

THE OLYMPIAD CORNER

No. 201

R.E. Woodrow

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As a first contest this number we give the Thirty-first Canadian Mathematical Olympiad 1999. The contest was held March 31, 1999 with 81 competitors from 48 schools in five Canadian provinces participating. Competitors are invited on the basis of their performance on other contests. Each question was marked out of 7 marks for a total possible score of 35.

First prize went to Jimmy Chui, Second to Adrian Chan, Third to David Pritchard, and Honourable Mentions go to Edmond Choi, Masoud Kamgarpour, Jessie Lei, Pierre LeVan, Dave Nicholson, and Yannick Solari. Congratulations!

My thanks go to Daryl Tingley, Chair of the Canadian Mathematical Olympiad Committee for furnishing us with the contest as well as selected solutions by the contestants, which appear at the end of this number of the *Corner*. But try them first!

THE THIRTY-FIRST CANADIAN MATHEMATICAL OLYMPIAD 1999 March 31, 1999

1. Find all real solutions to the equation $4x^2 - 40[x] + 51 = 0$. Here, if x is a real number, then $[x]$ denotes the greatest integer that is less than or equal to x .
2. Let ABC be an equilateral triangle of altitude 1. A circle with radius 1 and centre on the same side of AB as C rolls along the segment AB . Prove that the arc of the circle that is inside the triangle always has the same length.
3. Determine all positive integers n with the property that $n = (d(n))^2$. Here $d(n)$ denotes the number of positive divisors of n .
4. Suppose a_1, a_2, \dots, a_8 are eight distinct integers from $\{1, 2, \dots, 16, 17\}$. Show that there is an integer $k > 0$ such that the equation $a_i - a_j = k$ has at least three different solutions. Also, find a specific set of 7 distinct integers from $\{1, 2, \dots, 16, 17\}$ such that the equation $a_i - a_j = k$ does not have three distinct solutions for any $k > 0$.

5. Let x , y , and z be non-negative real numbers satisfying $x + y + z = 1$. Show that

$$x^2y + y^2z + z^2x \leq \frac{4}{27},$$

and find when equality occurs.

Next we give the 28th United States of America Mathematical Olympiad. These problems are copyrighted by the committee on the American Mathematical Competitions of the Mathematical Association of America and may not be reproduced without permission. Solutions, and additional copies of the problems may be obtained from Professor Titu Andreescu, AMC Director, University of Nebraska, Lincoln, NE, USA 68588-0658. As always, we welcome your original, "nice" solutions and generalizations which differ from the published solutions.

28th UNITED STATES OF AMERICA MATHEMATICAL OLYMPIAD

Part I 9 a.m. — 12 noon

April 27, 1999

1. Some checkers placed on an $n \times n$ checkerboard satisfy the following conditions:

- (a) every square that does not contain a checker shares a side with one that does;
- (b) given any pair of squares that contain checkers, there is a sequence of squares containing checkers, starting and ending with the given squares, such that every two consecutive squares of the sequence share a side.

Prove that at least $(n^2 - 2)/3$ checkers have been placed on the board.

2. Let $ABCD$ be a cyclic quadrilateral. Prove that

$$|AB - CD| + |AD - BC| \geq 2|AC - BD|.$$

3. Let $p > 2$ be a prime and let a, b, c, d be integers not divisible by p , such that

$$\{ra/p\} + \{rb/p\} + \{rc/p\} + \{rd/p\} = 2$$

for any integer r not divisible by p . Prove that at least two of the numbers $a+b, a+c, a+d, b+c, b+d, c+d$ are divisible by p . (Note: $\{x\} = x - [x]$ denotes the fractional part of x .)

Part II 1 p.m. — 4 p.m.

April 27, 1999

4. Let a_1, a_2, \dots, a_n ($n > 3$) be real numbers such that

$$a_1 + a_2 + \dots + a_n \geq n \quad \text{and} \quad a_1^2 + a_2^2 + \dots + a_n^2 \geq n^2.$$

Prove that $\max(a_1, a_2, \dots, a_n) \geq 2$.

5. The Y2K Game is played on a 1×2000 grid as follows. Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy.

6. Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E . Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G . Prove that the triangle AFG is isosceles.

We next give selected problems of the Ukrainian Mathematical Olympiad of March 26–27, 1996. My thanks go to J.P. Grossman, Team Leader of the Canadian International Olympiad Team at Mumbai, India, for collecting these problems.

UKRAINIAN MATHEMATICAL OLYMPIAD**March 26–27, 1996****Selected Problems**

1. (8th grade) A regular polygon with 1996 vertices is given. What minimal number of vertices can we delete so that we do not have four vertices remaining, which form: (a) a square? (b) a rectangle?

2. (9th grade) Ivan has made the models of all triangles with integer lengths of sides and perimeters 1993 cm. Peter has made the models of all triangles with integer lengths of sides and perimeters 1996 cm. Who has more models?

3. (10th grade) Prove that $\sin(\pi/20) + \sin(2\pi/20) + \dots + \sin(9\pi/20) < 99/10 - (2/\pi) \arcsin(1/10) - (2/\pi) \arcsin(2/10) - \dots - (2/\pi) \arcsin(9/10)$.

4. (10th grade) Let S be the set of all points of the coordinate plane with integer coordinates. We shall say that a one-to-one correspondence of S preserves a distance x if any two points in S at distance x have the images at distance x . Is it true that a one-to-one correspondence necessarily preserves all positive distances if:

- (a) it preserves the distance 1?
 (b) it preserves the distance 2?
 (c) it preserves the distance 2 and the distance 3?

5. (10th grade) Let O be the centre of the parallelogram $ABCD$ with $\angle AOB > \pi/2$. We take the points A_1, B_1 on the half-lines OA, OB , respectively so that $A_1B_1 \parallel AB$ and $\angle A_1B_1C = \angle ABC/2$. Prove that $A_1D \perp B_1C$.

6. (11th grade) The sequence $\{a_n\}$, $n \geq 0$, is such that $a_0 = 1$, $a_{499} = 0$ and for $n \geq 1$, $a_{n+1} = 2a_1a_n - a_{n-1}$.

- (a) Prove that $|a_1| \leq 1$.
 (b) Find a_{1996} .

7. (11th grade) Does a function $f : \mathbb{R} \rightarrow \mathbb{R}$ exist which is not a polynomial and such that for all real x

$$(x-1)f(x+1) - (x+1)f(x-1) = 4x(x^2-1)?$$

8. (11th grade) Let M be the number of all positive integers which have n digits 1, n digits 2 and no other digits in their decimal representations. Let N be the number of all n -digit positive integers with only digits 1, 2, 3, 4 in the representation where the number of 1's equals the number of 2's. Prove that $M = N$.

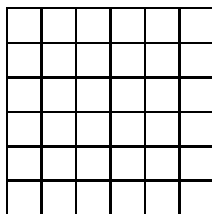
As another problem set we give the problems of the XII Italian Mathematical Olympiad, Cesenatico, 3 May, 1996. Again thanks for collecting these go to J.P. Grossman, Team Leader of the Canadian International Olympiad Team at Mumbai, India.

XII ITALIAN MATHEMATICAL OLYMPIAD Cesenatico, 3 May, 1996

- 1.** Among the triangles with an assigned side l and with given area S , determine all those for which the product of the three altitudes is maximum.
- 2.** Prove that the equation $a^2 + b^2 = c^2 + 3$ has infinitely many integer solutions (a, b, c) .
- 3.** Let A and B be opposite vertices of a cube with side 1. Find the radius of the sphere with centre interior to the cube, tangent to the three faces meeting in A and tangent to the three edges meeting in B .
- 4.** Given an alphabet with three letters a, b, c , find the number of words of n letters which contain an even number of a 's.

5. Let a circle C and a point A exterior to C be given. For each point P on C construct the square $APQR$, with anticlockwise ordering of the letters A, P, Q, R . Find the locus of the point Q when P runs over C .

6. What is the minimum number of squares that one needs to draw on a white sheet in order to obtain a full grid of size n ? (The picture shows a full grid of size 6).



As a relative newcomer to the *Corner*, we next give the problems of the South African Mathematics Olympiad, Third Round, 7 September 1995, Section A and B. Again thanks go to J.P. Grossman for collecting these while at the IMO at Mumbai, India as Canadian Team Leader.

SOUTH AFRICAN MATHEMATICS OLYMPIAD
Third Round — 7 September 1995
SECTION A

1. Prove that there are no integers m and n such that

$$419m^2 + 95mn + 2000n^2 = 1995.$$

2. ABC is a triangle with $\angle A > \angle C$, and D is the point on BC such that $\angle BAD = \angle ACB$. The perpendicular bisectors of AD and AC intersect in the point E . Prove that $\angle BAE = 90^\circ$.

3. Suppose that $a_1, a_2, a_3, \dots, a_n$ are the numbers $1, 2, 3, \dots, n$ but written in any order. Prove that

$$(a_1 - 1)^2 + (a_2 - 2)^2 + (a_3 - 3)^2 + \dots + (a_n - n)^2$$

is always even.

4. Three circles, with radii p, q, r , and centres A, B, C , respectively, touch one another externally at points D, E, F . Prove that the ratio of the areas of $\triangle DEF$ and $\triangle ABC$ equals

$$\frac{2pqr}{(p+q)(q+r)(r+p)}.$$

SECTION B

1. The convex quadrilateral $ABCD$ has area 1, and AB is produced to E , BC to F , CD to G and DA to H , such that $AB = BE$, $BC = CF$, $CD = DG$ and $DA = AH$. Find the area of the quadrilateral $EFGH$.

2. Find all pairs (m, n) of natural numbers with $m < n$ such that $m^2 + 1$ is a multiple of n and $n^2 + 1$ is a multiple of m .

3. The circumcircle of $\triangle ABC$ has radius 1 and centre O , and P is a point inside the triangle such that $OP = x$. Prove that

$$AP \cdot BP \cdot CP \leq (1+x)^2(1-x),$$

with equality only if $P = O$.

The next problems are those of the Taiwan Olympiad, 1996. Thanks go to J.P. Grossman, Team Leader for Canada at the IMO at Mumbai, India, for collecting them.

TAIWAN MATHEMATICAL OLYMPIAD 1996

1. Let the angles α, β, γ be such that $0 < \alpha, \beta, \gamma < \frac{\pi}{2}$ and $\alpha + \beta + \gamma = \frac{\pi}{4}$. Suppose that

$$\tan \alpha = \frac{1}{a}, \quad \tan \beta = \frac{1}{b}, \quad \tan \gamma = \frac{1}{c},$$

where a, b, c are positive integers. Determine the values of a, b, c .

2. Let a be a real number such that $0 < a \leq 1$ and $a \leq a_j \leq \frac{1}{a}$, for $j = 1, 2, \dots, 1996$. Show that for any non-negative real numbers λ_j ($j = 1, 2, \dots, 1996$), with

$$\sum_{j=1}^{1996} \lambda_j = 1,$$

one has

$$\left(\sum_{i=1}^{1996} \lambda_i a_i \right) \left(\sum_{j=1}^{1996} \lambda_j a_j^{-1} \right) \leq \frac{1}{4} \left(a + \frac{1}{a} \right)^2.$$

3. Let A and B be two fixed points on a fixed circle. Let a point P move on this circle and let M be a corresponding point such that either M is on the segment PA with $AM = MP + PB$ or M is on the segment PB with $AP + MP = PB$. Determine the locus of such points P .

4. Show that for any real numbers a_3, a_4, \dots, a_{85} , the roots of the equation

$$a_{85}x^{85} + a_{84}x^{84} + \dots + a_3x^3 + 3x^2 + 2x + 1 = 0$$

are not all real.

5. Find 99 integers $a_1, a_2, \dots, a_{99} = a_0$, satisfying

$$|a_{k-1} - a_k| \geq 1996 \quad \text{for all } k = 1, 2, \dots, 99,$$

so that the number

$$m = \max\{|a_{k-1} - a_k|; k = 1, 2, \dots, 99\}$$

is as small as possible, and determine the minimum value m^* of m .

6. Let q_0, q_1, q_2, \dots be a sequence of integers such that

(a) for any $m > n$, $m - n$ is a factor of $q_m - q_n$, and

(b) $|q_n| \leq n^{10}$ for all integers $n \geq 0$.

Show that there exists a polynomial $Q(x)$ satisfying $Q(n) = q_n$ for all n .

The next problems are those of the Croatian National Mathematics Competition, Kraljevica, May 16–19, 1996, IV Class and IMO Team Selection Competition problems. Thanks go to J.P. Grossman, Team Leader for Canada at the IMO at Mumbai, India, for collecting the problem set.

**CROATIAN NATIONAL MATHEMATICS
COMPETITION**
Kraljevica, May 16–19, 1996
IV CLASS

1. Is there any solution of the equation

$$[x] + [2x] + [4x] + [8x] + [16x] + [32x] = 12345?$$

($[x]$ denotes the greatest integer which does not exceed x .)

2. Determine all pairs of numbers $\lambda_1, \lambda_2 \in \mathbb{R}$ for which every solution of the equation $(x + i\lambda_1)^n + (x + i\lambda_2)^n = 0$ is real. Find the solutions.

3. Determine all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ continuous at 0, which satisfy the following relation $f(x) - 2f(tx) + f(t^2x) = x^2$ for all $x \in \mathbb{R}$, where $t \in (0, 1)$ is a given number.

4. Let α and β be positive irrational numbers such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and $\mathbb{A} = \{[n\alpha] \mid n \in \mathbb{N}\}$, $\mathbb{B} = \{[n\beta] \mid n \in \mathbb{N}\}$. Prove that $\mathbb{A} \cup \mathbb{B} = \mathbb{N}$ and $\mathbb{A} \cap \mathbb{B} = \emptyset$.

Remark: You can prove the following equivalent assertion: For a function $\pi : \mathbb{N} \rightarrow \mathbb{N}$ defined by

$$\pi(m) = \text{Card}\{k \mid k \in \mathbb{N}, k \leq m, k \in \mathbb{A}\} + \text{Card}\{k \mid k \in \mathbb{N}, k \leq m, k \in \mathbb{B}\}$$

one has $\pi(m) = m$, $\forall m \in \mathbb{N}$. ($[x]$ denotes the greatest integer which does not exceed x .)

**ADDITIONAL COMPETITION FOR SELECTION OF THE IMO TEAM
May 18, 1996**

1. (a) $n = 2k + 1$ points are given in the plane. Construct an n -gon such that these points are mid-points of its sides.

(b) Arbitrary $n = 2k$, $k > 1$, points are given in the plane. Prove that it is impossible to construct an n -gon, in each case, such that these points are mid-points of its sides.

2. The side-length of the square $ABCD$ equals a . Two points E and F are given on sides \overline{BC} and \overline{AB} such that the perimeter of the triangle BEF equals $2a$. Determine $\angle EDF$.

3. Find all pairs of consecutive integers the difference of whose cubes is a full square.

4. Let A_1, A_2, \dots, A_n be a regular n -gon inscribed in the circle of radius 1 with the centre at O . A point M is given on the ray OA_1 outside the n -gon. Prove that

$$\sum_{k=1}^n \frac{1}{|MA_k|} \geq \frac{n}{|OM|}.$$

We next turn to solutions. First an alternative solution to that given earlier this year to problem 5 of the Iranian Olympiad (1994) [1999: 142–143]

5. [1998: 6–7] [1999: 142–143] *Iranian Mathematical Olympiad (1994)*. Show that if D_1 and D_2 are two skew lines, then there are infinitely many straight lines such that their points have equal distance from D_1 and D_2 .

Comments by J. Chris Fisher, University of Regina, Regina, Saskatchewan; with alternative solution by Aart Blokhuis, Mathematics Department, Eindhoven University of Technology, the Netherlands.

Rename the lines l and m . Fix points A and B on l a unit distance apart. For each A' on m there are two points B' and B'' on m that are a unit distance from A' . There is a unique rotation that takes A and B to A' and B' , and another taking them to A' and B'' ; the points of the axes of these two rotations are equidistant from l and m since the perpendicular from an axis point to l is taken by the rotation to the perpendicular from that point to m . Each A' leads to a different pair of lines. (To see that the rotation exists as claimed, take as mirror 1 the plane of points equidistant from A and A' ; if B^* is the image of B under reflection in mirror 1 then take mirror 2 to be the perpendicular bisector (necessarily through A') of B^* and either B' or B'' . The product of reflections in these two mirrors is a rotation about their line of intersection.)

Comments. (1) The locus of points equidistant from the skew lines l and m is a ruled surface, namely the *hyperbolic paraboloid*. To see the parabolas, take the section of the surface by a plane through l : the locus of points in that plane that are equidistant from the point where it meets m and from the line l is a parabola.

(2) A slight generalization of the problem provides a simple construction of a *spread* (which is a collection of skew lines that completely cover the three dimensional space in the sense that every point of space is on exactly one of the lines of the spread): the locus of points whose distances from l and m are in the ratio $1 : k$ is a ruled quadratic. The proof of the original problem can be modified to a proof of the claim by taking B' and B'' to be k units from A' , and using a dilative rotation to take A and B to their primed mates.

To finish this number of the *Corner* we give participant or “official” solutions to the Canadian Mathematical Olympiad given at the beginning of this number. My thanks go to Daryl Tingley, Chair of the Canadian Mathematical Olympiad Committee for furnishing the following:

CANADIAN MATHEMATICAL OLYMPIAD 1999 SOLUTIONS

Most of the solutions to the problems of the 1999 CMO presented below are taken from students' papers. Some minor editing has been done — unnecessary steps have been eliminated and some wording has been changed to make the proofs clearer. But for the most part, the proofs are as submitted.

1. *Solution* — Adrian Chan, Upper Canada College, Toronto, Ontario. Rearranging the equation we get $4x^2 + 51 = 40[x]$. It is known that

$x \geq [x] > x - 1$, so

$$\begin{aligned} 4x^2 + 51 = 40[x] &> 40(x - 1), \\ 4x^2 - 40x + 91 &> 0, \\ (2x - 13)(2x - 7) &> 0. \end{aligned}$$

Hence $x > 13/2$ or $x < 7/2$. Also,

$$\begin{aligned} 4x^2 + 51 = 40[x] &\leq 40x, \\ 4x^2 - 40x + 51 &\leq 0, \\ (2x - 17)(2x - 3) &\leq 0. \end{aligned}$$

Hence $3/2 \leq x \leq 17/2$. Combining these inequalities gives $3/2 \leq x < 7/2$ or $13/2 < x \leq 17/2$.

Case 1: $3/2 \leq x < 7/2$.

For this case, the possible values for $[x]$ are 1, 2 and 3.

If $[x] = 1$ then $4x^2 + 51 = 40 \cdot 1$ so $4x^2 = -11$, which has no real solutions.

If $[x] = 2$ then $4x^2 + 51 = 40 \cdot 2$ so $4x^2 = 29$ and $x = \frac{\sqrt{29}}{2}$. Notice that $\frac{\sqrt{16}}{2} < \frac{\sqrt{29}}{2} < \frac{\sqrt{36}}{2}$ so $2 < x < 3$ and $[x] = 2$.

If $[x] = 3$ then $4x^2 + 51 = 40 \cdot 3$ and $x = \sqrt{69}/2$. But $\frac{\sqrt{69}}{2} > \frac{\sqrt{64}}{2} = 4$. So, this solution is rejected.

Case 2: $13/2 < x \leq 17/2$.

For this case, the possible values for $[x]$ are 6, 7 and 8.

If $[x] = 6$ then $4x^2 + 51 = 40 \cdot 6$, so that $x = \frac{\sqrt{189}}{2}$. Notice that $\frac{\sqrt{144}}{2} < \frac{\sqrt{189}}{2} < \frac{\sqrt{196}}{2}$, so that $6 < x < 7$ and $[x] = 6$.

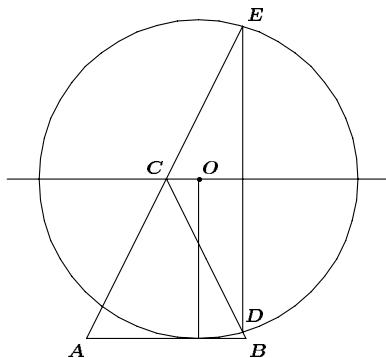
If $[x] = 7$ then $4x^2 + 51 = 40 \cdot 7$, so that $x = \frac{\sqrt{229}}{2}$. Notice that $\frac{\sqrt{196}}{2} < \frac{\sqrt{229}}{2} < \frac{\sqrt{256}}{2}$, so that $7 < x < 8$ and $[x] = 7$.

If $[x] = 8$ then $4x^2 + 51 = 40 \cdot 8$, so that $x = \frac{\sqrt{269}}{2}$. Notice that $\frac{\sqrt{256}}{2} < \frac{\sqrt{269}}{2} < \frac{\sqrt{324}}{2}$, so that $8 < x < 9$ and $[x] = 8$.

The solutions are $x = \frac{\sqrt{29}}{2}, \frac{\sqrt{189}}{2}, \frac{\sqrt{229}}{2}, \frac{\sqrt{269}}{2}$.

(*Editor:* Adrian then checks these four solutions.)

2. *Solution 1* — Keon Choi, A.Y. Jackson SS, North York, Ontario.



Let D and E be the intersections of BC and extended AC , respectively, with the circle.

Since $CO \parallel AB$ (because both the altitude and the radius are 1) $\angle BCO = 60^\circ$ and therefore $\angle ECO = 180^\circ - \angle ACB - \angle BCO = 60^\circ$.

Since a circle is always symmetric about its diameter and line CE is a reflection of line CB in CO , line segment CE is a reflection of line segment CD .

Therefore $CE = CD$.

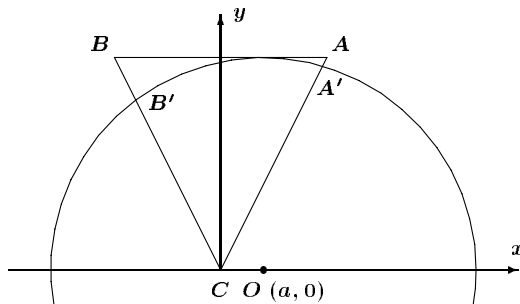
Therefore $\triangle CED$ is an isosceles triangle.

Therefore $\angle CED = \angle CDE$ and $\angle CED + \angle CDE = \angle ACB = 60^\circ$.

$\angle CED = 30^\circ$ regardless of the position of centre O . Since $\angle CED$ is also the angle subtended from the arc inside the triangle, if $\angle CED$ is constant, the arc length is also constant.

Editor's Note: This proof has had no editing.

Solution 2 — Jimmy Chui, Earl Haig SS, North York, Ontario.



Place C at the origin, point A at $(\frac{1}{\sqrt{3}}, 1)$ and point B at $(-\frac{1}{\sqrt{3}}, 1)$. Then $\triangle ABC$ is equilateral with altitude of length 1.

Let O be the centre of the circle. Because the circle has radius 1, and since it touches line AB , the locus of O is on the line through C parallel to AB (since C is length 1 away from AB); that is, the locus of O is on the x -axis.

Let point O be at $(a, 0)$. Then $-\frac{1}{\sqrt{3}} \leq a \leq \frac{1}{\sqrt{3}}$ since we have the restriction that the circle rolls along AB .

Now, let A' and B' be the intersection of the circle with CA and CB , respectively. The equation of CA is $y = \sqrt{3}x$, $0 \leq x \leq \frac{1}{\sqrt{3}}$, of CB is $y = -\sqrt{3}x$, $-\frac{1}{\sqrt{3}} \leq x \leq 0$, and of the circle is $(x - a)^2 + y^2 = 1$.

We solve for A' by substituting $y = \sqrt{3}x$ into $(x - a)^2 + y^2 = 1$ to get $x = \frac{a \pm \sqrt{4 - 3a^2}}{4}$.

Visually, we can see that solutions represent the intersection of AC extended and the circle, but we are only concerned with the greater x -value — this is the solution that is on AC , not on AC extended. Therefore

$$x = \frac{a + \sqrt{4 - 3a^2}}{4}, \quad y = \sqrt{3} \left(\frac{a + \sqrt{4 - 3a^2}}{4} \right).$$

Likewise we solve for B' , but we take the lesser x -value to get

$$x = \frac{a - \sqrt{4 - 3a^2}}{4}, \quad y = -\sqrt{3} \left(\frac{a - \sqrt{4 - 3a^2}}{4} \right).$$

Let us find the length of $A'B'$:

$$\begin{aligned} |A'B'|^2 &= \left(\frac{a + \sqrt{4 - 3a^2}}{4} - \frac{a - \sqrt{4 - 3a^2}}{4} \right)^2 + \\ &\quad \left(\left(\sqrt{3} \frac{a + \sqrt{4 - 3a^2}}{4} \right) - \left(-\sqrt{3} \frac{a - \sqrt{4 - 3a^2}}{4} \right) \right)^2 \\ &= \frac{4 - 3a^2}{4} + 3 \frac{a^2}{4} = 1, \end{aligned}$$

which is independent of a .

Consider the points O , A' and B' . $\triangle OA'B'$ is an equilateral triangle (because $A'B' = OA' = OB' = 1$).

Therefore $\angle A'OB' = \frac{\pi}{3}$ and arc $A'B' = \frac{\pi}{3}$, a constant.

3. Solution — Masoud Kamgarpour, Carson SS, North Vancouver, BC.

Note that $n = 1$ is a solution. For $n > 1$ write n in the form $n = P_1^{\alpha_1} P_2^{\alpha_2} \dots P_m^{\alpha_m}$ where the P_i 's, $1 \leq i \leq m$, are distinct prime numbers and $\alpha_i > 0$. Since $d(n)$ is an integer, n is a perfect square, so $\alpha_i = 2\beta_i$ for integers $\beta_i > 0$.

Using the formula for the number of divisors of n ,

$$d(n) = (2\beta_1 + 1)(2\beta_2 + 1) \dots (2\beta_m + 1),$$

which is an odd number. Now because $d(n)$ is odd, $(d(n))^2$ is odd, therefore n is odd as well, so $P_i \geq 3, 1 \leq i \leq m$. We get

$$P_1^{\alpha_1} \cdot P_2^{\alpha_2} \dots P_m^{\alpha_m} = [(\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_m + 1)]^2$$

or using $\alpha_i = 2\beta_i$

$$P_1^{\beta_1} P_2^{\beta_2} \dots P_m^{\beta_m} = (2\beta_1 + 1)(2\beta_2 + 1) \dots (2\beta_m + 1).$$

Now we prove a lemma:

Lemma: $P^t \geq 2t + 1$ for positive integers t and $P \geq 3$, with equality only when $P = 3$ and $t = 1$.

Proof: We use mathematical induction on t . The statement is true for $t = 1$ because $P \geq 3$. Now suppose $P^k \geq 2k + 1, k \geq 1$; then we have

$$P^{k+1} = P^k \cdot P \geq P^k(1+2) > P^k + 2 \geq (2k+1) + 2 = 2(k+1) + 1.$$

Thus $P^t \geq 2t + 1$ and equality occurs only when $P = 3$ and $t = 1$.

Let us say n has a prime factor $P_k > 3$; then (by the lemma) $P_k^{\beta_k} > 2\beta_k + 1$ and we have $P_1^{\beta_1} \dots P_m^{\beta_m} > (2\beta_1 + 1) \dots (2\beta_m + 1)$, a contradiction.

Therefore, the only prime factor of n is $P = 3$ and we have $3^\gamma = 2\gamma + 1$. By the lemma $\gamma = 1$.

The only positive integer solutions are 1 and 9.

4. *Solution 1 — David Nicholson, Fenelon Falls SS, Fenelon Falls, Ontario.*

Without loss of generality let $a_1 < a_2 < a_3 \dots < a_8$.

Assume that there is no such integer k . Let us just look at the seven differences $d_i = a_{i+1} - a_i$. Then amongst the d_i there can be at most two 1s, two 2s, and two 3s, which totals to 12.

Now $16 = 17 - 1 \geq a_8 - a_1 = d_1 + d_2 + \dots + d_7$ so the seven differences must be 1, 1, 2, 2, 3, 3, 4.

Now let us think of arranging the differences 1, 1, 2, 2, 3, 3, 4. Note that the sum of consecutive differences is another difference. (For example, $d_1 + d_2 = a_3 - a_1, d_1 + d_2 + d_3 = a_4 - a_1$)

We cannot place the two 1s side by side because that will give us another difference of 2. The 1s cannot be beside a 2 because then we have three 3s. They cannot both be beside a 3 because then we have three 4s! So we must have either 1, 4, -, -, -, 3, 1 or 1, 4, 1, 3, -, -, - (or their reflections).

In either case we have a 3, 1 giving a difference of 4 so we cannot put the 2s beside each other. Also we cannot have 2, 3, 2 because then (with the 1, 4) we will have three 5s. So all cases give a contradiction.

Therefore there will always be three differences equal.

One set of seven numbers satisfying the criteria is $\{1, 2, 4, 7, 11, 16, 17\}$. [Editor: There are many such sets.]

Solution 2 — The CMO committee.

Consider all the consecutive differences (that is, d_i above) as well as the differences $b_i = a_{i+2} - a_i$, $i = 1, \dots, 6$. Then the sum of these thirteen differences is $2 \cdot (a_8 - a_1) + (a_7 - a_2) \leq 2(17 - 1) + (16 - 2) = 46$. Now if no difference occurs more than twice, the smallest the sum of the thirteen differences can be is $2 \cdot (1 + 2 + 3 + 4 + 5 + 6) + 7 = 49$, giving a contradiction.

5. Solution 1 — The CMO committee.

Let $f(x, y, z) = x^2y + y^2z + z^2x$. We wish to determine where f is maximal. Since f is cyclic, without loss of generality we may assume that $x \geq y, z$. Since

$$\begin{aligned} f(x, y, z) - f(x, z, y) &= x^2y + y^2z + z^2x - x^2z - z^2y - y^2x \\ &= (y - z)(x - y)(x - z), \end{aligned}$$

we may also assume $y \geq z$. Then

$$\begin{aligned} f(x + z, y, 0) - f(x, y, z) &= (x + z)^2y - x^2y - y^2z - z^2x \\ &= z^2y + yz(x - y) + xz(y - z) \geq 0, \end{aligned}$$

so we may now assume $z = 0$. The rest follows from the arithmetic-geometric mean inequality:

$$f(x, y, 0) = \frac{2x^2y}{2} \leq \frac{1}{2} \left(\frac{x + x + 2y}{3} \right)^3 = \frac{4}{27}.$$

Equality occurs when $x = 2y$, hence at $(x, y, z) = (\frac{2}{3}, \frac{1}{3}, 0)$. (As well as $(0, \frac{2}{3}, \frac{1}{3})$ and $(\frac{1}{3}, 0, \frac{2}{3})$).

Solution 2 — The CMO committee.

With f as above, and $x \geq y, z$

$$f\left(x + \frac{z}{2}, y + \frac{z}{2}, 0\right) - f(x, y, z) = yz(x - y) + \frac{xz}{2}(x - z) + \frac{z^2y}{4} + \frac{z^3}{8},$$

so we may assume that $z = 0$. The rest follows as for solution 1.

That completes this number of the *Corner*. Send me your nice solutions as well as Olympiad materials for use in future issues.