frac{x+4}{4} + 4(x + 4) = 125$. Thus, $x = 16$, and consequently, $y = 24$, $z = 5$, $w = 80$.

2. A semi-circular piece of paper of radius 10 cm is formed into a conical paper cup as shown (the cup is inverted in the diagram):



Find the height of the paper cup, that is, the depth of water in the cup when it is full.

Solution. The height of the paper cup is $5\sqrt{3}$ cm.

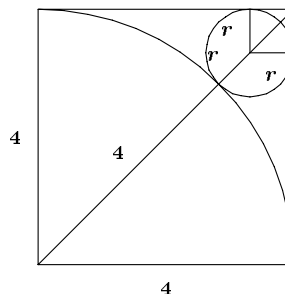
The base of the conical paper cup is a circle with circumference equal to the length of the given semicircle. Thus, if r is the radius of the base then $2\pi r = \frac{1}{2}(2\pi 10)$. Hence, $r = 5$ cm. The side length of the cone s is the same as the radius of the semicircle; thus $s = 10$ cm. Finally, the height of the cone is

$$h = \sqrt{s^2 - r^2} = \sqrt{10^2 - 5^2} = 5\sqrt{3} \text{ cm.}$$

3. In the diagram a quarter circle is inscribed in a square with side length 4, as shown. Find the radius of the small circle that is tangent to the quarter circle and two sides of the square.

Solution. The radius of the small circle is $12 - 8\sqrt{2}$.

The Pythagorean Theorem implies that the diagonal of a square with side a has length $a\sqrt{2}$. Thus, the diagonal of the larger square has length $4\sqrt{2}$. It is equal to the sum of the radius of the larger circle, 4, the radius of the smaller circle, r , and the diagonal of the smaller square, $r\sqrt{2}$. Hence, $4\sqrt{2} = 4 + r + r\sqrt{2}$. This gives



$$r = \frac{4\sqrt{2} - 4}{1 + \sqrt{2}} = \frac{4\sqrt{2} - 4}{1 + \sqrt{2}} \left(\frac{1 - \sqrt{2}}{1 - \sqrt{2}} \right) = 12 - 8\sqrt{2}.$$

4. Using the digits 1, 9, 9 and 8 *in that order* create expressions equal to 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. You may use any of the four basic operations (+, −, ×, ÷), the square root symbol ($\sqrt{\quad}$) and parentheses, as necessary. For example, valid expressions for 25 and 36 would be

$$\begin{aligned} 25 &= -1 + 9 + 9 + 8, \\ 36 &= 1 + 9 \times \sqrt{9} + 8. \end{aligned}$$

Note: You may place a negative sign in front of 1 to create -1 if you wish.

Solution. One of several possible solutions is:

$$\begin{aligned} 1 &= -1 + \sqrt{9} - 9 + 8, \\ 2 &= 1 \times \sqrt{9} - 9 + 8, \\ 3 &= -1 + \sqrt{9} + 9 - 8, \\ 4 &= 1 \times \sqrt{9} + 9 - 8, \\ 5 &= 1 + \sqrt{9} + 9 - 8, \\ 6 &= -1 - 9 \div 9 + 8, \\ 7 &= -1 + 9 - 9 + 8, \\ 8 &= -1 + 9 \div 9 + 8, \\ 9 &= -1 + 9 + 9 - 8, \\ 10 &= 1 + 9 \div 9 + 8. \end{aligned}$$

5. At 6 am one Saturday, you and a friend begin a recreational climb of Mt. Mystic. Two hours into your climb, you are overtaken by some scouts. As they pass, they inform you that they are attempting to set a record for ascending and descending the mountain. At 10 am they pass you again on their way down, crowing that they had not stopped once to rest, not even at the top.

You finally reach the summit at noon. Assuming that both you and the scouts travelled at a constant vertical rate, both climbing and descending, when did the scouts reach the top of Mt. Mystic?

Solution. The scouts reached the top of Mt. Mystic at 9:20 am.

Suppose that during the time period from 8:00 am to 10:00 am you have travelled from point A to point B and you climbed a distance of x kilometres. Then, since you have been climbing at a uniform rate and reached the top at noon, the distance from B to the top is also x kilometres. During the two hours you climbed x kilometres from A to B , the scouts climbed the distance of $3x$ kilometres: x from A to B , x from B to the top, and x on the way back to B from the top. Since their pace was uniform, they needed $\frac{2}{3}$ of an hour, that is 40 minutes, to get from the top to point B , where they met you at 10:00 am. This implies that they must have reached the top at 9:20 am.

That completes the *Corner* for this number. Send me contest materials and suggestions for the evolution of the *Skoliad Corner*.