THE ACADEMY CORNER

No. 28

Bruce Shawyer

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A Trial Balloon

Vedula N. Murty

The following problem on heights and distances was set in the I.I.T. Entrance Exam in 1979.

> *A balloon is observed simultaneously from three points, A, B and C, on a straight road directly beneath it. The angular elevation at B is twice that at A, and the angular elevation at C is three times that at A. If the distance between A and B is a, and the distance between B and C is b, find the height of the balloon in terms of a and b.*

Subsequently, this problem appeared in many textbooks in India with a solution which is straightforward. The height of the balloon is

$$y \;=\; \frac{a\sqrt{3b^2+2ab-a^2}}{2b}\,.$$

The conditions on a and b under which this solution is valid are not given in any of the solutions printed in the textbooks.

Professor M. Perisastry, a retired Professor of Mathematics at M.R. College, Viziahagaram, Andhra Pradesh, India, noted that

$$\begin{aligned} y > 0 \quad &\Longrightarrow \quad 3b^2+2ab-a^2 > 0 \\ &\Longrightarrow \quad 4b^2-(a-b)^2 > 0 \\ &\Longrightarrow \quad |a-b| < 2b \\ &\Longrightarrow \quad 0 < a < 3b\,. \end{aligned}$$

Moreover, it is easily seen that

$$\frac{a}{b} \;=\; \frac{\sin(3\alpha)}{\sin\alpha} \;=\; 3-4\sin^2\alpha\,,$$

where α is the angle of elevation at A.

Since the angle of elevation at C is $3\alpha < \pi/2$, this implies that

$$\begin{aligned} 0 < \alpha < \frac{\pi}{6} \quad &\Longrightarrow \quad 0 < \sin\alpha < \frac{1}{2} \\ &\Longrightarrow \quad \frac{a}{b} > 2 \\ &\Longrightarrow \quad a > 2b\,. \end{aligned}$$

Hence, $$\frac{a}{3} \;<\; b \;<\; \frac{a}{2}\,.$$

Readers of ***CRUX with MAYHEM*** may be interested in the above problem, and teachers should pay attention to the conditions under which a given solution is valid.

BAD CANCELLATIONS

$$\left| \frac{\cancel{\tan^{-1}}\,(2^{\cancel{n}+1})}{\cancel{\tan^{-1}}\,(2^{\cancel{n}})} \cdot \frac{\cancel{n}^{\cancel{3/2}}}{(\cancel{n}+1)^{\cancel{3/2}}} \right| = 2$$

QUESTIONS on MATHEMATICIANS

The year 1796 was the turning point in a (future) mathematician's career. Who was he?

And which mathematician was born in 1796?

What do Wilhelm Ackermann, Pavel Sergeevich Aleksandrov, Lester R. Ford, Ronald Martin Foster, Valeriĭ Ivanovich Glivenko, Kazimierz Kurtowski, and Carl Ludwig Seigel have in common?

What is the difference between Gustav Magnus Mittag-Leffler and Gaspard Monge?

THE OLYMPIAD CORNER

No. 201

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

As a first contest this number we give the Thirty-first Canadian Mathematical Olympiad 1999. The contest was held March 31, 1999 with 81 competitors from 48 schools in five Canadian provinces participating. Competitors are invited on the basis of their performance on other contests. Each question was marked out of 7 marks for a total possible score of 35.

First prize went to Jimmy Chui, Second to Adrian Chan, Third to David Pritchard, and Honourable Mentions go to Edmond Choi, Masoud Kamgarpour, Jessie Lei, Pierre LeVan, Dave Nicholson, and Yannick Solari. Congratulations!

My thanks go to Daryl Tingley, Chair of the Canadian Mathematical Olympiad Committee for furnishing us with the contest as well as selected solutions by the contestants, which appear at the end of this number of the *Corner*. But try them first!

THE THIRTY-FIRST CANADIAN MATHEMATICAL OLYMPIAD 1999

March 31, 1999

1. Find all real solutions to the equation $4x^2 - 40[x] + 51 = 0$. Here, if x is a real number, then $[x]$ denotes the greatest integer that is less than or equal to x.

2. Let ABC be an equilateral triangle of altitude 1. A circle with radius 1 and centre on the same side of AB as C rolls along the segment AB. Prove that the arc of the circle that is inside the triangle always has the same length.

3. Determine all positive integers n with the property that $n = (d(n))^2$. Here $d(n)$ denotes the number of positive divisors of n.

4. Suppose a_1, a_2, …, a_8 are eight distinct integers from $\{1, 2, \ldots, 16, 17\}$. Show that there is an integer $k > 0$ such that the equation $a_i - a_j = k$ has at least three different solutions. Also, find a specific set of 7 distinct integers from $\{1, 2, \ldots, 16, 17\}$ such that the equation $a_i - a_j = k$ does not have three distinct solutions for any $k > 0$.

5. Let x, y, and z be non-negative real numbers satisfying $x + y + z = 1$. Show that

$$x^2y + y^2z + z^2x \le \frac{4}{27},$$

and find when equality occurs.

Next we give the 28th United States of America Mathematical Olympiad. These problems are copyrighted by the committee on the American Mathematical Competitions of the Mathematical Association of America and may not be reproduced without permission. Solutions, and additional copies of the problems may be obtained from Professor Titu Andreescu, AMC Director, University of Nebraska, Lincoln, NE, USA 68588–0658. As always, we welcome your original, "nice" solutions and generalizations which differ from the published solutions.

28th UNITED STATES OF AMERICA MATHEMATICAL OLYMPIAD

Part I 9 a.m. — 12 noon

April 27, 1999

1. Some checkers placed on an $n \times n$ checkerboard satisfy the following conditions:

(a) every square that does not contain a checker shares a side with one that does;

(b) given any pair of squares that contain checkers, there is a sequence of squares containing checkers, starting and ending with the given squares, such that every two consecutive squares of the sequence share a side.

Prove that at least $(n^2 - 2)/3$ checkers have been placed on the board.

2. Let $ABCD$ be a cyclic quadrilateral. Prove that

$$|AB - CD| + |AD - BC| \ge 2|AC - BD|.$$

3. Let $p > 2$ be a prime and let a, b, c, d be integers not divisible by p, such that

$$\{ra/p\} + \{rb/p\} + \{rc/p\} + \{rd/p\} = 2$$

for any integer r not divisible by p. Prove that at least two of the numbers $a+b$, $a+c$, $a+d$, $b+c$, $b+d$, $c+d$ are divisible by p. (Note: $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x.)

Part II 1 p.m. — 4 p.m.

April 27, 1999

4. Let a_1, a_2, …, a_n ($n > 3$) be real numbers such that

$$a_1 + a_2 + \cdots + a_n \geq n \quad \text{and} \quad a_1^2 + a_2^2 + \cdots + a_n^2 \geq n^2 .$$

Prove that $\max(a_1, a_2, \ldots, a_n) \geq 2$.

5. The **Y2K** Game is played on a 1×2000 grid as follows. Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy.

6. Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles.

We next give selected problems of the Ukrainian Mathematical Olympiad of March 26–27, 1996. My thanks go to J.P. Grossman, Team Leader of the Canadian International Olympiad Team at Mumbai, India, for collecting these problems.

UKRAINIAN MATHEMATICAL OLYMPIAD

March 26–27, 1996

Selected Problems

1. (8th grade) A regular polygon with 1996 vertices is given. What minimal number of vertices can we delete so that we do not have four vertices remaining, which form: (a) a square? (b) a rectangle?

2. (9th grade) Ivan has made the models of all triangles with integer lengths of sides and perimeters 1993 cm. Peter has made the models of all triangles with integer lengths of sides and perimeters 1996 cm. Who has more models?

3. (10th grade) Prove that $\sin(\pi/20)+\sin(2\pi/20)+\cdots+\sin(9\pi/20) < 99/10-(2/\pi)\arcsin(1/10)-(2/\pi)\arcsin(2/10)-\cdots-(2/\pi)\arcsin(9/10)$.

4. (10th grade) Let S be the set of all points of the coordinate plane with integer coordinates. We shall say that a one-to-one correspondence of S preserves a distance x if any two points in S at distance x have the images at distance x. Is it true that a one-to-one correspondence necessarily preserves all positive distances if:

(a) it preserves the distance 1?

(b) it preserves the distance 2?

(c) it preserves the distance 2 and the distance 3?

5. (10th grade) Let O be the centre of the parallelogram $ABCD$ with $\angle AOB > \pi/2$. We take the points A_1, B_1 on the half-lines OA, OB, respectively so that $A_1B_1 \parallel AB$ and $\angle A_1B_1C = \angle ABC/2$. Prove that $A_1D \perp B_1C$.

6. (11th grade) The sequence $\{a_n\}$, $n \geq 0$, is such that $a_0 = 1$, $a_{499} = 0$ and for $n \geq 1$, $a_{n+1} = 2a_1a_n - a_{n-1}$.

(a) Prove that $|a_1| \leq 1$.

(b) Find a_{1996}.

7. (11th grade) Does a function $f : \mathbb{R} \to \mathbb{R}$ exist which is not a polynomial and such that for all real x

$$(x-1)f(x+1) - (x+1)f(x-1) = 4x(x^2-1)?$$

8. (11th grade) Let M be the number of all positive integers which have n digits 1, n digits 2 and no other digits in their decimal representations. Let N be the number of all n–digit positive integers with only digits 1, 2, 3, 4 in the representation where the number of 1's equals the number of 2's. Prove that $M = N$.

As another problem set we give the problems of the XII Italian Mathematical Olympiad, Cesenatico, 3 May, 1996. Again thanks for collecting these go to J.P. Grossman, Team Leader of the Canadian International Olympiad Team at Mumbai, India.

XII ITALIAN MATHEMATICAL OLYMPIAD
Cesenatico, 3 May, 1996

1. Among the triangles with an assigned side l and with given area S, determine all those for which the product of the three altitudes is maximum.

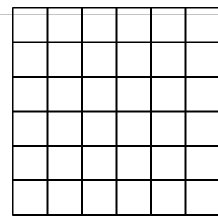
2. Prove that the equation $a^2 + b^2 = c^2 + 3$ has infinitely many integer solutions (a, b, c).

3. Let A and B be opposite vertices of a cube with side 1. Find the radius of the sphere with centre interior to the cube, tangent to the three faces meeting in A and tangent to the three edges meeting in B.

4. Given an alphabet with three letters a, b, c, find the number of words of n letters which contain an even number of a's.

5. Let a circle C and a point A exterior to C be given. For each point P on C construct the square $APQR$, with anticlockwise ordering of the letters A, P, Q, R. Find the locus of the point Q when P runs over C.

6. What is the minimum number of squares that one needs to draw on a white sheet in order to obtain a full grid of size n? (The picture shows a full grid of size 6).



As a relative newcomer to the *Corner*, we next give the problems of the South African Mathematics Olympiad, Third Round, 7 September 1995, Section A and B. Again thanks go to J.P. Grossman for collecting these while at the IMO at Mumbai, India as Canadian Team Leader.

SOUTH AFRICAN MATHEMATICS OLYMPIAD
Third Round — 7 September 1995
SECTION A

1. Prove that there are no integers m and n such that

$$419m^2 + 95mn + 2000n^2 = 1995 .$$

2. ABC is a triangle with $\angle A > \angle C$, and D is the point on BC such that $\angle BAD = \angle ACB$. The perpendicular bisectors of AD and AC intersect in the point E. Prove that $\angle BAE = 90°$.

3. Suppose that $a_1, a_2, a_3, \ldots, a_n$ are the numbers $1, 2, 3, \ldots, n$ but written in any order. Prove that

$$(a_1 - 1)^2 + (a_2 - 2)^2 + (a_3 - 3)^2 + \cdots + (a_n - n)^2$$

is always even.

4. Three circles, with radii p, q, r, and centres A, B, C, respectively, touch one another externally at points D, E, F. Prove that the ratio of the areas of $\triangle DEF$ and $\triangle ABC$ equals

$$\frac{2pqr}{(p+q)(q+r)(r+p)} .$$

SECTION B

1. The convex quadrilateral $ABCD$ has area 1, and AB is produced to E, BC to F, CD to G and DA to H, such that $AB = BE$, $BC = CF$, $CD = DG$ and $DA = AH$. Find the area of the quadrilateral $EFGH$.

2. Find all pairs (m, n) of natural numbers with $m < n$ such that $m^2 + 1$ is a multiple of n and $n^2 + 1$ is a multiple of m.

3. The circumcircle of $\triangle ABC$ has radius 1 and centre O, and P is a point inside the triangle such that $OP = x$. Prove that

$$AP \cdot BP \cdot CP \;\le\; (1+x)^2(1-x)\,,$$

with equality only if $P = O$.

The next problems are those of the Taiwan Olympiad, 1996. Thanks go to J.P. Grossman, Team Leader for Canada at the IMO at Mumbai, India, for collecting them.

TAIWAN MATHEMATICAL OLYMPIAD 1996

1. Let the angles α, β, γ be such that $0 < \alpha, \beta, \gamma < \frac{\pi}{2}$ and $\alpha + \beta + \gamma = \frac{\pi}{4}$. Suppose that

$$\tan\alpha \;=\; \frac{1}{a}, \quad \tan\beta \;=\; \frac{1}{b}\,, \quad \tan\gamma \;=\; \frac{1}{c}\,,$$

where a, b, c are positive integers. Determine the values of a, b, c.

2. Let a be a real number such that $0 < a \le 1$ and $a \le a_j \le \frac{1}{a}$, for $j = 1, 2, \ldots, 1996$. Show that for any non-negative real numbers λ_j ($j = 1, 2, \ldots, 1996$), with

$$\sum_{j=1}^{1996} \lambda_j \;=\; 1\,,$$

one has

$$\left(\sum_{i=1}^{1996} \lambda_i a_i\right)\left(\sum_{j=1}^{1996} \lambda_j a_j^{-1}\right) \;\le\; \frac{1}{4}\left(a + \frac{1}{a}\right)^2\,.$$

3. Let A and B be two fixed points on a fixed circle. Let a point P move on this circle and let M be a corresponding point such that either M is on the segment PA with $AM = MP + PB$ or M is on the segment PB with $AP + MP = PB$. Determine the locus of such points P.

4. Show that for any real numbers a_3, a_4, $\ldots$, a_{85}, the roots of the equation

$$a_{85}x^{85} + a_{84}x^{84} + \cdots + a_3x^3 + 3x^2 + 2x + 1 \;=\; 0$$

are not all real.

5. Find 99 integers a_1, a_2, $\ldots$, $a_{99} = a_0$, satisfying

$$|a_{k-1} - a_k| \;\geq\; 1996 \quad \text{for all} \quad k = 1, 2, \ldots, 99,$$

so that the number

$$m = \max\{|a_{k-1} - a_k|; \quad k = 1, 2, \ldots, 99\}$$

is as small as possible, and determine the minimum value m^* of m.

6. Let q_0, q_1, q_2, $\ldots$ be a sequence of integers such that

(a) for any $m > n$, $m - n$ is a factor of $q_m - q_n$, and

(b) $|q_n| \leq n^{10}$ for all integers $n \geq 0$.

Show that there exists a polynomial $Q(x)$ satisfying $Q(n) = q_n$ for all n.

The next problems are those of the Croatian National Mathematics Competition, Kraljevica, May 16–19, 1996, IV Class and IMO Team Selection Competition problems. Thanks go to J.P. Grossman, Team Leader for Canada at the IMO at Mumbai, India, for collecting the problem set.

CROATIAN NATIONAL MATHEMATICS COMPETITION

Kraljevica, May 16–19, 1996

IV CLASS

1. Is there any solution of the equation

$$[x] + [2x] + [4x] + [8x] + [16x] + [32x] \;=\; 12345?$$

($[x]$ denotes the greatest integer which does not exceed x.)

2. Determine all pairs of numbers λ_1, $\lambda_2 \in \mathbb{R}$ for which every solution of the equation $(x + i\lambda_1)^n + (x + i\lambda_2)^n \;=\; 0$ is real. Find the solutions.

3. Determine all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous at 0, which satisfy the following relation $f(x) - 2f(tx) + f(t^2x) \;=\; x^2$ for all $x \in \mathbb{R}$, where $t \in (0, 1)$ is a given number.

4. Let α and β be positive irrational numbers such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and $\mathbb{A} = \{[n\alpha] \mid n \in \mathbb{N}\}$, $\mathbb{B} = \{[n\beta] \mid n \in \mathbb{N}\}$. Prove that $\mathbb{A} \cup \mathbb{B} = \mathbb{N}$ and $\mathbb{A} \cap \mathbb{B} = \emptyset$.

Remark: You can prove the following equivalent assertion: For a function $\pi : \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(m) \;=\; \text{Card}\{k \mid k \in \mathbb{N}, k \le m, k \in \mathbb{A}\} + \text{Card}\{k \mid k \in \mathbb{N}, k \le m, k \in \mathbb{B}\}$$

one has $\pi(m) = m$, $\forall m \in \mathbb{N}$. ($[x]$ denotes the greatest integer which does not exceed x.)

ADDITIONAL COMPETITION FOR SELECTION OF THE IMO TEAM
May 18, 1996

1. (a) $n = 2k + 1$ points are given in the plane. Construct an n–gon such that these points are mid-points of its sides.

(b) Arbitrary $n = 2k$, $k > 1$, points are given in the plane. Prove that it is impossible to construct an n–gon, in each case, such that these points are mid-points of its sides.

2. The side-length of the square $ABCD$ equals a. Two points E and F are given on sides $\overline{BC}$ and $\overline{AB}$ such that the perimeter of the triangle BEF equals $2a$. Determine $\angle EDF$.

3. Find all pairs of consecutive integers the difference of whose cubes is a full square.

4. Let A_1, A_2, $\ldots$, A_n be a regular n–gon inscribed in the circle of radius 1 with the centre at O. A point M is given on the ray OA_1 outside the n–gon. Prove that

$$\sum_{k=1}^{n} \frac{1}{|MA_k|} \;\ge\; \frac{n}{|OM|}\,.$$

We next turn to solutions. First an alternative solution to that given earlier this year to problem 5 of the Iranian Olympiad (1994) [1999: 142–143]

5. [1998: 6–7] [1999: 142–143] *Iranian Mathematical Olympiad (1994)*.
Show that if D_1 and D_2 are two skew lines, then there are infinitely many straight lines such that their points have equal distance from D_1 and D_2.

Comments by J. Chris Fisher, University of Regina, Regina, Saskatchewan; with alternative solution by Aart Blokhuis, Mathematics Department, Eindhoven University of Technology, the Netherlands.

Rename the lines l and m. Fix points A and B on l a unit distance apart. For each A' on m there are two points B' and B'' on m that are a unit distance from A'. There is a unique rotation that takes A and B to A' and B', and another taking them to A' and B''; the points of the axes of these two rotations are equidistant from l and m since the perpendicular from an axis point to l is taken by the rotation to the perpendicular from that point to m. Each A' leads to a different pair of lines. (To see that the rotation exists as claimed, take as mirror 1 the plane of points equidistant from A and A'; if B^* is the image of B under reflection in mirror 1 then take mirror 2 to be the perpendicular bisector (necessarily through A') of B^* and either B' or B''. The product of reflections in these two mirrors is a rotation about their line of intersection.)

Comments. (1) The locus of points equidistant from the skew lines l and m is a ruled surface, namely the *hyperbolic paraboloid*. To see the parabolas, take the section of the surface by a plane through l: the locus of points in that plane that are equidistant from the point where it meets m and from the line l is a parabola.

(2) A slight generalization of the problem provides a simple construction of a *spread* (which is a collection of skew lines that completely cover the three dimensional space in the sense that every point of space is on exactly one of the lines of the spread): the locus of points whose distances from l and m are in the ratio $1 : k$ is a ruled quadratic. The proof of the original problem can be modified to a proof of the claim by taking B' and B'' to be k units from A', and using a dilative rotation to take A and B to their primed mates.

To finish this number of the *Corner* we give participant or "official" solutions to the Canadian Mathematical Olympiad given at the beginning of this number. My thanks go to Daryl Tingley, Chair of the Canadian Mathematical Olympiad Committee for furnishing the following:

CANADIAN MATHEMATICAL OLYMPIAD 1999
SOLUTIONS

Most of the solutions to the problems of the 1999 CMO presented below are taken from students' papers. Some minor editing has been done — unnecessary steps have been eliminated and some wording has been changed to make the proofs clearer. But for the most part, the proofs are as submitted.

1. *Solution — Adrian Chan, Upper Canada College, Toronto, Ontario.* Rearranging the equation we get $4x^2 + 51 = 40[x]$. It is known that

$x \geq [x] > x - 1$, so

$$\begin{aligned} 4x^2 + 51 = 40[x] &> 40(x-1)\,, \\ 4x^2 - 40x + 91 &> 0\,, \\ (2x-13)(2x-7) &> 0\,. \end{aligned}$$

Hence $x > 13/2$ or $x < 7/2$. Also,

$$\begin{aligned} 4x^2 + 51 = 40[x] &\leq 40x\,, \\ 4x^2 - 40x + 51 &\leq 0\,, \\ (2x-17)(2x-3) &\leq 0\,. \end{aligned}$$

Hence $3/2 \leq x \leq 17/2$. Combining these inequalities gives $3/2 \leq x < 7/2$ or $13/2 < x \leq 17/2$.

Case 1: $3/2 \leq x < 7/2$.

For this case, the possible values for $[x]$ are 1, 2 and 3.

If $[x] = 1$ then $4x^2 + 51 = 40 \cdot 1$ so $4x^2 = -11$, which has no real solutions.

If $[x] = 2$ then $4x^2 + 51 = 40 \cdot 2$ so $4x^2 = 29$ and $x = \frac{\sqrt{29}}{2}$. Notice that $\frac{\sqrt{16}}{2} < \frac{\sqrt{29}}{2} < \frac{\sqrt{36}}{2}$ so $2 < x < 3$ and $[x] = 2$.

If $[x] = 3$ then $4x^2 + 51 = 40 \cdot 3$ and $x = \sqrt{69}/2$. But $\frac{\sqrt{69}}{2} > \frac{\sqrt{64}}{2} = 4$. So, this solution is rejected.

Case 2: $13/2 < x \leq 17/2$.

For this case, the possible values for $[x]$ are 6, 7 and 8.

If $[x] = 6$ then $4x^2 + 51 = 40 \cdot 6$, so that $x = \frac{\sqrt{189}}{2}$. Notice that $\frac{\sqrt{144}}{2} < \frac{\sqrt{189}}{2} < \frac{\sqrt{196}}{2}$, so that $6 < x < 7$ and $[x] = 6$.

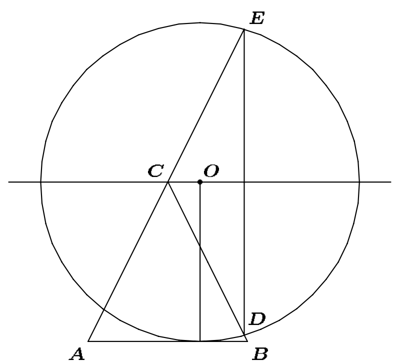
If $[x] = 7$ then $4x^2 + 51 = 40 \cdot 7$, so that $x = \frac{\sqrt{229}}{2}$. Notice that $\frac{\sqrt{196}}{2} < \frac{\sqrt{229}}{2} < \frac{\sqrt{256}}{2}$, so that $7 < x < 8$ and $[x] = 7$.

If $[x] = 8$ then $4x^2 + 51 = 40 \cdot 8$, so that $x = \frac{\sqrt{269}}{2}$. Notice that $\frac{\sqrt{256}}{2} < \frac{\sqrt{269}}{2} < \frac{\sqrt{324}}{2}$, so that $8 < x < 9$ and $[x] = 8$.

The solutions are $x = \dfrac{\sqrt{29}}{2}, \dfrac{\sqrt{189}}{2}, \dfrac{\sqrt{229}}{2}, \dfrac{\sqrt{269}}{2}$.

(*Editor*: Adrian then checks these four solutions.)

2. *Solution 1 — Keon Choi, A.Y. Jackson SS, North York, Ontario.*



Let D and E be the intersections of BC and extended AC, respectively, with the circle.

Since $CO \parallel AB$ (because both the altitude and the radius are 1) $\angle BCO = 60^\circ$ and therefore $\angle ECO = 180^\circ - \angle ACB - \angle BCO = 60^\circ$.

Since a circle is always symmetric about its diameter and line CE is a reflection of line CB in CO, line segment CE is a reflection of line segment CD.

Therefore $CE = CD$.

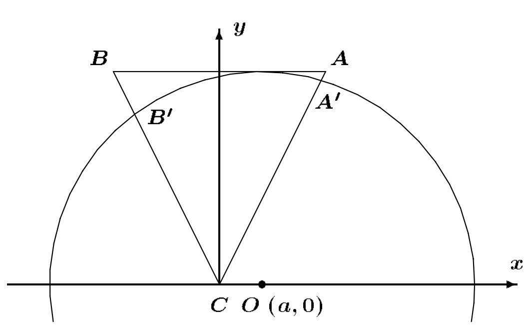
Therefore $\triangle CED$ is an isosceles triangle.

Therefore $\angle CED = \angle CDE$ and $\angle CED + \angle CDE = \angle ACB = 60^\circ$.

$\angle CED = 30^\circ$ regardless of the position of centre O. Since $\angle CED$ is also the angle subtended from the arc inside the triangle, if CED is constant, the arc length is also constant.

Editor's Note: This proof has had no editing.

Solution 2 — Jimmy Chui, Earl Haig SS, North York, Ontario.



Place C at the origin, point A at $\left(\frac{1}{\sqrt{3}}, 1\right)$ and point B at $\left(-\frac{1}{\sqrt{3}}, 1\right)$. Then $\triangle ABC$ is equilateral with altitude of length 1.

Let O be the centre of the circle. Because the circle has radius 1, and since it touches line AB, the locus of O is on the line through C parallel to AB (since C is length 1 away from AB); that is, the locus of O is on the x–axis.

Let point O be at $(a, 0)$. Then $-\frac{1}{\sqrt{3}} \le a \le \frac{1}{\sqrt{3}}$ since we have the restriction that the circle rolls along AB.

Now, let A' and B' be the intersection of the circle with CA and CB, respectively. The equation of CA is $y = \sqrt{3}\, x$, $0 \le x \le \frac{1}{\sqrt{3}}$, of CB is $y = -\sqrt{3}\, x$, $-\frac{1}{\sqrt{3}} \le x \le 0$, and of the circle is $(x-a)^2 + y^2 = 1$.

We solve for A' by substituting $y = \sqrt{3}\, x$ into $(x-a)^2 + y^2 = 1$ to get $x = \dfrac{a \pm \sqrt{4-3a^2}}{4}$.

Visually, we can see that solutions represent the intersection of AC extended and the circle, but we are only concerned with the greater x–value — this is the solution that is on AC, not on AC extended. Therefore

$$x = \frac{a + \sqrt{4-3a^2}}{4}, \qquad y = \sqrt{3}\left(\frac{a + \sqrt{4-3a^2}}{4}\right).$$

Likewise we solve for B', but we take the lesser x–value to get

$$x = \frac{a - \sqrt{4-3a^2}}{4}, \qquad y = -\sqrt{3}\left(\frac{a - \sqrt{4-3a^2}}{4}\right).$$

Let us find the length of $A'B'$:

$$\begin{aligned} |A'B'|^2 &= \left(\frac{a + \sqrt{4-3a^2}}{4} - \frac{a - \sqrt{4-3a^2}}{4}\right)^2 + \\ &\quad \left(\left(\sqrt{3}\,\frac{a + \sqrt{4-3a^2}}{4}\right) - \left(-\sqrt{3}\,\frac{a - \sqrt{4-3a^2}}{4}\right)\right)^2 \\ &= \frac{4-3a^2}{4} + 3\,\frac{a^2}{4} = 1, \end{aligned}$$

which is independent of a.

Consider the points O, A' and B'. $\triangle OA'B'$ is an equilateral triangle (because $A'B' = OA' = OB' = 1$).

Therefore $\angle A'OB' = \frac{\pi}{3}$ and $\text{arc } A'B' = \frac{\pi}{3}$, a constant.

3. *Solution — Masoud Kamgarpour, Carson SS, North Vancouver, BC.*

Note that $n = 1$ is a solution. For $n > 1$ write n in the form $n = P_1^{\alpha_1} P_2^{\alpha_2} \ldots P_m^{\alpha_m}$ where the P_i's, $1 \le i \le m$, are distinct prime numbers and $\alpha_i > 0$. Since $d(n)$ is an integer, n is a perfect square, so $\alpha_i = 2\beta_i$ for integers $\beta_i > 0$.

Using the formula for the number of divisors of n,

$$d(n) \;=\; (2\beta_1+1)(2\beta_2+1)\ldots(2\beta_m+1)\,,$$

which is an odd number. Now because $d(n)$ is odd, $(d(n))^2$ is odd, therefore n is odd as well, so $P_i \geq 3, 1 \leq i \leq m$. We get

$$P_1^{\alpha_1}\cdot P_2^{\alpha_2}\ldots P_m^{\alpha_m} \;=\; [(\alpha_1+1)(\alpha_2+1)\ldots(\alpha_m+1)]^2$$

or using $\alpha_i = 2\beta_i$

$$P_1^{\beta_1}P_2^{\beta_2}\ldots P_m^{\beta_m} \;=\; (2\beta_1+1)(2\beta_2+1)\ldots(2\beta_m+1)\,.$$

Now we prove a lemma:

Lemma: $P^t \geq 2t+1$ for positive integers t and $P \geq 3$, with equality only when $P = 3$ and $t = 1$.

Proof: We use mathematical induction on t. The statement is true for $t = 1$ because $P \geq 3$. Now suppose $P^k \geq 2k+1$, $k \geq 1$; then we have

$$P^{k+1} \;=\; P^k\cdot P \;\geq\; P^k(1+2) \;>\; P^k+2 \;\geq\; (2k+1)+2 \;=\; 2(k+1)+1\,.$$

Thus $P^t \geq 2t+1$ and equality occurs only when $P = 3$ and $t = 1$.

Let us say n has a prime factor $P_k > 3$; then (by the lemma) $P_k^{\beta_k} > 2\beta_k+1$ and we have $P_1^{\beta_1}\ldots P_m^{\beta_m} > (2\beta_1+1)\ldots(2\beta_m+1)$, a contradiction.

Therefore, the only prime factor of n is $P = 3$ and we have $3^\gamma = 2\gamma+1$. By the lemma $\gamma = 1$.

The only positive integer solutions are 1 and 9.

4. *Solution 1 — David Nicholson, Fenelon Falls SS, Fenelon Falls, Ontario.*

Without loss of generality let $a_1 < a_2 < a_3 \ldots < a_8$.

Assume that there is no such integer k. Let us just look at the seven differences $d_i = a_{i+1} - a_i$. Then amongst the d_i there can be at most two 1s, two 2s, and two 3s, which totals to 12.

Now $16 = 17 - 1 \geq a_8 - a_1 = d_1 + d_2 + \ldots + d_7$ so the seven differences must be $1, 1, 2, 2, 3, 3, 4$.

Now let us think of arranging the differences $1, 1, 2, 2, 3, 3, 4$. Note that the sum of consecutive differences is another difference. (For example, $d_1 + d_2 = a_3 - a_1$, $d_1 + d_2 + d_3 = a_4 - a_1$)

We cannot place the two 1s side by side because that will give us another difference of 2. The 1s cannot be beside a 2 because then we have three 3s. They cannot both be beside a 3 because then we have three 4s! So we must have either $1, 4, -, -, -, 3, 1$ or $1, 4, 1, 3, -, -, -$ (or their reflections).

In either case we have a 3, 1 giving a difference of 4 so we cannot put the 2s beside each other. Also we cannot have 2, 3, 2 because then (with the 1, 4) we will have three 5s. So all cases give a contradiction.

Therefore there will always be three differences equal.

One set of seven numbers satisfying the criteria is $\{1, 2, 4, 7, 11, 16, 17\}$. [*Editor*: There are many such sets.]

Solution 2 — The CMO committee.

Consider all the consecutive differences (that is, d_i above) as well as the differences $b_i = a_{i+2} - a_i$, $i = 1, \dots, 6$. Then the sum of these thirteen differences is $2 \cdot (a_8 - a_1) + (a_7 - a_2) \leq 2(17-1) + (16-2) = 46$. Now if no difference occurs more than twice, the smallest the sum of the thirteen differences can be is $2 \cdot (1+2+3+4+5+6) + 7 = 49$, giving a contradiction.

5. *Solution 1 — The CMO committee.*

Let $f(x,y,z) = x^2y + y^2z + z^2x$. We wish to determine where f is maximal. Since f is cyclic, without loss of generality we may assume that $x \geq y, z$. Since

$$\begin{aligned} f(x,y,z) - f(x,z,y) &= x^2y + y^2z + z^2x - x^2z - z^2y - y^2x \\ &= (y-z)(x-y)(x-z)\,, \end{aligned}$$

we may also assume $y \geq z$. Then

$$\begin{aligned} f(x+z,y,0) - f(x,y,z) &= (x+z)^2y - x^2y - y^2z - z^2x \\ &= z^2y + yz(x-y) + xz(y-z) \geq 0\,, \end{aligned}$$

so we may now assume $z = 0$. The rest follows from the arithmetic-geometric mean inequality:

$$f(x,y,0) = \frac{2x^2y}{2} \leq \frac{1}{2}\left(\frac{x+x+2y}{3}\right)^3 = \frac{4}{27}\,.$$

Equality occurs when $x = 2y$, hence at $(x,y,z) = (\frac{2}{3}, \frac{1}{3}, 0)$. (As well as $(0, \frac{2}{3}, \frac{1}{3})$ and $(\frac{1}{3}, 0, \frac{2}{3})$).

Solution 2 — The CMO committee.

With f as above, and $x \geq y, z$

$$f\left(x + \frac{z}{2}, y + \frac{z}{2}, 0\right) - f(x,y,z) = yz(x-y) + \frac{xz}{2}(x-z) + \frac{z^2y}{4} + \frac{z^3}{8}\,,$$

so we may assume that $z = 0$. The rest follows as for solution 1.

That completes this number of the *Corner*. Send me your nice solutions as well as Olympiad materials for use in future issues.

BOOK REVIEWS

ALAN LAW

Calculus, The Dynamics of Change edited by A. Wayne Roberts,
published by The Mathematical Association of America, 1996.
ISBN # 0-88385-098-2, softcover, 166+ pages, $34.95 (U.S.).
Reviewed by **Jack W. Macki**, *University of Alberta, Edmonton, Alberta.*

This volume is number 39 in the outstanding Mathematical Association of America series on mathematics education, with the emphasis on calculus and calculus reform. There are four main sections: I. Visions; II. Planning; III. Assessment; IV. Connections, each of which has a number of articles by various authors. In addition, there is a prefatory section on how to think about and plan a modern calculus course and three final sections on, respectively, resources, calculus on the Internet, and a historical and philosophical section "Calculus for a New Century". A total of 17 different authors were involved in the various articles.

The prefatory section is a good "how to" guide to effective organization of a course in calculus—every beginning instructor (and all old fogeys) should read it. There are ideas you may not accept, but overall any instructor will get a refreshingly succinct and useful guide to preparing an interesting and useful course.

There are six articles in the Visions section. Sharon Cutler Ross in "Visions of Calculus" gives a historical discussion of the development of reform, with a very balanced discussion of the most important issues. This is a good way to bring yourself to an understanding of what are the issues, what has worked, and which ideas are still in question. A short article by Thomas Tucker shows, with one simple example, how one can solve calculus problems using numerical or verbal techniques. Mai Gehrke and David Pengelley report on their programs at New Mexico State University, and give practical advice for conducting reform courses, and for getting colleagues on side. Deborah Hughes Hallett contributes a very personal essay on her experiences with calculus teaching and calculus reform. David A. Smith contributes an essay which emphasizes 10 active verbs which characterize his ideas about teaching.

There are four articles in the Planning section. Martin Flashman's article reports on the full history of the introduction of reform at six institutions. Morton Brown reports on the Michigan program. The last two articles are in fact an outline for change followed by a checklist.

Part III, Assessment, has an introductory article by David Bressoud, followed by a large set of final examinations for each of Calculus I, II and III. Each examination had at least one pleasant surprise for me. Bressoud's article has lots of good thoughts, and begins with an apocryphal quote of

Richard Feynman, *The biggest problem with being a student is that you're always too busy getting an education to learn anything*, and the rest of the article is just as interesting in its insights. Assessment is probably the weakest area of most mathematics courses, and this section will help most of us learn to think more deeply about the problem.

Part IV, Connections, is concerned with what in many ways is the most overlooked area in Mathematics departments. The degree to which we think of our individual courses in isolation, rather than as an integrated whole, is almost criminal. I am not thinking of the "analysis sequence" or the "algebra sequence", but of how all of our courses fit (or rather, do not) into some kind of integrated system. John Dossey's article on secondary school mathematics reform is not terribly relevant to Canada—this nation is quite far ahead of the U.S. in designing effective high school mathematics programs—whether students take them and are attracted to the subject by them is another matter! Robert Borrelli and Courtney Coleman of Harvey Mudd report on their experiments with modifying the introductory differential equations course. The article gives four examples to emphasize how modern ideas can be brought into a first course. David Carlson and Wayne Roberts give a very brief report on their experiences with post-calculus linear algebra and analysis. Sheldon Gordon recorded a round-table discussion between mathematicians, electrical engineers, ecologists, physicists, biologists, chemists and chemical engineers.

Martin Flashman's article on the Internet gives a couple of key, central, sites. Since this book appeared in 1996, there are by now hordes of other first class sites.

Recent reports from colleagues indicate that at many schools the mathematics department is more or less completely isolated from its clients, in particular from the faculty of engineering and the department of physics. At many schools these faculties and departments keep their students away from the mathematics department as much as possible. The reports indicate an arrogance and lack of respect for other disciplines which is truly amazing. These mathematics departments are recognized as leaders in our profession, yet they have no research in teaching, learning, or assessment, nor would they be valued if they did. This book and others like it show that in our profession there is a large group of committed professionals trying to get us all to think about the teaching and learning side of our work. Thank goodness. And thank the MAA.

THE SKOLIAD CORNER

No. 41

R.E. Woodrow

This issue we give the Final Round Parts A and B of the 1998 British Columbia Colleges Senior High School Mathematics Contest. My thanks go to Jim Totten, University College of the Cariboo, one of the organizers, for forwarding the materials for use in the *Corner*.

BRITISH COLUMBIA COLLEGES SENIOR HIGH SCHOOL MATHEMATICS CONTEST

Final Round 1998

Part A

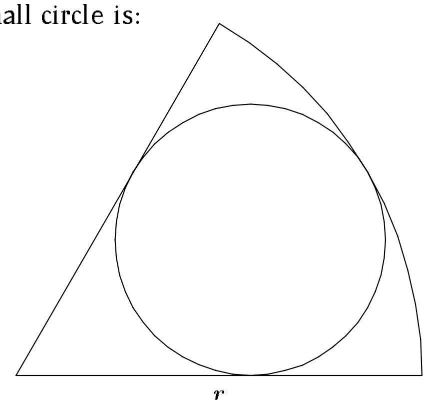
1. If $\left(r+\frac{1}{r}\right)^2=3$, then $r^3+\frac{1}{r^3}=$

(a) 0 **(b)** 1 **(c)** 2 **(d)** $2\sqrt{3}$ **(e)** $4\sqrt{3}$

2. Kevin has five pairs of socks in his drawer, all of different colours and patterns and, being a typical teenage boy, they are not folded and have been thoroughly mixed up. On the first day of school Kevin reaches into his sock drawer without looking and pulls out three socks. What is the probability that two of the socks match?

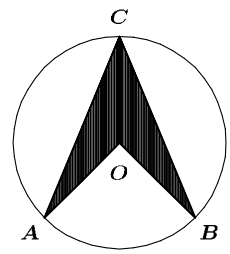
(a) $\frac{3}{10}$ **(b)** $\frac{3}{5}$ **(c)** $\frac{1}{3}$ **(d)** $\frac{1}{24}$ **(e)** $\frac{1}{15}$

3. A small circle is drawn within a $\frac{1}{6}$ sector of a circle of radius r, as shown. The small circle is tangent to the two radii and the arc of the sector. The radius of the small circle is:



(a) $\frac{r}{2}$ **(b)** $\frac{r}{3}$ **(c)** $\frac{2\sqrt{3}\,r}{3}$ **(d)** $\frac{\sqrt{2}\,r}{2}$ **(e)** none of these

4. In the accompanying diagram, the circle has radius one, the central angle AOB is a right angle and AC and BC are of equal length. The shaded area is:



(a) $\frac{\pi}{2}$ (b) $\frac{\sqrt{2}}{2}$ (c) $\frac{\pi-\sqrt{2}}{2}$ (d) $\frac{\sqrt{2}+1}{2}$ (e) $\frac{1}{2}$

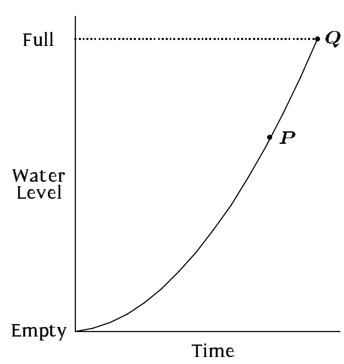
5. The side, front and bottom faces of a rectangular solid have areas $2x$, $\frac{y}{2}$, and xy square centimetres, respectively. The volume of the solid is:

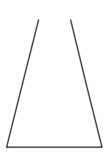
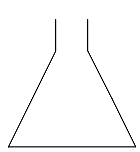
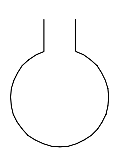
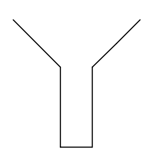
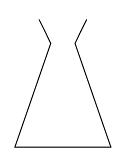
(a) xy (b) $2xy$ (c) x^2y^2 (d) $4xy$ (e) impossible to determine from the given information

6. The numbers from 1 to 25 are each written on separate slips of paper which are placed in a pile. You draw slips from the pile without replacing any slip you have chosen. You can continue drawing until the *product* of two numbers on any pair of slips you have chosen is a perfect square. The maximum number of slips you can choose before you will be forced to quit is:

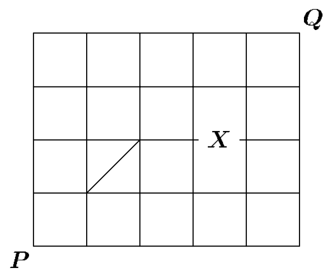
(a) 13 (b) 14 (c) 15 (d) 16 (e) 17

7. A container is completely filled from a tap running at a constant rate. The accompanying graph shows the level of the water in the container at any time while the container is being filled. The segment PQ is a straight line. The shape of the container which corresponds with the graph is:



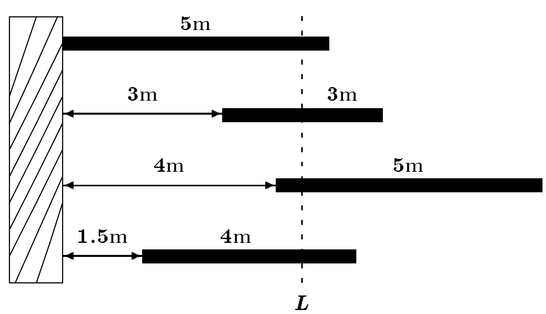
(a)  (b)  (c)  (d)  (e) 

8. The accompanying diagram is a road plan of a small city. All the roads go east-west or north-south, with the exception of the one short diagonal road shown. Due to repairs one road is impassable at the point X. Of all the possible routes from P to Q, there are several shortest routes. The total number of shortest routes is:



(a) 4 (b) 7 (c) 9 (d) 14 (e) 16

9. Four pieces of timber with the lengths shown are placed in the parallel positions shown. A single cut is made along the line L perpendicular to the lengths of timber so that the total length of timber on each side of L is the same. The length, in metres, of the longest piece of timber remaining is:



(a) 4.85 (b) 4.50 (c) 4.75 (d) 3.75 (e) none of the above

10. The positive integers are written in order with one appearing once, two appearing twice, three appearing three times, ..., ten appearing ten times, and so on, so that the beginning of the sequence looks like this:

$$1, \ 2, \ 2, \ 3, \ 3, \ 3, \ 4, \ 4, \ 4, \ 4$$

The number of 9's appearing in the first 1998 *digits* of the sequence is:

(a) 57 (b) 96 (c) 113 (d) 145 (e) 204

Part B

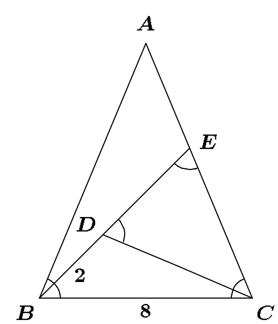
1. A right triangle has an area of 5 and its hypotenuse has length 5. Determine the lengths of the other two sides.

2. Find a set of three consecutive positive integers such that the smallest is a multiple of 5, the second is a multiple of 7 and the largest is a multiple of 9.

3. In the diagram, $BD = 2$, $BC = 8$ and the marked angles are all equal; that is,

$$\angle ABC = \angle BCA = \angle CDE = \angle DEC .$$

Find AB.



4. The ratio of male to female voters in an election was $a : b$. If c fewer men and d fewer women had voted, then the ratio would have been $e : f$. Determine the total number of voters who cast ballots in the election in terms of a, b, c, d, e and f.

5. Three neighbours named Penny, Quincy and Rosa took part in a local recycling drive. Each spent a Saturday afternoon collecting all the aluminum cans and glass bottles he or she could. At the end of the afternoon each person counted up what he or she had gathered, and they discovered that even though Penny had collected three times as many cans as Quincy, and Quincy had collected four times as many bottles as Rosa, each had collected exactly the same number of items, and the three as a group had collected exactly as many cans as bottles. In total, the three collected fewer than 200 items in all. Assuming that each person collected at least one can and one bottle, how many cans and bottles did each person collect?

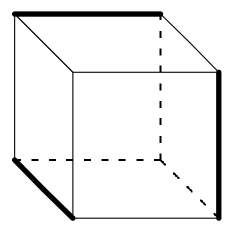
Last issue we gave the Final Round Parts A and B of the 1998 British Columbia Colleges Junior High School Mathematics Contest. My thanks go to Jim Totten, University College of the Cariboo, one of the organizers, for forwarding the "official solutions" which follow.

BRITISH COLUMBIA COLLEGES
JUNIOR HIGH SCHOOL MATHEMATICS CONTEST
Final Round 1998
Part A

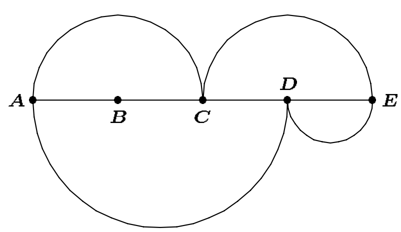
1. Each edge of a cube is coloured either red or black. If every face of the cube has at least one black edge, the smallest possible number of black edges is:

(a) 6 (b) 5 (c) 4 (d) 3 (e) 2

Answer: The correct answer is **(d)**.
Suppose that every face of a cube has at least one black edge. Since every edge belongs to exactly two faces, and there are six faces, the cube has at least three black edges. On the other hand, three black edges suffice to satisfy the requirement, as we can see on the diagram. The black edges are represented by the thicker lines.



2. Line AE is divided into four equal parts by the points B, C and D. Semicircles are drawn on segments AC, CE, AD and DE creating semicircular regions as shown. The ratio of the area enclosed above the line AE to the area enclosed below the line is:

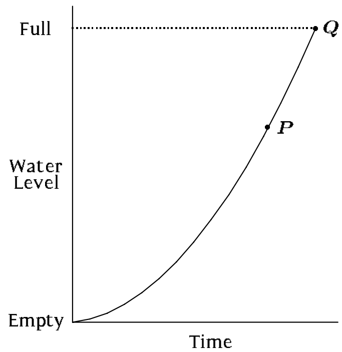


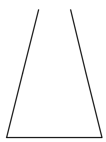
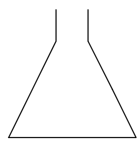
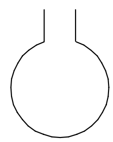
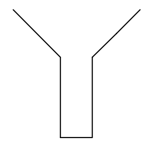
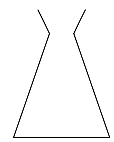
(a) $4 : 5$ (b) $5 : 4$ (c) $1 : 1$ (d) $8 : 9$ (e) $9 : 8$

Answer: The correct answer is **(a)**.

Suppose that AB has length 1. Then both semicircles lying above AE have radii of 1, while the semicircles below AE have radii of $1\frac{1}{2}$ and $\frac{1}{2}$. The ratio of the enclosed areas is $[\frac{1}{2}\pi(1)^2 + \frac{1}{2}\pi(1)^2] \div [\frac{1}{2}\pi(1\frac{1}{2})^2 + \frac{1}{2}\pi(\frac{1}{2})^2] = \frac{4}{5}$.

3. A container is completely filled from a tap running at a uniform rate. The accompanying graph shows the level of the water in the container at any time while the container is being filled. The segment PQ is a straight line. The shape of the container which corresponds with the graph is:



(a)  (b)  (c)  (d)  (e) 

Readers may have noticed that this is the same as problem 7 in part A of the British Columbia Colleges **Senior** High School Mathematics Contest Final Round 1998, printed above. For this reason, we delay publishing this official solution until the next issue.

4. The digits 1, 9, 9, and 8 are placed on four cards. Two of the cards are selected at random. The probability that the sum of the numbers on the cards selected is a multiple of 3 is:

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$ (e) $\frac{3}{4}$

Answer: The correct answer is **(b)**.

Let a, b, c, d denote the cards with digits 1, 9, 9, and 8, respectively. There are six possible choices of two cards from the set of four: $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$. For exactly two of these, $\{b, c\}$ and $\{a, d\}$, the corresponding sums, $9 + 9$ and $1 + 8$, are divisible by 3. This gives the probability of $\frac{2}{6} = \frac{1}{3}$.

5. The surface areas of the six faces of a rectangular solid are 4, 4, 8, 8, 18 and 18 square centimetres. The volume of the solid, in cubic centimetres is:

(a) 24 (b) 48 (c) 60 (d) 324 (e) 576

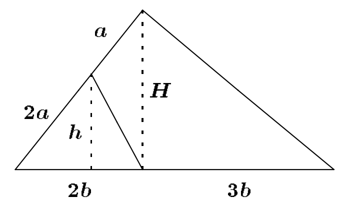
Answer: The correct answer is **(a)**.

If the edges of a rectangular solid have lengths a, b, and c, then the areas of its nonparallel faces are ab, bc, and ac. Its volume $abc = \sqrt{(ab)(bc)(ac)}$ .In our case $abc = \sqrt{4 \cdot 8 \cdot 18} = 24$.

6. The area of the small triangle in the diagram is 8 square units. The area of the large triangle, in square units, is:

(a) 18 (b) 20 (c) 24 (d) 28 (e) 30

Answer: The correct answer is (e). The length of the base of the larger triangle is $5b$, while the length of the base of the smaller triangle is $2b$. This gives the ratio of $\frac{5}{2}$. Similarly, the ratio of the corresponding perpendicular heights, $H : h$, is $3a : 2a = \frac{3}{2}$. Hence, the area of the larger triangle is $\frac{5}{2}(\frac{3}{2})(8) = 30$.



7. At 6:15 the hands of the clock form two positive angles with a sum of $360°$. The difference of the degree measures of these two angles is:

(a) 165 (b) 170 (c) 175 (d) 180 (e) 185

Answer: The correct answer is (a).

At 6:15 the minute hand points at 3, while the hour hand is $\frac{1}{4}$ of the way from 6 to 7. The smaller angle between the hands is $[90 + \frac{1}{4}(\frac{360}{12})]° = 97.5°$, while the larger is $(360 - 97.5)° = 262.5°$. This gives the difference of $(262.5 - 97.5)° = 165°$.

8. The last digit of the number 8^{26} is:

(a) 0 (b) 2 (c) 4 (d) 6 (e) 8

Answer: The correct answer is (c).

By inspecting the last digit of the numbers in the sequence 8^1, 8^2, 8^3, 8^4, ..., we discover a repeating pattern of length four: 8, 4, 2, 6. Since $8^{26} = 8^{4(6)+2}$, we conclude that the last digit of 8^{26} is the same as the last digit of 8^2, that is 4.

9. For the equation $\frac{A}{x+3} + \frac{B}{x-3} = \frac{-x+9}{x^2-9}$ to be true for all values of x for which the expressions in the equation make sense, the value of AB is:

(a) 2 (b) -1 (c) -2 (d) -3 (e) -6

Answer: The correct answer is (c).

The expression makes sense for all values of x, except ± 3. By multiplying both sides of the equation by the common denominator $x^2 - 9 = (x-3)(x+3)$, we get $A(x-3) + B(x+3) = -x + 9$. After multiplying out and collecting the like terms on the left hand side of this equation we get $(A+B)x + 3B - 3A = -x + 9$. Clearly, the polynomials on both sides must be identical; therefore $A + B = -1$ and $3B - 3A = 9$. This system of two equations can be solved in any standard way. For example, we can find $B = 3 + A$ from the second equation and substitute this for B in the first equation. In that way we find $A = -2$ and $B = 1$.

10. A hungry hunter came upon two shepherds, Joe and Frank. Joe had three small loaves of bread and Frank five loaves of the same size. The loaves were divided equally among the three people, and the hunter paid $8 for his share. If the shepherds divide the money so that each gets an equitable share based on the amount of bread given to the hunter, the amount of money that Joe receives is:

(a) $1 (b) $1.50 (c) $2 (d) $2.50 (e) $3

Answer: The correct answer is (a).

Divide each loaf into 3 parts and distribute equally to each of the three persons. Each person receives 8 parts. The two shepherds start with 9 and 15 parts each, so (after removing their own 8 parts) they contribute 1 and 7 parts, respectively, to the hunter and should receive compensation from the hunter in that ratio. Thus the hunter who originally had 3 loaves should receive $1.

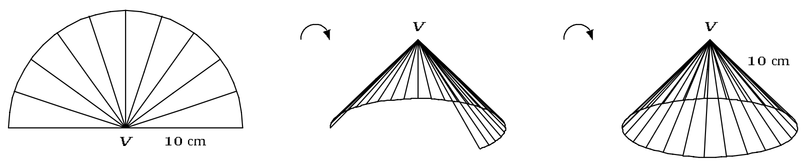
Part B

1. Four positive integers sum to 125. If the first of these numbers is increased by 4, the second is decreased by 4, the third is multiplied by 4 and the fourth is divided by 4, you produce four equal numbers. What are the four original numbers?

Solution. The numbers are 16, 24, 5 and 80.

If x, y, z, and w are the numbers then $x + y + z + w = 125$ and $x + 4 = y - 4 = 4z = \frac{w}{4}$. Hence, $y = x + 8$, $z = \frac{x+4}{4}$, $w = 4(x + 4)$. By substituting these expressions to the first equation, we get $x + (x + 8) + \frac{x+4}{4} + 4(x + 4) = 125$. Thus, $x = 16$, and consequently, $y = 24$, $z = 5$, $w = 80$.

2. A semi-circular piece of paper of radius 10 cm is formed into a conical paper cup as shown (the cup is inverted in the diagram):



Find the height of the paper cup, that is, the depth of water in the cup when it is full.

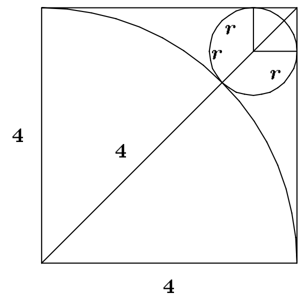
Solution. The height of the paper cup is $5\sqrt{3}$ cm.

The base of the conical paper cup is a circle with circumference equal to the length of the given semicircle. Thus, if r is the radius of the base then $2\pi r = \frac{1}{2}(2\pi 10)$. Hence, $r = 5$ cm. The side length of the cone s is the same as the radius of the semicircle; thus $s = 10$ cm. Finally, the height of the cone is

$$h = \sqrt{s^2 - r^2} = \sqrt{10^2 - 5^2} = 5\sqrt{3}\text{ cm}.$$

3. In the diagram a quarter circle is inscribed in a square with side length 4, as shown. Find the radius of the small circle that is tangent to the quarter circle and two sides of the square.

Solution. The radius of the small circle is $12 - 8\sqrt{2}$.



The Pythagorean Theorem implies that the diagonal of a square with side a has length $a\sqrt{2}$. Thus, the diagonal of the larger square has length $4\sqrt{2}$. It is equal to the sum of the radius of the larger circle, 4, the radius of the smaller circle, r, and the diagonal of the smaller square, $r\sqrt{2}$. Hence, $4\sqrt{2} = 4 + r + r\sqrt{2}$. This gives

$$r = \frac{4\sqrt{2} - 4}{1 + \sqrt{2}} = \frac{4\sqrt{2} - 4}{1 + \sqrt{2}} \left(\frac{1 - \sqrt{2}}{1 - \sqrt{2}} \right) = 12 - 8\sqrt{2}\,.$$

4. Using the digits 1, 9, 9 and 8 *in that order* create expressions equal to $1, 2, 3, 4, 5, 6, 7, 8, 9$, and 10. You may use any of the four basic operations ($+$, $-$, $\times$, $\div$), the square root symbol ($\sqrt{\ }$) and parentheses, as necessary. For example, valid expressions for 25 and 36 would be

$$\begin{aligned} 25 &= -1 + 9 + 9 + 8\,, \\ 36 &= 1 + 9 \times \sqrt{9} + 8\,. \end{aligned}$$

Note: You may place a negative sign in front of 1 to create -1 if you wish.

Solution. One of several possible solutions is:

$$\begin{aligned} 1 &= -1 + \sqrt{9} - 9 + 8\,, \\ 2 &= 1 \times \sqrt{9} - 9 + 8\,, \\ 3 &= -1 + \sqrt{9} + 9 - 8\,, \\ 4 &= 1 \times \sqrt{9} + 9 - 8\,, \\ 5 &= 1 + \sqrt{9} + 9 - 8\,, \\ 6 &= -1 - 9 \div 9 + 8\,, \\ 7 &= -1 + 9 - 9 + 8\,, \\ 8 &= -1 + 9 \div 9 + 8\,, \\ 9 &= -1 + 9 + 9 - 8\,, \\ 10 &= 1 + 9 \div 9 + 8\,. \end{aligned}$$

5. At 6 am one Saturday, you and a friend begin a recreational climb of Mt. Mystic. Two hours into your climb, you are overtaken by some scouts. As they pass, they inform you that they are attempting to set a record for ascending and descending the mountain. At 10 am they pass you again on their way down, crowing that they had not stopped once to rest, not even at the top.

You finally reach the summit at noon. Assuming that both you and the scouts travelled at a constant vertical rate, both climbing and descending, when did the scouts reach the top of Mt. Mystic?

Solution. The scouts reached the top of Mt. Mystic at 9:20 am.

Suppose that during the time period from 8:00 am to 10:00 am you have travelled from point A to point B and you climbed a distance of x kilometres. Then, since you have been climbing at a uniform rate and reached the top at noon, the distance from B to the top is also x kilometres. During the two hours you climbed x kilometres from A to B, the scouts climbed the distance of $3x$ kilometres: x from A to B, x from B to the top, and x on the way back to B from the top. Since their pace was uniform, they needed $\frac{2}{3}$ of an hour, that is 40 minutes, to get from the top to point B, where they met you at 10:00 am. This implies that they must have reached the top at 9:20 am.

That completes the *Corner* for this number. Send me contest materials and suggestions for the evolution of the *Skoliad Corner*.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to Mathematical Mayhem, Department of Mathematics, University of Toronto, 100 St. George St., Toronto, Ontario, Canada. M5S 3G3. The electronic address is still

mayhem@math.toronto.edu

Shreds and Slices

A Combinatorial Proof of a Trigonometric Identity

Douglass Grant

A friend who attended university in Germany once told me that his course in first year calculus was made memorable by the fact that the basic definition of the sine and cosine functions used by his professor were the Maclaurin series for the two functions. At the time, I made the flippant remark that such an approach would make it a challenge even to prove that $\sin^2 x + \cos^2 x = 1$. The details of that proof, in fact, involve some identities more commonly encountered in discrete mathematics or combinatorics than in calculus.

Since

$$\sin x \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1}\,,$$

squaring the series and shifting the index by one on the inner series yields

$$\begin{aligned}\sin^2 x &= \sum_{k=0}^{\infty}\left(\sum_{\substack{n+m=k\\ n,m\geq 0}} \frac{(-1)^n(-1)^m}{(2n+1)!(2m+1)!}\right) x^{2k+2}\\ &= \sum_{k=1}^{\infty}\left(\sum_{\substack{n+m=k-1\\ n,m\geq 0}} \frac{(-1)^{k-1}}{(2n+1)!(2m+1)!}\right) x^{2k}\\ &= \sum_{k=1}^{\infty}\left(\sum_{n=0}^{k-1} \frac{(-1)^{k-1}}{(2n+1)!(2k-2n-1)!}\right) x^{2k}\,.\end{aligned}$$

Similarly,

$$\cos^2 x = \sum_{k=0}^{\infty}\left(\sum_{n=0}^{k} \frac{(-1)^k}{(2n)!(2k-2n)!}\right) x^{2k}\,.$$

Since the constant term in the series for $\cos^2 x$ is clearly unity, it suffices to show that the sum of the coefficients of x^{2k} for $\sin^2 x$ and $\cos^2 x$ is zero for $k \geq 1$. Note that the integers whose factorials appear in the denominators of both inner summations sum to $2k$.

For $k \geq 1$, let

$$S_k = \sum_{n=0}^{k-1} \frac{(-1)^{k-1}}{(2n+1)!(2k-2n-1)!} + \sum_{n=0}^{k} \frac{(-1)^k}{(2n)!(2k-2n)!}\,.$$

Then

$$(2k)!S_k = \sum_{n=0}^{2k} \frac{(-1)^n(2k)!}{(2n)!(2k-2n)!} = \sum_{n=0}^{2k} (-1)^n \binom{2k}{n}\,.$$

But by the Binomial Theorem,

$$(a-b)^{2k} = \sum_{n=0}^{2k} (-1)^n \binom{2k}{n} a^{2k-n} b^n\,,$$

so letting $a = b = 1$, we obtain

$$0 = \sum_{n=0}^{2k} (-1)^n \binom{2k}{n} = (2k)!S_k\,,$$

whence $S_k = 0$ for $k \geq 1$, as required.

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Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from this issue be submitted in time for issue 8 of 2000.

High School Problems

Editor: Adrian Chan, 229 Old Yonge Street, Toronto, Ontario, Canada. M2P 1R5 `<all238@sprint.com>`

We correct problem H253 again, which we tried to correct in Issue 5.

H253. Find all real solutions to the equation

$$\sqrt{3x^2-18x+52}+\sqrt{2x^2-12x+162} = \sqrt{-x^2+6x+280}\,.$$

H261. Solve for x:

$$\left(\sqrt{7-\sqrt{48}}\right)^x+\left(\sqrt{7+\sqrt{48}}\right)^x = 14\,.$$

H262. *Proposed by Mohammed Aassila, CRM, Montréal, Québec.*
Solve the equation

$$x-\frac{x}{\sqrt{x^2-1}} = \frac{91}{60}\,.$$

H263. Let ABC be an acute-angled triangle such that $a = 14$, $\sin B = 12/13$, and c, a, b form an arithmetic sequence (in that order). Find $\tan A+\tan B+\tan C$.

H264. Find all values of a such that $x^3-6x^2+11x+a-6=0$ has exactly three integer solutions.

Advanced Problems

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A237. Show that for any sequence of decimal digits that does not begin with 0, there is a Fibonacci number whose decimal representation begins with this sequence. (The Fibonacci sequence is the sequence F_n generated by the initial conditions $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.)

A238. Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at P and Q. A line through P intersects $\mathcal{C}_1$ and $\mathcal{C}_2$ again at A and B, respectively, and X is the mid-point of AB. The line through Q and X intersects $\mathcal{C}_1$ and $\mathcal{C}_2$ again at Y and Z, respectively. Prove that X is the mid-point of YZ.

(1997 Baltic Way)

A239. *Proposed by Mohammed Aassila, CRM, Montréal, Québec.*

Let a_1, a_2, ... , a_n be n distinct numbers, $n \geq 3$. Prove that

$$\sum_{i=1}^{n} \left(a_i \cdot \prod_{j \neq i} \frac{1}{a_i - a_j} \right) \;=\; 0\,.$$

A240. *Proposed by Mohammed Aassila, CRM, Montréal, Québec.*

Let a, b, and c be integers, not all equal to 0. Show that

$$\frac{1}{4a^2 + 3b^2 + 2c^2} \;\leq\; \left| \sqrt[3]{4}a + \sqrt[3]{2}b + c \right|\,.$$

Challenge Board Problems

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C89. *Proposed by Tal Kubo, Brown University.*

Show that the formal power series (in x and y) $\sum_{n=0}^{\infty} (xy)^n$ cannot be expressed as a finite sum $\sum_{i=1}^{m} f_i(x) g_i(y)$, where $f_i(x)$ and $g_i(y)$ are formal power series in x and y, respectively, $1 \leq i \leq m$.

C90. *Proposed by Noam Elkies, Harvard University.*

Let S_1, S_2, and S_3 be three spheres in $\mathbb{R}^3$ whose centres are not collinear. Let $k \le 8$ be the number of planes which are tangent to all three spheres. Let A_i, B_i, and C_i be the point of tangency between the i^{th} such tangent plane, $1 \le i \le k$, and S_1, S_2, and S_3, respectively, and let O_i be the circumcentre of triangle $A_iB_iC_i$. Prove that all the O_i are collinear. (If $k = 0$, then this statement is vacuously true.)

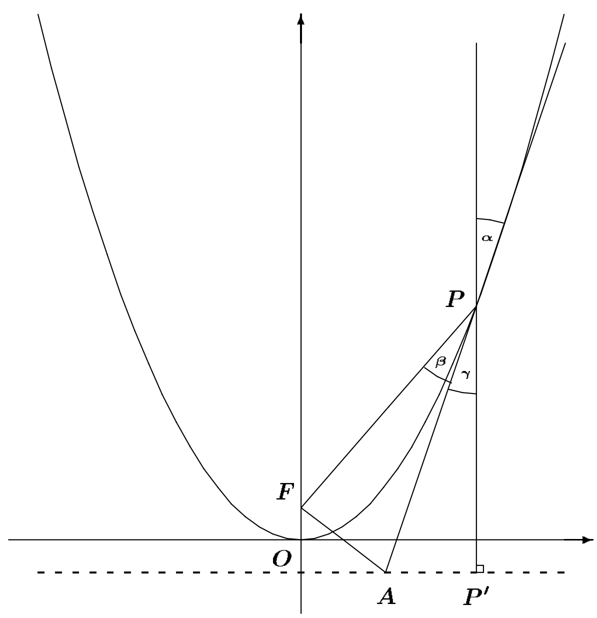
Problem of the Month

Jimmy Chui, student, University of Toronto

Problem. The point $P(a, b)$ is on the parabola $x^2 = 4y$. The tangent at P meets the line $y = -1$ at the point A. For the point $F(0, 1)$, prove that $\angle AFP = 90°$ for all positions of P, except $(0, 0)$.

(Descartes 1998, C3)

Solution. This question can be done using calculus; however, here we show a clever method that makes the problem fall apart quite easily.



First we note that the point F is the focus and the line $y = -1$ is the directrix of the given parabola.

Drop the perpendicular from P to the line $y = -1$, and call that point P'. Now, let α, β, and γ be the angles as in the diagram; that is, let $\beta = \angle APF$, let $\gamma = APP'$, and let α represent the angle opposite $\angle APP'$.

Observe that $PF = PP'$, because any point on a parabola is equidistant to both the focus and directrix.

From the opposite angle theorem we get $\alpha = \gamma$.

Now, it is known that a ray, parallel to the axis of symmetry in a parabola, will pass through the focus when it reflects off the interior of the parabola. The line $P'P$ extended is parallel to the line $x = 0$ (the axis of symmetry) and can be taken as the ray of incidence. The ray PF is then the ray of reflection. The line AP extended is the tangent at the point P of the parabola, and the angle of incidence equals the angle of reflection, or $\alpha = \beta$. So we have $\beta = \gamma$.

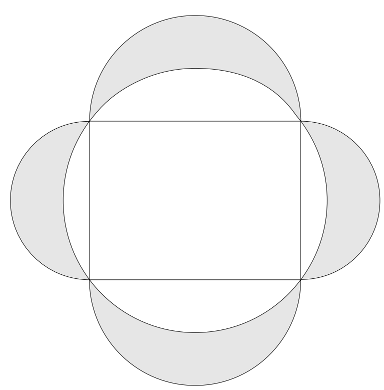
Hence, from SAS congruency, we have that $\triangle APF \equiv \triangle APP'$. This implies that $\angle AFP = \angle AP'P = 90°$, QED.

J.I.R. McKnight Problems Contest 1991

1. (a) The vertices of a right-angled triangle ABC are $A(4, 6)$, $B(3, -3)$ and $C(8, y)$. Find all possible values of y.

   (b) Simplify:
$$\frac{5^{3x+1} - 5^{3x-1} + 24}{(2.4)5^{3x} + 12}.$$

2. (a) A rectangle has been inscribed in a circle and semi-circles have been drawn on its sides (as shown in the diagram). Determine the ratio of the sum of the area of the four lunes (shaded regions) to the area of the rectangle.



(b) Find the value of $(2^{2^0}+1)(2^{2^1}+1)(2^{2^2}+1)\cdots(2^{2^{50}}+1)$.

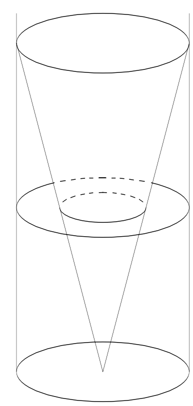
3. (a) Find the sum of the first one thousand terms for the series: $1\cdot 4+2\cdot 7+3\cdot 10+\cdots$.

   (b) Solve for x, $y \in \mathbb{R}$:

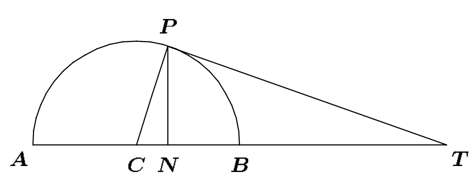
$$\begin{aligned} x+y+\sqrt{xy} &= 19 \\ x^2+y^2+xy &= 133 \end{aligned}$$

4. The two equations $a^2x^2+192bx+1991a=0$ and $bx^2+192a^2x+1991b=0$ have a common root. If $a^2\neq b$, determine all possible values of the common root.

5. A metal cone of height 12 cm and radius 4 cm just fits into a cylinder of radius 4 cm and height 100 cm which is filled with water to a depth of 50 cm. The cone is lowered at a constant rate of 4 cm per second (with respect to the walls of the cylinder). At what rate is the water in the cylinder rising when the vertex of the cone is immersed to a depth of 6 cm?



6. A semi-circle has centre C and diameter AB. The point N is on CB and AB is produced to T so that $AT:AC=AN:CN$. The tangent from T meets the semi-circle at P. Prove that $\angle CNP=90^\circ$.



7. (a) Prove that the product of 4 consecutive positive integers cannot be a perfect square.

   (b) What must be added to the product of 4 consecutive terms of any arithmetic sequence to produce a perfect square?

   (c) What must be added to the product of 4 consecutive terms of any geometric sequence to produce a perfect square?

8. Triangle ABC has sides $AB = 6$, $AC = 4$ and $BC = 5$. Point P is on AB and point Q is on AC such that PQ bisects the area of triangle ABC. Prove that the minimum length of PQ is $\sqrt{42}/2$.
9. Given $b > 0$, $b \neq 1$, find the set of values of k for which the equation $\log_{b^2}(x^2 - b^2) = \log_b(x - bk)$ has real solutions.
10. Six points are chosen in space such that no three are collinear and no four are coplanar. The 15 line segments joining the points in pairs are drawn then painted, some red and some blue. Prove that some triangle has all its sides the same colour.

IMO Report

Jimmy Chui
student, University of Toronto

The 1999 Canadian IMO team members commenced their summer with one and a half weeks of training at the University of Waterloo. During this time, they managed to hike twice, and it was a shame that no one was lost this year. On the 13th of July, after an exhausting full day of travel, the team found itself in Bucharest, Romania, ready for the 40th International Mathematical Olympiad.

The members of this year's team were David "Pippy" Arthur, Jimmy "The Squeeze" Chui, James "Roadkill" Lee, Jessie "Pyromaniac" Lei, David "Monkey Matrix" Nicholson, and David "23 Across" Pritchard. Team leader Dr. Ed "81" Barbeau gave incessant lectures on continuity while deputy leader Dr. Arthur "Put down that math and deal!" Baragar could be found playing Tetris on some particular portable game machine. Meanwhile, the deputy leader observer, Dr. Dorette "Maybe this works..." Pronk, dutifully took pictures of the other team members while they were not looking. The team is also grateful to Dr. Ed Wang and Richard Hoshino for their wise words in combinatorics and inequalities, and to Dr. Christopher Small for sharing his functional equations knowledge, as well as for his outstanding hospitality in Elora.

This year's contest was immensely challenging, and it continued the low medal cut-off scores the last few IMOs have seen. Considering the difficulty of the questions, Canada performed respectably and brought home 3 bronzes. The scores were as follows:

CAN 1	David Arthur	18	Bronze Medal
CAN 2	Jimmy Chui	16	Bronze Medal
CAN 3	James Lee	6	
CAN 4	Jessie Lei	9	
CAN 5	David Nicholson	8	
CAN 6	David Pritchard	17	Bronze Medal

Unofficially, Canada's total score of 74 was enough for 32^{nd} place out of the 83 competing countries. Best of luck to CAN 3 and 5 as they pursue their university studies at the University of Waterloo, and to CAN 2 and 4 as they move on to the University of Toronto. The remaining two members are still eligible for next year's team. Hopefully there will be shouts of "We like to party!" after the competition next year!

Special thanks must also go to Dr. Graham Wright of the Canadian Mathematical Society for once again supplying the funds for the team, and again to team leader Dr. Ed Barbeau for his continual efforts training IMO potentials through the CMS's correspondence program.

It was the first flight to Europe for many of us, and quite an experience it turned out to be. We were surprised at the endless supply of cheese that the cafeteria managed to put on our plates. We found it a great object to ward off stray dogs. Visits to several museums, including the infamous Transylvania Castle, were a real treat to the competitors, but it was a shame that Dracula was nowhere to be found. However, the cheese did find its way along with us. With all that said and done, the IMO was once again a success. We wish the best of luck to all hopefuls for the 2000 Canadian IMO team, bound for the Republic of Korea for the 41^{st} IMO.

Stan Wagon's e–mail problem of the week

For the benefit of those readers of ***CRUX with MAYHEM*** who make use of Stan Wagon's *e-mail Problem of the Week*, and for the information of those who do not know about it, the Math Forum at Swarthmore has just taken over the handling of the e-list. The instructions for subscribing are:

> to subscribe to the Problem of the Week send a message to <<majordomo@forum.swarthmore.edu>>. Body of message should read simply SUBSCRIBE MACPOW. Macalester students should NOT subscribe to the e–list, but get printed postings instead.

Here is a sample of the type of problem that you can expect:

Problem 887 Square Division

Show how to divide a unit square into two rectangles so that the smaller rectangle can be placed on the larger with every vertex of the smaller on exactly one of the edges of the larger.

Source: 1994 Dutch Mathematical Olympiad; as reported in Crux Mathematicorum, Sept 1998, Vol. 24, No. 5, p. 264.

An Identity of a Tetrahedron

Murat Aygen

Problem. Let $ABCD$ be a tetrahedron with sides $a = BC$, $b = AC$, $c = AB$, $a_1 = AD$, $b_1 = BD$, and $c_1 = CD$ (see Figure 1(a)). Let V and R denote the volume and circumradius of the tetrahedron, respectively. Show that $6VR$ equals the area of the triangle with sides aa_1, bb_1, and cc_1.

Solution. Consider the plane of triangle ABC and the parallel plane through D. These intersect the circumsphere of $ABCD$ in two circles; let their radii be r_1 and r_2, respectively (see Figure 1(b)). Let h be the distance between the two planes.

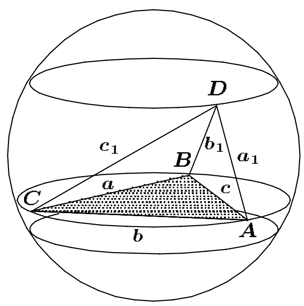


Figure 1(a).

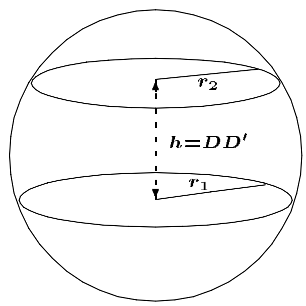


Figure 1(b).

Let O be the circumcentre of triangle ABC, let D' be the projection of D onto the plane of triangle ABC, and let $2\theta = \angle D'OC$ (see Figure 2).

By the Cosine Law on triangle $D'OC$,

$$\begin{aligned} (CD')^2 &= r_1^2 + r_2^2 - 2r_1r_2\cos 2\theta \\ &= r_1^2 + r_2^2 - 2r_1r_2(1 - 2\sin^2\theta) \\ &= (r_1 - r_2)^2 + 4r_1r_2\sin^2\theta\,. \end{aligned}$$

Since $DD' = h$, we obtain similarly that

$$\begin{aligned} a_1^2 &= AD^2 = (DD')^2 + (AD')^2 \\ &= h^2 + (r_1 - r_2)^2 + 4r_1r_2\sin^2(B - \theta)\,, \\ b_1^2 &= BD^2 = (DD')^2 + (BD')^2 \\ &= h^2 + (r_1 - r_2)^2 + 4r_1r_2\sin^2(A + \theta)\,, \\ c_1^2 &= CD^2 = (DD')^2 + (CD')^2 \\ &= h^2 + (r_1 - r_2)^2 + 4r_1r_2\sin^2\theta\,. \end{aligned}$$

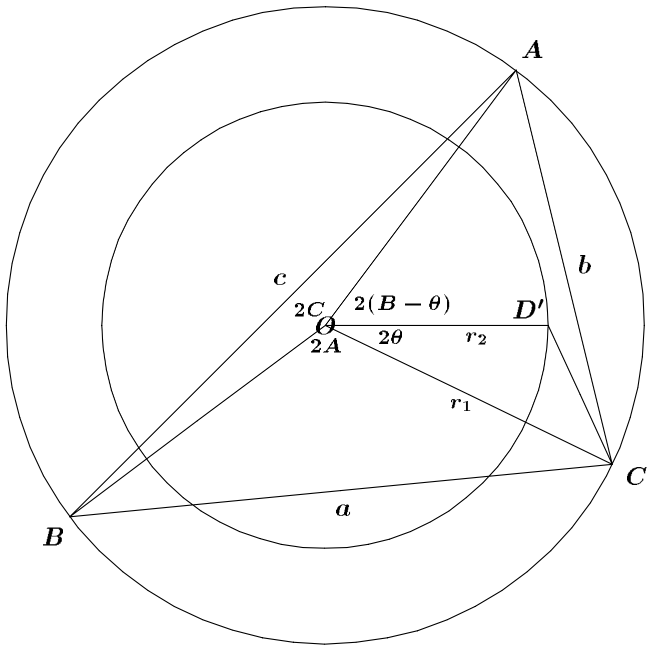


Figure 2.

We now present a geometrical derivation of these lengths. Let K and K_1 denote the area of the triangle with sides a, b, and c, and the triangle with sides aa_1, bb_1, and cc_1, respectively. We will derive a relationship between K and K_1.

Let P_0 be the plane containing triangle ABC. Consider another plane P_1, passing through A, and meeting P_0 at an angle of ϕ. Let ψ be the angle between AB and the intersection of P_0 and P_1 (see Figure 3), where ϕ and ψ are angles that will be specified later.

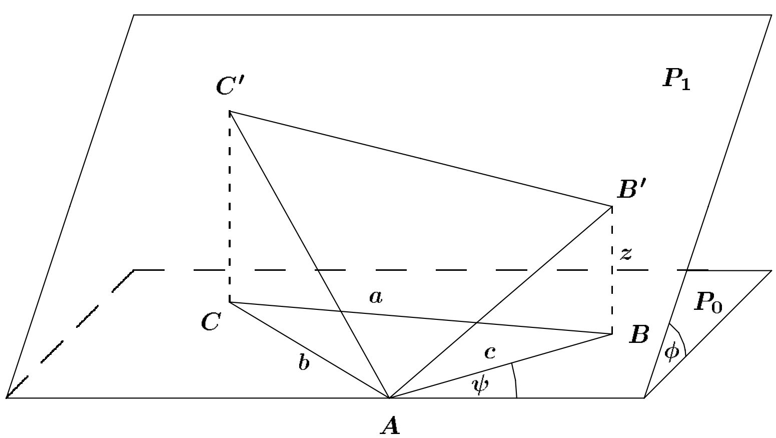


Figure 3.

Let B' and C' be the points in P_1 that project to B and C in P_0, respectively. Then by some elementary trigonometry,

$$\begin{aligned}(B'C')^2 &= a^2 + a^2 \tan^2\phi \, \sin^2(B-\psi)\,, \\ (AC')^2 &= b^2 + b^2 \tan^2\phi \, \sin^2(A+\psi)\,, \\ (AB')^2 &= c^2 + c^2 \tan^2\phi \, \sin^2\psi\,.\end{aligned}$$

Now, assign ψ and ϕ such that

$$\psi = \theta \quad \text{and} \quad \tan^2\phi = \frac{4r_1r_2}{h^2+(r_1-r_2)^2}\,.$$

Then the equations above become

$$\begin{aligned}(B'C')^2 &= a^2(1+\tan^2\phi \, \sin^2\theta) \\ &= \frac{a^2[h^2+(r_1-r_2)^2+4r_1r_2\sin^2\theta]}{h^2+(r_1-r_2)^2} \\ &= \frac{a^2a_1^2}{h^2+(r_1-r_2)^2}\,, \\ (AC')^2 &= b^2[1+\tan^2\phi \, \sin^2(B-\theta)] \\ &= \frac{b^2b_1^2}{h^2+(r_1-r_2)^2}\,, \\ (AB')^2 &= c^2[1+\tan^2\phi \, \sin^2(A+\theta)] \\ &= \frac{c^2c_1^2}{h^2+(r_1-r_2)^2}\,.\end{aligned}$$

Hence, $AB'C'$ is proportional to the triangle with sides aa_1, bb_1, and cc_1, with ratio of areas

$$\frac{1}{h^2+(r_1-r_2)^2}\,.$$

Projecting from P_1 to P_0 scales the area by a further factor of $1/\cos\phi$. Hence,

$$K = \frac{K_1\cos\phi}{h^2+(r_1-r_2)^2}\,. \tag{1}$$

We also know that

$$V = \frac{1}{3}Kh\,, \tag{2}$$

$$h = \sqrt{R^2-r_1^2}+\sqrt{R^2-r_2^2}\,. \tag{3}$$

Squaring both sides of (3), we obtain

$$h^2+r_1^2+r_2^2-2R^2 = 2\sqrt{(R^2-r_1^2)(R^2-r_2^2)}\,.$$

Therefore,

$$(h^2 + r_1^2 + r_2^2)^2 - 4R^2h^2 - 4R^2r_1^2 - 4R^2r_2^2 + 4R^4$$
$$= 4R^4 - 4R^2r_1^2 - 4R^2r_2^2 + 4r_1^2r_2^2 \,,$$

yielding

$$\begin{aligned} 4R^2h^2 &= h^4 + r_1^4 + r_2^4 + 2h^2r_1^2 + 2h^2r_2^2 - 2r_1^2r_2^2 \\ &= h^4 + r_1^4 + r_2^4 + 2h^2r_1^2 + 2h^2r_2^2 + 2r_1^2r_2^2 - 4r_1^2r_2^2 \\ &= (h^2 + r_1^2 + r_2^2)^2 - (2r_1r_2)^2 \\ &= [h^2 + (r_1 + r_2)^2][h^2 + (r_1 - r_2)^2] \,. \qquad (4) \end{aligned}$$

Finally, note that

$$\cos^2\phi = \frac{1}{1 + \tan^2\phi} = \frac{h^2 + (r_1 - r_2)^2}{h^2 + (r_1 + r_2)^2} \,. \qquad (5)$$

Therefore,

$$\begin{aligned} (6VR)^2 &= 4K^2R^2h^2 \qquad \text{(from (2))} \\ &= \frac{4K_1^2R^2h^2\cos^2\phi}{[h^2 + (r_1 - r_2)^2]^2} \qquad \text{(from (1))} \\ &= \frac{K_1^2[h^2 + (r_1 + r_2)^2][h^2 + (r_1 - r_2)^2]^2}{[h^2 + (r_1 + r_2)^2][h^2 + (r_1 - r_2)^2]^2} \qquad \text{(from (4) and (5))} \\ &= K_1^2 \,, \end{aligned}$$

so that

$$6VR = K_1 \,.$$

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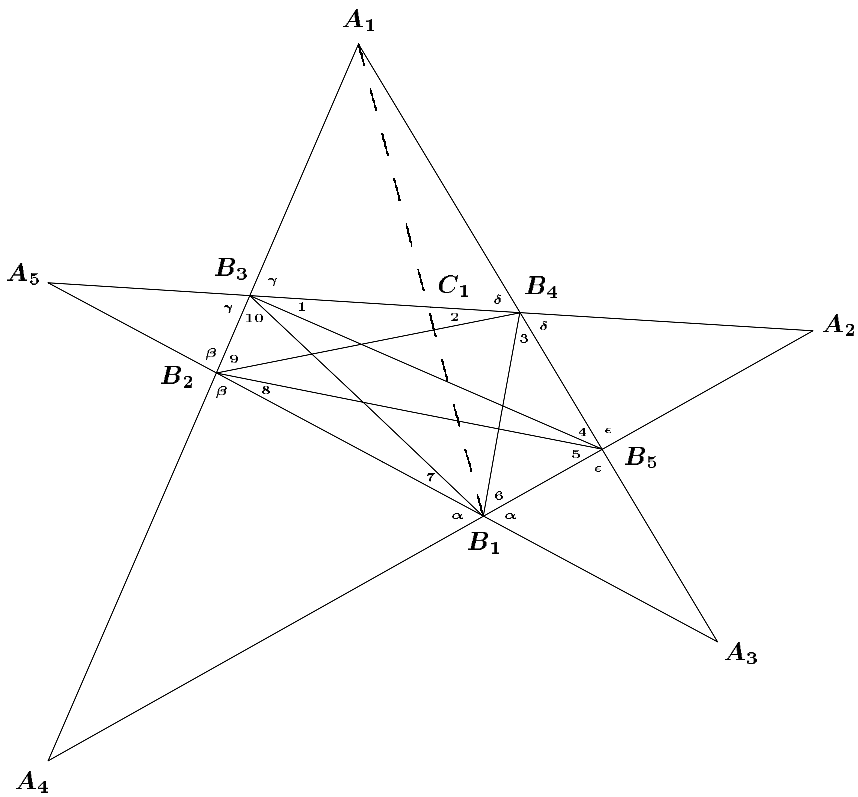
A Simple Proof of a Pentagram Theorem

Geoffrey A. Kandall

We will give a short, transparent proof of Eiji Konishi's pentagram theorem, which was communicated by Hiroshi Kotera [1]. The proof is really just an exercise in the Law of Sines.

Theorem. Let $A_1A_2A_3A_4A_5$ be a pentagram with pentagon $B_1B_2B_3B_4B_5$ as shown in the figure. Let C_k be the intersection of line segment A_kB_k and side $B_{k+2}B_{k+3}$, for $k = 1, 2, \ldots, 5$. Then

$$\frac{B_3C_1}{C_1B_4} \cdot \frac{B_4C_2}{C_2B_5} \cdot \frac{B_5C_3}{C_3B_1} \cdot \frac{B_1C_4}{C_4B_2} \cdot \frac{B_2C_5}{C_5B_3} = 1 .$$



Proof. Let $[P]$ denote the area of polygon P. Then we have that

$$\begin{aligned}\frac{B_3C_1}{C_1B_4} &= \frac{[A_1B_3B_1]}{[A_1B_4B_1]}\\ &= \frac{A_1B_3 \cdot B_3B_1 \cdot \sin\angle A_1B_3B_1}{A_1B_4 \cdot B_4B_1 \cdot \sin\angle A_1B_4B_1}\\ &= \frac{\sin\delta \cdot B_3B_1 \cdot \sin\theta_{10}}{\sin\gamma \cdot B_4B_1 \cdot \sin\theta_3}\,.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{B_4C_2}{C_2B_5} &= \frac{\sin\epsilon \cdot B_4B_2 \cdot \sin\theta_2}{\sin\delta \cdot B_5B_2 \cdot \sin\theta_5}\,,\\ \frac{B_5C_3}{C_3B_1} &= \frac{\sin\alpha \cdot B_5B_3 \cdot \sin\theta_4}{\sin\epsilon \cdot B_1B_3 \cdot \sin\theta_7}\,,\\ \frac{B_1C_4}{C_4B_2} &= \frac{\sin\beta \cdot B_1B_4 \cdot \sin\theta_6}{\sin\alpha \cdot B_2B_4 \cdot \sin\theta_9}\,,\\ \frac{B_2C_5}{C_5B_3} &= \frac{\sin\gamma \cdot B_2B_5 \cdot \sin\theta_8}{\sin\beta \cdot B_3B_5 \cdot \sin\theta_1}\,.\end{aligned}$$

Multiplying these five equations together, we obtain

$$\begin{aligned}&\frac{B_3C_1}{C_1B_4}\cdot\frac{B_4C_2}{C_2B_5}\cdot\frac{B_5C_3}{C_3B_1}\cdot\frac{B_1C_4}{C_4B_2}\cdot\frac{B_2C_5}{C_5B_3}\\ &\quad= \frac{\sin\theta_{10}\cdot\sin\theta_2\cdot\sin\theta_4\cdot\sin\theta_6\cdot\sin\theta_8}{\sin\theta_7\cdot\sin\theta_9\cdot\sin\theta_1\cdot\sin\theta_3\cdot\sin\theta_5}\\ &\quad= \frac{B_1B_2}{B_2B_3}\cdot\frac{B_2B_3}{B_3B_4}\cdot\frac{B_3B_4}{B_4B_5}\cdot\frac{B_4B_5}{B_5B_1}\cdot\frac{B_5B_1}{B_1B_2}\\ &\quad= 1\,.\end{aligned}$$

Reference

[1] H. Kotera, *The Pentagram Theorem*, ***CRUX with MAYHEM*** 24:5 (1998), 291–295.

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PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}$"×11" or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 April 2000**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in* LaTeX*). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

Last month's problem section asked, in error, for solutions by 1 January 2000. That was a Y2K bug! It should have read **1 March 2000**.

2452. Correction. *Proposed by Antal E. Fekete, Memorial University of Newfoundland, St. John's, Newfoundland.*

Establish the following equalities:

(a) $$\sum_{n=0}^{\infty} \frac{(2n+1)^2}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(2n+2)^2}{(2n+2)!}\,.$$

(b) and (c) are correct as originally printed.

2453. Correction. *Proposed by Antal E. Fekete, Memorial University of Newfoundland, St. John's, Newfoundland.*

Establish the following equalities:

(a) $$\sum_{n=0}^{\infty}(-1)^n \frac{(2n+1)^3}{(2n+1)!} = -3\sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!}\,.$$

(b) $$\sum_{n=0}^{\infty}(-1)^n \frac{(2n)^3}{(2n)!} = -3\sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n)!}\,.$$

(c) $$\left(\sum_{n=0}^{\infty}(-1)^n \frac{(2n+1)^2}{(2n+1)!}\right)^2 + \left(\sum_{n=0}^{\infty}(-1)^n \frac{(2n)^2}{(2n)!}\right)^2 = 2\,.$$

2476. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

Let n be a positive integer and consider the set $\{1, 2, 3, \ldots, 2n\}$. Give a **combinatorial** proof that the number of subsets A such that

1. A has exactly n elements, and
2. the sum of all elements in A is divisible by n,

is equal to

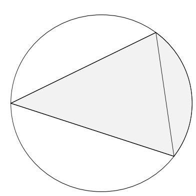
$$\frac{1}{n}\sum_{d|n}(-1)^{n+d}\phi\left(\frac{n}{d}\right)\binom{d}{2d},$$

where ϕ is the Euler function.

Note: When n is prime, proving the formula is problem 6 of the 1995 IMO. A non-combinatorial proof of the formula is due to Roberto Dvornicich and Nikolay Nikolov.

2477. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Given a non-degenerate $\triangle ABC$ with circumcircle Γ, let r_A be the inradius of the region bounded by BA, AC and $\operatorname{arc}(CB)$ (so that the region includes the triangle).



Similarly, define r_B and r_C. As usual, r and R are the inradius and circumradius of $\triangle ABC$.

Prove that

(a) $\frac{64}{27}r^3 \;\leq\; r_A r_B r_C \;\leq\; \frac{32}{27}Rr^2$;

(b) $\frac{16}{3}r^2 \;\leq\; r_B r_C + r_C r_A + r_A r_B \;\leq\; \frac{8}{3}Rr$;

(c) $4r \;\leq\; r_A + r_B + r_C \;\leq\; \frac{4}{3}(R+r)$,

with equality occurring in all cases if and only if $\triangle ABC$ is equilateral.

2478. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

For $n \in \mathbb{N}$, evaluate $\displaystyle\sum_{k=0}^{n}\frac{n-k}{k+1}\binom{2k}{k}\binom{2n-2k}{n-k}$.

2479. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

Writing $\tau(n)$ for the number of divisors of n, and $\omega(n)$ for the number of distinct prime factors of n, prove that

$$\sum_{k=1}^{n} (\tau(k))^2 \;=\; \sum_{k=1}^{n} 2^{\omega(k)} \sum_{j=1}^{\lfloor n/k \rfloor} \left\lfloor \frac{\lfloor n/k \rfloor}{j} \right\rfloor .$$

2480. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

Writing $\phi(n)$ for Euler's totient function, evaluate

$$\sum_{d|n} d \sum_{k|d} \frac{\phi(k)\phi(d/k)}{k} .$$

2481. *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that A, B, C are 2×2 commutative matrices. Prove that

$$\det\left((A+B+C)(A^3+B^3+C^3-3ABC)\right) \;\geq\; 0 .$$

2482. *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that p, q, r are complex numbers. Prove that

$$|p+q| + |q+r| + |r+p| \;\leq\; |p| + |q| + |r| + |p+q+r| .$$

2483. *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Suppose that $0 \leq A$, B, C and $A+B+C \leq \pi$. Show that

$$0 \;\leq\; A - \sin A - \sin B - \sin C + \sin(A+B) + \sin(A+C) \;\leq \pi .$$

There are, of course, similar inequalities with the angles permuted cyclically.

[The proposer notes that this came up during an attempt to generalise problem 2383.]

2484. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given a square $ABCD$, suppose that E is a point on AB produced beyond B, that F is a point on AD produced beyond D, and that $EF = 2AB$. Let P and Q be the intersections of EF with BC and CD, respectively. Prove that

(a) $\triangle APQ$ is acute-angled;

(b) $\angle PAQ \geq 45^\circ$.

2485. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a convex quadrilateral with $AB = BC = CD$. Let P be the intersection of the diagonals AC and BD. Suppose that $AP : BD = DP : AC$.

Prove that either $BC \parallel AD$ or $AB \perp CD$.

2486. *Proposed by Joe Howard, New Mexico Highlands University, Las Vegas, NM, USA.*

It is well-known that $\cos(20°)\cos(40°)\cos(80°) = \frac{1}{8}$.

Show that $\sin(20°)\sin(40°)\sin(80°) = \frac{\sqrt{3}}{8}$.

2487. *Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain.*

If a, b, c, d are distinct real numbers, prove that

$$\frac{a^4+1}{(a-b)(a-c)(a-d)} + \frac{b^4+1}{(b-a)(b-c)(b-d)}$$
$$+ \frac{c^4+1}{(c-a)(c-b)(c-d)} + \frac{d^4+1}{(d-a)(d-b)(d-c)} \;=\; a+b+c+d\,.$$

2488. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $S_n = \mathbf{A}_1\mathbf{A}_2\ldots\mathbf{A}_{n+1}$ be a simplex in $\mathbb{E}^n$, and M a point in S_n. It is known that there are real positive numbers λ_1, λ_2, $\ldots$, λ_{n+1} such that $\sum\limits_{j=1}^{n+1}\lambda_j = 1$ and $M = \sum\limits_{j=1}^{n+1}\lambda_j\mathbf{A}_j$ (here, by a point $\mathbf{P}$, we mean the position vector $\overrightarrow{\mathbf{OP}}$). Suppose also that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and let $\mathbf{B}_k = \dfrac{1}{k}\sum\limits_{j=1}^{k}\mathbf{A}_j$.

Prove that

$$M \;\in\; \textbf{convex cover of } \{\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_{n+1}\}\,;$$

that is, there are real positive numbers μ_1, μ_2, $\ldots$, μ_{n+1} such that

$$M \;=\; \sum_{k=1}^{n+1}\mu_k\mathbf{B}_k\,.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2370. [1998: 364] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Determine the exact values of the roots of the polynomial equation

$$x^5 - 55x^4 + 330x^3 - 462x^2 + 165x - 11 \;=\; 0\,.$$

Solution by Joe Howard, New Mexico Highlands University, Las Vegas, NM, USA.

Let $c = \cos\theta$, $s = \sin\theta$, where $\theta = \frac{k\pi}{11}$, $k = 1, 3, 5, 7, 9$. Then, by DeMoivre's formula, we have

$$(c+is)^{11} \;=\; \left(e^{i\theta}\right)^{11} \;=\; e^{11i\theta} \;=\; e^{k\pi i} \;=\; -1\,.$$

Equating the imaginary parts, we get

$$\binom{11}{1}c^{10}s - \binom{11}{3}c^8s^3 + \binom{11}{5}c^6s^5 - \binom{11}{7}c^4s^7 + \binom{11}{9}c^2s^9 - \binom{11}{11}s^{11} \;=\; 0\,.$$

Since $s \neq 0$, we obtain

$$s^{10} - 55s^8c^2 + 330s^6c^4 - 462s^4c^6 + 165s^2c^8 - 11c^{10} \;=\; 0\,.$$

Since $c \neq 0$, dividing the last equation by c^{10} yields

$$\tan^{10}\theta - 55\tan^8\theta + 330\tan^6\theta - 462\tan^4\theta + 165\tan^2\theta - 11 \;=\; 0\,.$$

Therefore, the roots of the given polynomial equation are given by $x = \tan^2\left(\frac{k\pi}{11}\right)$, $k = 1, 3, 5, 7, 9$.

Also solved by MANUEL BENITO MUÑOZ and EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; DOUGLASS L. GRANT, University College of Cape Breton, Sydney, Nova Scotia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece (2 solutions); GERRY LEVERSHA, St. Paul's School, London, England; KEE-WAI LAU, Hong Kong; VEDULA N. MURTY, Visakhapatnam, India; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ARAM TANGBOONDOUANGJIT, Carnegie Mellon University, Pittsburgh, PA, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

All the submitted solutions are more or less equivalent to the one given above. Almost all the solvers gave the answers $x = \tan^2\left(\frac{k\pi}{11}\right)$, $k = 1, 2, 3, 4, 5$. *Benito and Fernández gave the answer* $x = \cot^2\left(\frac{k\pi}{22}\right)$, $k = 1, 3, 5, 7, 9$. *It is easy to see that all these expressions are the same as the one obtained by Howard.*

2371. [1998: 364] *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

For n an integer greater than 4, let $f(n)$ be the number of five-element subsets, S, of $\{1, 2, \ldots, n\}$ which have *no isolated points*, that is, such that if $s \in S$, then either $s-1$ or $s+1$ (**not** taken $\mathrm{modulo}\ n$) is in S.

Find a "nice" formula for $f(n)$.

I. *Solution by Manuel Benito and Emilio Fernandez, I.B. Praxedes Mateo Sagasta, Logroño, Spain.*

Let us represent any such subset S as $[a_1, a_2, a_3, a_4, a_5]$, where $a_1 < a_2 < a_3 < a_4 < a_5$. It is clear that the number a_3 can take on each of the $n-4$ values from 3 to $n-2$, but let us ask: how many times can it take on each one of these values?

Well, a_1 is not isolated, so that $a_2 = a_1+1$, and a_5 is not isolated either so $a_4 = a_5 - 1$. Then, for each fixed k from 3 to $n-2$ we can enumerate the subsets S_k for which $a_3 = k$, in the following manner:

- type $[k-2, k-1, k, k+1, k+2]$ .... 1 subset;
- from $a_1 = 1$ to $a_1 = k-3$, type $[a_1, a_2, k, k+1, k+2]$ .... $k-3$ subsets;
- from $a_5 = k+3$ to $a_5 = n$, type $[k-2, k-1, k, a_5-1, a_5]$ .... $n-(k+3)+1$ subsets;

which gives a total of $n-4$ subsets S_k for each k. Since there are $n-4$ values of k, this gives

$$f(n) \;=\; (n-4)^2\,.$$

II. *Solution by Kathleen E. Lewis, SUNY Oswego, Oswego, NY, USA.*

The characteristic function for such a subset will contain $n-5$ zeros and five ones. To be non-isolated, the ones must either consist of a single block or be broken into a block of size two and a block of size three (in either order). Looking at the case of two blocks, we can think of this as a problem of placing two distinct items plus $n-5$ indistinguishable items (the zeros) in a sequence, which can be done in $(n-3)(n-4)$ ways. But if the two blocks of ones end up adjacent to each other, we get the case of all five ones together. Since it does not matter in this case which block of ones is first, we have double-counted these arrangements. There are $n-4$ ways to put all the ones together among the $n-5$ zeros. Therefore, when we compensate for double-counting this case earlier, we get a final answer of

$$(n-3)(n-4) - (n-4) \;=\; (n-4)^2\,.$$

III. *Solution by Jeremy Young, student, Nottingham High School, Nottingham, UK.*

Given $f(n)$, consider $f(n+1)$ and every corresponding subset S. There are by definition $f(n)$ subsets S with $n+1 \notin S$. If $n+1 \in S$, $n \in S$ too.

Consider subsets of this form. Now if $n-1 \in S$, the remaining two elements must be adjacent; that is, k and $k+1$ with $1 \leq k \leq n-3$, and there are $n-3$ such subsets. If $n-1 \notin S$, the three remaining elements must be adjacent; that is, $k, k+1, k+2$ with $1 \leq k \leq n-4$, and there are $n-4$ such subsets.

We have thus established the recurrence

$$f(n+1) \;=\; f(n)+(n-3)+(n-4) \;=\; f(n)+2n-7\,,$$

with $f(5) = 1$ (obviously) giving $f(6) = 4$, $f(7) = 9$, $f(8) = 16$, ... Note that $f(k) = (k-4)^2$ is true for $k = 5$ and gives

$$f(k+1) \;=\; (k-4)^2+2k-7 \;=\; k^2-6k+9 \;=\; (k-3)^2\,.$$

Therefore $f(n) = (n-4)^2$ by induction, for all integers $n \geq 5$.

Also solved by SAM BAETHGE, Nordheim, TX, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong, China; GERRY LEVERSHA, St. Paul's School, London, England; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer. One incorrect solution was received.

Bradley's solution was the same as Lewis's, and Lau, Leversha and Perz all had solutions similar to Young's.

2372. [1998: 365] *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

For n and k positive integers, let $f(n,k)$ be the number of k–element subsets S of $\{1, 2, \ldots, n\}$ satisfying:

(i) $1 \in S$ and $n \in S$; and

(ii) whenever $s \in S$ with $s < n-1$, then either $s+2 \in S$ or $s+3 \in S$.

Prove that $f(n,k) = f(4k-2-n, k)$ for all n and k; that is, the sequence

$$f(k,k)\,,\, f(k+1,k)\,,\, f(k+2,k)\,,\,\ldots\,,\, f(3k-2,k)$$

of non-zero values of $\{f(n,k)\}_{n=1}^{\infty}$ is a palindrome for every k.

Solution by Gerry Leversha, St. Paul's School, London, England.

I shall call sets which satisfy the condition *permissible*.

In general, it is clear that $f(k,k) = 1$. Also $f(3k-2,k) = 1$, since the only possible set is $\{1, 4, 7, \ldots, 3k-5, 3k-2\}$, an arithmetic progression with common difference 3 and k terms. Since this is the "sparsest" possible k–element permissible set, it is clear that $f(n,k) = 0$ for $n \geq 3k-1$. Equally obviously $f(n,k) = 0$ for $n < k$. So we need only consider n such that $k < n < 3k-2$. For $n = 2k-1$, we have $4k-2-n = 2k-1$, so the palindromic condition is trivial here. It remains to show that for $k < n < 2k - 1$, $f(n,k) = f(4k - 2 - n, k)$.

Consider a permissible set $S \subset \{1, 2, 3, \ldots, n\}$ which has k elements. Let T be the sequence of *differences* between successive elements of S; then

- T is an ordered $(k-1)$–tuple each of whose elements is either 1, 2 or 3;
- The sum of the elements of T is $n - 1$;
- T cannot contain the elements 1, 3 in that order.

Any such T will yield a permissible set S. The restriction on subsequences of the form 1, 3 ensures that we cannot have a, $a + 1$, $a + 4$ appearing in S, which would contravene the condition since neither $a + 2$ nor $a + 3$ would be in S. Hence

$f(n,k)$ is equal to the number of such sequences T.

Now from any such sequence T construct a sequence T^* as follows:

- Replace every element x of T by $4 - x$;
- Reverse the sequence so formed.

Then the new sequence T^* has the following properties:

- It is an ordered $(k-1)$–tuple formed from the elements 1, 2 and 3;
- The sum of the elements is $4(k - 1) - (n - 1) = 4k - n - 3$;
- It cannot contain a subsequence 1, 3 (since there would have to have been such a subsequence in T in the first place).

Thus there is a one-to-one correspondence between sequences of type T and those of type T^*. But sequences of type T^* correspond in turn to permissible k–element sets $S \subset \{1, 2, \ldots, 4k - n - 2\}$. Hence we have shown that

$$f(n,k) \;=\; f(4k - n - 2, k)\,.$$

I shall illustrate this process in the case of $f(7,5)$ and $f(11,5)$. The four columns below show, respectively,

- a permissible 5–element subset of $\{1, 2, 3, 4, 5, 6, 7\}$,
- the corresponding T sequence,
- the corresponding T^* sequence,
- the permissible 5–element subset of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$.

$\{1,2,3,5,7\}$	$(1,1,2,2)$	$(2,2,3,3)$	$\{1,3,5,8,11\}$
$\{1,2,4,5,7\}$	$(1,2,1,2)$	$(2,3,2,3)$	$\{1,3,6,8,11\}$
$\{1,2,4,6,7\}$	$(1,2,2,1)$	$(3,2,2,3)$	$\{1,4,6,8,11\}$
$\{1,3,4,5,7\}$	$(2,1,1,2)$	$(2,3,3,2)$	$\{1,3,6,9,11\}$
$\{1,3,4,6,7\}$	$(2,1,2,1)$	$(3,2,3,2)$	$\{1,4,6,9,11\}$
$\{1,3,5,6,7\}$	$(2,2,1,1)$	$(3,3,2,2)$	$\{1,4,7,9,11\}$
$\{1,4,5,6,7\}$	$(3,1,1,1)$	$(3,3,3,1)$	$\{1,4,7,10,11\}$

Also solved by MANUEL BENITO and EMILIO FERNANDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

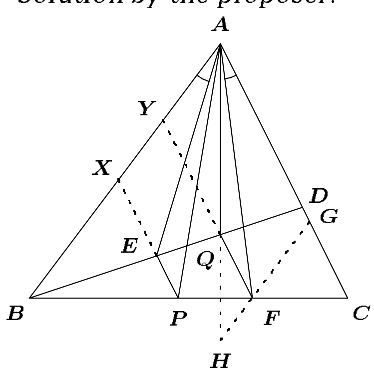
Lambrou, Lewis and Young gave "combinatorial" solutions similar to Leversha's.

2375. [1998: 365] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let D be a point on side AC of triangle ABC. Let E and F be points on the segments BD and BC, respectively, such that $\angle BAE = \angle CAF$. Let P and Q be points on BC and BD, respectively, such that $EP \parallel DC$ and $FQ \parallel CD$.

Prove that $\angle BAP = \angle CAQ$.

Solution by the proposer.



Let PE and FQ meet AB at X and Y, respectively.

Since $PX \parallel FY$ and FP, QE and YX concur at B, we have

$$\frac{XE}{XP} = \frac{YQ}{YF}. \qquad (1)$$

Let the line through F parallel to AB meet AC and AQ at G and H, respectively.

Since $AY \parallel FH$ and $QF \parallel AG$, we get

$$\frac{YQ}{YF} = \frac{AQ}{AH} = \frac{GF}{GH}. \qquad (2)$$

From (1) and (2), we now have

$$\frac{XE}{XP} = \frac{GF}{GH}. \qquad (3)$$

Since $XE \parallel AG$ and $AX \parallel GF$, we have

$$\angle AXP = 180^\circ - \angle XAG = \angle AGF. \qquad (4)$$

Since $\angle XAE = \angle BAE = \angle CAF = \angle GAF$, we get from (4) that

$$\triangle AXE \sim \triangle AGF . \qquad (5)$$

From (3) and (5), we have that

$$\triangle AXP \sim \triangle AGH .$$

Therefore, we obtain that $\angle XAP = \angle GAH$, and this implies that $\angle BAP = \angle CAQ$.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; MANUEL BENITO MUÑOZ and EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; C. FESTRAETS-HAMOIR, Brussels, Belgium; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; and D.J. SMEENK, Zaltbommel, the Netherlands.

No-one other than the proposer used pure geometric methods. Lambrou made use of vectors, and all the other solvers made use of trigonometry.

2376. [1998: 424] *Proposed by Albert White, St. Bonaventure University, St. Bonaventure, NY, USA.*

Suppose that ABC is a right-angled triangle with the right angle at C. Let D be a point on hypotenuse AB, and let M be the mid-point of CD. Suppose that $\angle AMD = \angle BMD$. Prove that

1. $\overline{AC}^{\,2}\,\overline{MC}^{\,2} + 4[ABC]\,[BCD] = \overline{AC}^{\,2}\,\overline{MB}^{\,2}$;
2. $4\overline{AC}^{\,2}\,\overline{MC}^{\,2} - \overline{AC}^{\,2}\overline{BD}^{\,2} = 4[ACD]^2 - 4[BCD]^2$,

where $[XYZ]$ denotes the area of $\triangle XYZ$.

(This is a continuation of problem **1812**, [1993: 48].)

Solution by Gerry Leversha, St. Paul's School, London, England.

We do not make use of the assumption that $\angle AMD = \angle BMD$. Let $AB = c, BC = a, CA = b, BD = \lambda c, \angle BCD = \theta$, and let X be the foot of the perpendicular from D to BC.

$$\begin{aligned} MB^2 - MC^2 &= BC^2 - 2MC \cdot BC \cos\theta \\ &= a^2 - CD \cdot a \cos\theta \\ &= a^2 - a \cdot CX \;=\; a \cdot BX . \end{aligned}$$

But by similar triangles $BX = \lambda a$ and so $MB^2 - MC^2 = \lambda a^2$. Now $[ABC] = \frac{1}{2}ab$ and $[BCD] = \frac{1}{2}\lambda ab$ and so

$$4[ABC][BCD] = \lambda a^2 b^2 = AC^2(MB^2 - MC^2) ,$$

which is part 1 of the problem. In a similar vein

$$\begin{aligned} AC^2(4MC^2 - BD^2) &= AC^2(CD^2 - BD^2) \\ &= b^2(BC^2 - 2BC \cdot BD \cos B) \\ &= b^2(a^2 - 2a \cdot \lambda c \cos B) \;=\; a^2 b^2 (1 - 2\lambda) \\ &= 4((1-\lambda)^2 - \lambda^2)[ABC]^2 \\ &= 4[ACD]^2 - 4[BCD]^2\,, \end{aligned}$$

as required for part 2.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; D.J. SMEENK, Zaltbommel, the Netherlands; PARAGIOU THEOKLITOS, Limassol, Cyprus; and the proposer.

Smeenk's solution is virtually identical to our featured solution, while that of Lambrou is quite similar although he avoids the cosine law by using Cartesian coordinates. All submitted solutions avoided the unnecessary condition involving the pair of angles at M; the proposer evidently arrived at his problem while investigating problem **1812** [1993: 48, 1994: 20-22], *and he failed to notice that his result is valid for any choice of D on AB.*

2377. [1998: 425] *Proposed by Nikolaos Dergiades, Thessaloniki, Greece.*

Let ABC be a triangle and P a point inside it. Let $BC = a$, $CA = b$, $AB = c$, $PA = x$, $PB = y$, $PC = z$, $\angle BPC = \alpha$, $\angle CPA = \beta$ and $\angle APB = \gamma$.

Prove that $ax = by = cz$ if and only if $\alpha - A = \beta - B = \gamma - C = \frac{\pi}{3}$.

Solution by Jun-hua Huang, the Middle School Attached To Hunan Normal University, Changsha, China.

Let M, N, L be the feet of the perpendiculars from P to CA, AB, BC, respectively.

Thus $ax = 2R\sin A \cdot x = 2R \cdot MN$, where R is the circumradius of $\triangle ABC$. Similarly $by = 2R \cdot NL$ and $cz = 2R \cdot ML$. So $ax = by = cz \iff MN = NL = ML$. Now $\alpha - A = \angle ABP + \angle ACP = \angle NLP + \angle MLP = \angle NLM$ and $\beta - B = \angle LMN$. $\gamma - C = \angle MNL$. So

$$MN = NL = ML \iff \alpha - A = \beta - B = \gamma - C = \frac{\pi}{3}\,,$$

which completes the proof.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; G. TSINTSIFAS, Thessaloniki, Greece; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

2380. [1998: 425] *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

When the price of a certain book in a store is reduced by $1/3$ and rounded to the nearest cent, the cents and dollars are switched. For example, if the original price was \$43.21, the new price would be \$21.43 (this does not satisfy the "reduced by $1/3$" condition, of course). What was the original price of the book? [For the benefit of readers unfamiliar with North American currency, there are 100 cents in one dollar.]

Solution by Gerry Leversha, St. Paul's School, London, England.

Let the price be a dollars and b cents. Then

$$\frac{2}{3}(100a + b) \;=\; 100b + a + \frac{x}{3}\,,$$

where $x \in \{-1, 0, 1\}$ is included to deal with any possible rounding. This simplifies to

$$197a \;=\; 298b + x\,.$$

Now 197 and 298 are coprime, so the smallest solution in the case $x = 0$ is $a = 298$, $b = 197$, which is impossible since $0 \le b \le 99$. The usual Euclidean algorithm procedure yields

$$59 \times 197 \;-\; 39 \times 298 \;=\; 1\,,$$

and this shows that we should take $a = 59$, $b = 39$ and $x = 1$. Hence the price of the book was **\$59.39**.

Also solved by CHARLES ASHBACHER, Cedar Rapids, IA, USA; MIGUEL CARRIÓN ÁLVAREZ, Universidad Complutense de Madrid, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; SHAWN GODIN, Cairine Wilson S.S., Orleans, Ontario; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; J.A. McCALLUM, Medicine Hat, Alberta; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; D.J. SMEENK, Zaltbommel, the Netherlands; DIGBY SMITH, Mount Royal College, Calgary, Alberta; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer. Two incorrect solutions were received, at least one of which was due to misunderstanding the problem.

Many solvers noted that the answer is unique, as can be seen from the above solution.

Diminnie opines that "in the U.S. there would be no nontrivial solution to this problem, since U.S. stores seem to insist on rounding up in all circumstances!".

2381. [1998: 425] *Proposed by Angel Dorito, Geld, Ontario.*

Solve the equation $\log_2 x \;=\; \log_4(x+1)$.

Solution by Chris Cappadocia, student, St. Joseph Scollard Hall SS, North Bay, Ontario.

Let both of them equal y. Then

$$2^y = x \qquad \text{and} \qquad 4^y = 2^{2y} = x + 1\,.$$

And so

$$2^y \;=\; \frac{2^{2y}}{2^y} \;=\; \frac{x+1}{x}\,.$$

Comparing, we get $x = (x+1)/x$, and solving for x and throwing away the negative value, we get a final answer of

$$x \;=\; \frac{1+\sqrt{5}}{2}\,.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts, USA; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; BOOKERY PROBLEM GROUP, Walla Walla, Washington, USA; MIGUEL CARRIÓN ÁLVAREZ, Universidad Complutense de Madrid, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; RUSSELL EULER and JAWAD SADEK, NW Missouri State University, Maryville, MO, USA; C. FESTRAETS-HAMOIR, Brussels, Belgium; SHAWN GODIN, Cairine Wilson S. S., Orleans, Ontario; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; JOE HOWARD, New Mexico Highlands University, Las Vegas, NM, USA; JUN-HUA HUANG, the Middle School Attached To Hunan Normal University, Changsha, China; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ANGEL JOVAL ROQUET, Instituto Español de Andorra, Andorra; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; J.A. McCALLUM, Medicine Hat, Alberta; JOHN GRANT McLOUGHLIN, Faculty of Education, Memorial University, St. John's, Newfoundland; HENRY J. RICARDO, Medger Evers College (CUNY), Brooklyn, New York, USA; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; MAX SHVARAYEV, Tucson, Arizona, USA; SKIDMORE COLLEGE PROBLEM GROUP, Saratoga Springs, New York, USA; D.J. SMEENK, Zaltbommel, the Netherlands; DIGBY SMITH, Mount Royal College, Calgary, Alberta; ARAM TANGBOONDOUANGJIT, Carnegie Mellon University, Pittsburgh, PA, USA; PARAGIOU THEOKLITOS, Limassol, Cyprus; PANOS E. TSAOUSSOGLOU, Athens, Greece; UNIVERSITY OF ARIZONA PROBLEM SOLVING GROUP, Tucson, Arizona, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

Joval's solution must be the first one ever received by this journal from Andorra! Welcome — and can we now hear from Liechtenstein and Monaco?

2382. [1998: 425] *Proposed by Mohammed Aassila, Université Louis Pasteur, Strasbourg, France.*

If $\triangle ABC$ has inradius r and circumradius R, show that

$$\cos^2\left(\frac{B-C}{2}\right) \;\geq\; \frac{2r}{R}\,.$$

Solution by Vedula N. Murty, Dover, PA, USA.

We have

$$\left(\cos\frac{B-C}{2} - 2\sin\frac{A}{2}\right)^2 \;\geq\; 0\,,$$

so

$$\begin{aligned}\cos^2\left(\frac{B-C}{2}\right) &\geq 4\cos\frac{B-C}{2}\sin\frac{A}{2} - 4\sin^2\frac{A}{2} \\ &= 4\sin\frac{A}{2}\left[\cos\frac{B-C}{2} - \cos\frac{B+C}{2}\right] \\ &= 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{2r}{R},\end{aligned}$$

since $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ [see for example, Roger A. Johnson, *Modern Geometry* (1929) **298a**].

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; JUN-HUA HUANG, the Middle School Attached To Hunan Normal University, Changsha, China; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; ISAO NAOI and HIDETOSHI FUKAGAWA, Gifu, Japan; ISTVÁN REIMAN, Budapest, Hungary; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; ARAM TANGBOONDOUANGJIT, Carnegie Mellon University, Pittsburgh, PA, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; G. TSINTSIFAS, Thessaloniki, Greece; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; and the proposer.

Arslanagić, Bellot Rosado, Janous, Naoi and Fukagawa, and Seiffert all note that we have equality precisely when $2a = b + c$. Konečný, Smeenk and Yiu provide the equivalent condition, $\tan\frac{B}{2}\tan\frac{C}{2} = \frac{1}{3}$.

Iftimie Simion (Math Teacher, Stuyvesant HS, New York, NY, USA) points out that this problem appears in a 10th grade textbook [Matematică: Geometrie si trigonometrie, *A. Cota et al., p. 106*] *used in Romania.*

Bellot Rosado refers us to two notes on this inequality by Dan Plaesu (Iasi) and Gheorge Marchidan (Suceava) in the Romanian journal Gazeta matematica *(1991) nos. 6 and 7. In the second note the authors prove the related inequality*

$$\cos^2\frac{B-C}{2} \geq \frac{a^2bc}{R^2(b+c)^2}.$$

Bellot Rosado also shows that

$$\frac{a^2bc}{R^2(b+c)^2} > \frac{2r}{R}$$

precisely when $\frac{a}{s} \in (3-\sqrt{5}, 1)$.

2383. [1998: 425] *Proposed by Mohammed Aassila, Université Louis Pasteur, Strasbourg, France.*

Suppose that three circles, each of radius 1, pass through the same point in the plane. Let A be the set of points which lie inside at least two of the circles. What is the least area that A can have?

Solution by Kathleen E. Lewis, SUNY Oswego, Oswego, NY, USA.

Let the point of intersection of the circles be labelled O and the centres be C_1, C_2 and C_3. Consider first the area of intersection of the first two circles as a function of the angle C_1OC_2, which we will call θ. If D is the other point of intersection of these circles, then the area of intersection is twice the difference between the area of the sector created by the angle OC_1D and the area of the triangle OC_1D. The angle OC_1D has measure $\pi-\theta$, so the sector has area $(\pi-\theta)/2$ and the triangle has area $\sin(\theta/2)\cos(\theta/2) = (1/2)\sin\theta$. Therefore the area of overlap of the two circles is $\pi-\theta-\sin\theta$, which we call $f(\theta)$. Since $f'(\theta) = -1-\cos\theta$, which is never positive, the area decreases as θ increases from 0 to π.

Suppose that the circles with centres at C_1 and C_2 are in fixed position, with an angle of $\theta \leq \pi$ between them, and we wish to place the third circle in a way that minimizes the total area of overlap. Since we want large angles between the centres of the circles, we want to put the third circle so that the new angles formed at O divide the angle $2\pi-\theta$ rather than θ. If we call these angles ϕ and ρ, then the total area of overlap is

$$(\pi-\theta-\sin\theta)+(\pi-\phi-\sin\phi)+(\pi-\rho-\sin\rho)\,.$$

Since θ is fixed and π is constant, we need to minimize

$$\begin{aligned}-\phi-\sin\phi-\rho-\sin\rho &= -\phi-\sin\phi-(2\pi-\theta-\phi)-\sin(2\pi-\theta-\phi)\\ &= \theta-2\pi-\sin\phi-\sin(2\pi-\theta-\phi)\end{aligned}$$

as a function of ϕ. Taking the derivative, we get $-\cos\phi+\cos(2\pi-\theta-\phi)$, which is zero if and only if $\cos\phi = \cos(2\pi-\theta-\phi)$. Since the sum of these two angles is strictly less than 2π, the cosines can only be equal if the angles are. Thus, the minimum area occurs when $\phi = \rho$. This argument could equally well be used to argue that if any two of the angles are unequal, then the area could be reduced by moving the circle between those angles to equalize them (leaving the other two circles fixed). The minimum area therefore occurs when all three angles are equal to $2\pi/3$, giving an overlap area of $3[\pi/3-\sin(2\pi/3)] = \pi-3\sqrt{3}/2$.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; MAX SHKARAYEV, Tuscon, AZ, USA; THE UNIVERSITY OF ARIZONA PROBLEM SOLVING GROUP, Tuscon, AZ, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer. There were one incomplete and three incorrect solutions.

2384. [1998: 425] *Proposed by Paul Bracken, CRM, Université de Montréal, Québec.*

Prove that $2(3n-1)^n \geq (3n+1)^n$ for all $n \in \mathbb{N}$.

Solution by Michel Bataille, Rouen, France.

The inequality is obvious for $n = 0$, so we may assume $n \geq 1$. We have to prove

$$\left(\frac{3n-1}{3n+1}\right)^n \geq \frac{1}{2}\,,$$

or, equivalently,

$$n \ln \frac{3n-1}{3n+1} \;\geq\; \ln \frac{1}{2}\,. \tag{1}$$

To this aim, we introduce the function

$$f(x) \;=\; x \ln \frac{3x-1}{3x+1}$$

defined on $[1, \infty)$. We compute

$$f'(x) \;=\; \ln \frac{3x-1}{3x+1} + \frac{6x}{9x^2-1} \qquad \text{and} \qquad f''(x) \;=\; \frac{-12}{(9x^2-1)^2}\,.$$

Since $f''(x) < 0$, $f'(x)$ is strictly decreasing on $[1, \infty)$. Moreover,

$$\lim_{x \to \infty} f'(x) = 0\,,$$

so $f'(x) > 0$ for all $x \in [1, \infty)$. Hence f is increasing on $[1, \infty)$ and, since $f(1) = \ln \frac{1}{2}$, the inequality (1) follows.

Also solved by MIGUEL CARRIÓN ÁLVAREZ, Universidad Complutense de Madrid, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; KEITH EKBLAW, Walla Walla, WA, USA; RUSSELL EULER and JAWAD SADEK, NW Missouri State University, Maryville, MO, USA; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JUN-HUA HUANG, the Middle School Attached To Hunan Normal University, Changsha, China; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece (three solutions); NORVALD MIDTTUN, Royal Norwegian Navy Academy, Norway; VEDULA N. MURTY, Visakhapatnam, India; VICTOR OXMAN, University of Haifa, Haifa, Israel; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; MAX SHKARAYEV, Tucson, AZ, USA; DIGBY SMITH, Mount Royal College, Calgary, Alberta; PANOS E. TSAOUSSOGLOU, Athens, Greece; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer. There were also three incorrect solutions submitted.

Janous and Seiffert have proved the more general inequality

$$\frac{a+b}{a-b}(ax-b)^x \geq (ax+b)^x$$

for all real $x \in [1, \infty)$ and real a and b such that $a > b > 0$.

2385. [1998: 426] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

A die is thrown $n \geq 3$ consecutive times. Find the probability that the sum of its n outcomes is greater than or equal to $n+6$ and less than or equal to $6n-6$.

Solution by Jeremy Young, student, Nottingham High School, Nottingham, UK, modified by the editor.

Consider the number of ways of scoring a sum of $n+k$ for some k such that $0 \le k \le 5$. This is the number of ways of expressing $n+k$ as a sum of n positive integers, where the order of the summands matters. This number is well-known to be $\binom{n+k-1}{n-1}$. [*Ed*.: see Theorem 1.5.3 on p. 142 of [1].] Furthermore, the number of ways of scoring a sum of $6n-k$ for some k such that $0 \le k \le 5$ is the same as that of scoring a sum of $n+k$. This can be seen by replacing each outcome λ_j by $7-\lambda_j$ for $j = 1, 2, \ldots, n$, since $n \le \sum_{j=1}^{n} \lambda_j \le n+5$ if and only if $6n-5 \le \sum_{j=1}^{n}(7-\lambda_j) \le 6n$.

The total number of possible scoring sequences is 6^n. Hence, the required probability is

$$\begin{aligned} p_n &= \frac{1}{6^n}\left(6^n - 2\sum_{k=0}^{5}\binom{n+k-1}{n-1}\right) \\ &= 1 - \frac{2}{6^n}\sum_{k=0}^{5}\binom{n+k-1}{n-1} = 1 - \frac{2}{6^n}\binom{n+5}{n} \end{aligned}$$

by the well-known combinatorial identity $\sum_{m=r}^{n}\binom{m}{r} = \binom{n+1}{r+1}$, where $0 \le r \le n$. [*Ed*.: See Theorem 1.6.4 on p. 156 of [1].]

Reference:

[1]. H. Joseph Straight, *Combinatorics, An Invitation*, Brooks/Cole, 1993.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; NORVALD MIDTTUN, Royal Norwegian Navy Academy, Norway; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. There were two partially incorrect solutions.

Janous, Leversha and Midttun all obtained the answer $1 - \frac{2}{6^n}\sum_{k=0}^{5}\binom{5}{5-k}\binom{n}{k}$ *first, and then stated or showed that* $\sum_{k=0}^{5}\binom{5}{5-k}\binom{n}{k} = \binom{n+5}{5}$, *which, of course, is just a special case of the well-known and easy-to-prove Vandermonde's Identity:*

$$\sum_{k=0}^{l}\binom{m}{k}\binom{n}{l-k} = \binom{m+n}{l} \qquad \text{for integers } l,\ m,\ n \text{ such that } 0 \le l \le m,\ n.$$

[Ed.: *See Ex. 26 on p. 167 of* [1].]

Janous remarked that, in general, for $t \in \{0, 1, 2, \ldots, \lfloor 5n/2 \rfloor\}$, *we have*

$$p(n+t \le X \le 6n-t) = 1 - \frac{2}{6^n}\binom{n+t+1}{t-1},$$

where X *is the random variable denoting the sum of the* n *outcomes.*

Leversha based his solution on the fact that the probability generating function for X *is* $G(t) = \frac{1}{6^n}\left(t+t^2+t^3+t^4+t^5+t^6\right)^n$.

2388 [1998, 503; Correction 1999, 171]. *Proposed by Daniel Kupper, Büllingen, Belgium.*

Suppose that $n \geq 1 \in \mathbb{N}$ is given and that, for each integer $k \in \{0, 1, \ldots, n-1\}$, the numbers $a_k, b_k, z_k \in \mathbb{C}$ are given, with the z_k^2 distinct. Suppose that the polynomials

$$A_n(z) = z^n + \sum_{k=0}^{n-1} a_k z^k \quad \text{and} \quad B_n(z) = z^n + \sum_{k=0}^{n-1} b_k z^k$$

satisfy $A_n(z_j) = B_n(z_j^2) = 0$ for all $j \in \{0, 1, \ldots, n-1\}$.

Find an expression for $b_0, b_1, \ldots, b_{n-1}$ in terms of $a_0, a_1, \ldots, a_{n-1}$.

Solution by Kee-Wai Lau, Hong Kong.

We have $A_n(z) \equiv \prod_{k=1}^{n} (z - z_k)$ and $B_n(z) = \prod_{k=1}^{n} (z - z_k^2)$.

Hence, $B_n\left(z^2\right) = \prod_{k=1}^{n} (z - z_k)(z + z_k) \equiv (-1)^n A_n(z) A_n(-z)$.

Thus, $\sum_{k=0}^{n} b_k z^{2k} \equiv (-1)^n \left(\sum_{k=0}^{n} a_k z^k\right)\left(\sum_{k=0}^{n} (-1)^k a_k z^k\right)$.

It follows that, for $k = 0, 1, 2, \ldots, n$,

$$b_k = \sum_{j=\max\{0,2k-n\}}^{\min\{n,2k\}} (-1)^{n-j} a_j a_{2k-j}\,.$$

Also solved by MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; JOSÉ LUIS DIAZ, Universidad Politécnica de Cataluña, Terrassa, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. There was one incorrect solution.

We thank Michael Lambrou for the correction to the statement of this problem.

The answer obtained by Seiffert is the same as the one given above, while those given by Lambrou, Leversha and the proposer are minor variations thereof. On the other hand, the answer obtained by Luis has a different "appearance":

$$b_k = (-1)^{n-k}\left(a_k^2 + 2\sum_{j=1}^{n-k+1} (-1)^j a_{k-j} a_{k+j}\right), \qquad k = 0, 1, \ldots, n$$

with $a_j = 0$ if $j < 0$ or $j > n$.

2392. [1998: 504] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*
Suppose that x_i, y_i, $(1 \le i \le n)$ are positive real numbers. Let

$$A_n = \sum_{i=1}^{n} \frac{x_i y_i}{x_i + y_i}, \qquad B_n = \frac{\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{\sum_{i=1}^{n}(x_i + y_i)},$$

$$C_n = \frac{\left(\sum_{i=1}^{n} x_i\right)^2 + \left(\sum_{i=1}^{n} y_i\right)^2}{\sum_{i=1}^{n}(x_i + y_i)}, \qquad D_n = \sum_{i=1}^{n} \frac{x_i^2 + y_i^2}{x_i + y_i}.$$

Prove that

1. $A_n \le C_n$,
2. $B_n \le D_n$,
3. $2A_n \le 2B_n \le C_n \le D_n$.'

Solution by Heinz-Jürgen Seiffert, Berlin, Germany.
It is sufficient to prove the inequalities in the third part.

Let

$$X_n = \sum_{j=1}^{n} x_j\,, \qquad Y_n = \sum_{j=1}^{n} y_j\,, \quad \text{and} \quad Z_n = \sum_{j=1}^{n} \frac{(x_j - y_j)^2}{x_j + y_j}\,.$$

The Cauchy-Schwarz Inequality gives

$$\begin{aligned}\left(\sum_{j=1}^{n}(x_j - y_j)\right)^2 &= \left(\sum_{j=1}^{n} \sqrt{x_j + y_j}\,\frac{x_j - y_j}{\sqrt{x_j + y_j}}\right)^2 \\ &\le \left(\sum_{j=1}^{n}(x_j + y_j)\right)\left(\sum_{j=1}^{n} \frac{(x_j - y_j)^2}{x_j + y_j}\right)\,,\end{aligned}$$

or

$$(X_n - Y_n)^2 \le (X_n + Y_n)\, Z_n\,. \qquad (1)$$

Using the easily verified identities

$$4A_n = X_n + Y_n - Z_n\,, \qquad B_n = \frac{X_n Y_n}{X_n + Y_n}\,,$$

$$C_n = \frac{X_n^2 + Y_n^2}{X_n + Y_n}\,, \quad \text{and} \quad 2D_n = X_n + Y_n + Z_n\,,$$

we see that the inequalities $A_n \le B_n$ and $C_n \le D_n$ both follow from (1), while the inequality $2B_n \le C_n$ is an immediate consequence of the AM–GM Inequality.

Also solved by MICHEL BATAILLE, Rouen, France; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; PANOS E. TSAOUSSOGLOU, Athens, Greece; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

All the solvers, except Tsaoussoglou and the proposer, had solutions similar to the one given above. Tsaoussoglou proved stronger inequalities than the proposed 1. *and* 2., *to obtain his solution.*

2393. [1998: 504] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*
Suppose that a, b, c and d are positive real numbers. Prove that

1. $$\Big((a+b)(b+c)(c+d)(d+a)\Big)^{3/2} \;\geq\; 4abcd(a+b+c+d)^2,$$

2. $$\Big((a+b)(b+c)(c+d)(d+a)\Big)^{3} \;\geq\; 16(abcd)^2 \prod_{\substack{a,b,c,d\\ \text{cyclic}}} (2a+b+c).$$

Solution by Jun-hua Huang, the Middle School Attached To Hunan Normal University, Changsha, China.

1. The inequality is equivalent to

$$[(a+b)(b+c)(c+d)(d+a)]^3 \;\geq\; 16(abcd)^2(a+b+c+d)^4\,,$$

or, dividing by $(abcd)^6$,

$$\left[\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{c}+\frac{1}{d}\right)\left(\frac{1}{d}+\frac{1}{a}\right)\right]^3 \;\geq\; 16\left(\frac{1}{bcd}+\frac{1}{acd}+\frac{1}{abd}+\frac{1}{abc}\right)^4\,.$$

Let $x=a^{-1}$, $y=b^{-1}$, $z=c^{-1}$, and $w=d^{-1}$. The inequality becomes

$$[(x+y)(y+z)(z+w)(w+x)]^3 \;\geq\; 16(xyz+yzw+zwx+wxy)^4\,.$$

We prove it as follows. Applying the Geometric Mean–Arithmetic Mean Inequality, we obtain

$$\begin{aligned}
&4(xyz+yzw+zwx+wxy)^2\\
&\quad= 4\left[xw(y+z)+yz(x+w)\right]^2\\
&\quad= 4\left[\sqrt{xw}\sqrt{xw}(y+z)+\sqrt{yz}\sqrt{yz}(x+w)\right]^2\\
&\quad\leq \left[\sqrt{xw}(x+w)(y+z)+\sqrt{yz}(y+z)(x+w)\right]^2\\
&\quad= (x+w)^2(y+z)^2(\sqrt{xw}+\sqrt{yz})^2\\
&\quad= (x+w)^2(y+z)^2(xw+yz+2\sqrt{xwyz})\\
&\quad\leq (x+w)^2(y+z)^2(xw+yz+xy+wz)\\
&\quad= (x+w)^2(y+z)^2(x+z)(w+y)\,.
\end{aligned}$$

Hence

$$4(xyz+yzw+zwx+wxy)^2 \;\le\; (x+w)^2(y+z)^2(x+z)(w+y)\,.$$

Similarly,

$$4(xyz+yzw+zwx+wxy)^2 \;\le\; (x+w)(y+z)(x+z)^2(w+y)^2\,.$$

The result follows by multiplying the last two inequalities.

2. The inequality follows easily from the first one, because

$$\prod_{\text{cyclic}}(2a+b+c) \;\le\; \left(\frac{1}{4}\sum_{\text{cyclic}}(2a+b+c)\right)^4 \;=\; (a+b+c+d)^4\,,$$

by the Geometric Mean–Arithmetic Mean Inequality.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.