SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2324. [1998: 109, 1999: 50] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Find the exact value of $\sum_{n=1}^{\infty} \frac{1}{u_n}$, where u_n is given by the recurrence

$$u_n = n! + \left(\frac{n-1}{n}\right) u_{n-1},$$

with the initial condition $u_1 = 2$.

Solution by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We first show that $u_n = n! + (n-1)!$. This is true for $n = 1$, since $u_1 = 2$. Suppose that $u_n = n! + (n-1)!$ for some $n \geq 1$. Then

$$\begin{aligned} u_{n+1} &= (n+1)! + \frac{n}{n+1}(n! + (n-1)!) \\ &= (n+1)! + \frac{n}{n+1}(n-1)!(n+1) \\ &= (n+1)! + n!, \end{aligned}$$

completing the induction.

Hence, for all $m \geq 1$, we have

$$\begin{aligned} \sum_{n=1}^{m} \frac{1}{u_n} &= \sum_{n=1}^{m} \frac{1}{n! + (n-1)!} = \sum_{n=1}^{m} \frac{1}{(n-1)!(n+1)} \\ &= \sum_{n=1}^{m} \frac{n+1-1}{(n+1)!} = \sum_{n=1}^{m} \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right) = 1 - \frac{1}{(m+1)!}, \end{aligned}$$

from which it follows that $\sum_{n=1}^{\infty} \frac{1}{u_n} = \lim_{m\to\infty} \sum_{n=1}^{m} \frac{1}{u_n} = 1.$

Also solved by the proposer.

2339. [1998: 234] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

A rhombus $ABCD$ has incircle Γ, and Γ touches AB at T. A tangent to Γ meets sides AB, AD at P, S respectively, and the line PS meets BC, CD at Q, R respectively. Prove that

(a) $\dfrac{1}{PQ} + \dfrac{1}{RS} = \dfrac{1}{BT}$,

and

(b) $\dfrac{1}{PS} - \dfrac{1}{QR} = \dfrac{1}{AT}$.

Solution by Michael Lambrou, University of Crete, Crete, Greece.

If PS meets Γ at U we show, slightly generalizing (and correcting) the situation, that, depending on the position of U, we have

$$\frac{1}{PQ} \pm \frac{1}{RS} \;=\; \pm\frac{1}{BT}\,,$$

$$\frac{1}{PS} \pm \frac{1}{QR} \;=\; \pm\frac{1}{AT}\,,$$

with an appropriate choice of $\pm$ each time.

Using as coordinate axes the two (perpendicular) diagonals, meeting at O say, we may assume that A, B, C, D have coordinates $A(a,0)$, $B(0,b)$, $C(-a,0)$, $D(0,-b)$ respectively, where a, $b > 0$. The radius of Γ is OT which, being perpendicular to AB is the altitude of OAB. Writing $OT = h$ we have $h \cdot AB = OA \cdot OB = ab$, and the coordinates of U are of the form $(h\cos\theta, h\sin\theta)$. As $PS \perp OU$, the equation of PS is clearly

$$y\sin\theta \;=\; -x\cos\theta + h\,.$$

Line AB is $\frac{x}{a} + \frac{y}{b} = 1$, so the coordinates of P, being on AB and PS, are easily seen to be

$$(x_P, y_P) \;=\; \frac{1}{b\sin\theta - a\cos\theta}\Big(a(b\sin\theta - h), b(h - a\cos\theta)\Big)\,. \qquad (1)$$

Similarly (or quicker, by replacing a by $-a$) we find the coordinates of Q as

$$(x_Q, y_Q) \;=\; \frac{1}{b\sin\theta + a\cos\theta}\Big(-a(b\sin\theta - h), b(h + a\cos\theta)\Big)\,. \qquad (2)$$

Assume now that the position of U is such that PS cuts, say, AB and BC internally (the rest of the cases follow by a trivial adaptation of our argument here) and so the coordinates of P are both positive [and Q has a negative x–coordinate and a positive y–coordinate]. By (1) we have $b\sin\theta > h > a\cos\theta$. Thus

$$PQ \;=\; \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2} \;=\; \frac{2ab(b\sin\theta - h)}{(b\sin\theta)^2 - (a\cos\theta)^2}\,.$$

Similarly

$$RS \;=\; \frac{2ab(b\sin\theta + h)}{(b\sin\theta)^2 - (a\cos\theta)^2}\,,$$

and so

$$\begin{aligned}\frac{1}{RS}-\frac{1}{PQ} &= \frac{(b\sin\theta)^2-(a\cos\theta)^2}{2ab}\left(\frac{1}{b\sin\theta+h}-\frac{1}{b\sin\theta-h}\right)\\ &= -\frac{(b\sin\theta)^2-(a\cos\theta)^2}{2ab}\cdot\frac{2h}{(b\sin\theta)^2-h^2}\,.\end{aligned}$$

Writing $h^2=\frac{a^2b^2}{(AB)^2}=\frac{a^2b^2}{a^2+b^2}$, this simplifies to $-\frac{(a^2+b^2)h}{ab^3}$ which equals $-\frac{1}{BT}$ as $BT\cdot AB=OB^2=b^2$, as required.

The proof of $\frac{1}{PS}\pm\frac{1}{QR}=\pm\frac{1}{AT}$ is similar and routine.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK.

Bradley and Lambrou were the only readers who recognized that the problem was incorrect as stated. There were eleven partial solutions.

2346. [1998: 236] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

The angles of $\triangle ABC$ satisfy $A>B\geq C$. Suppose that H is the foot of the perpendicular from A to BC, that D is the foot of the perpendicular from H to AB, that E is the foot of the perpendicular from H to AC, that P is the foot of the perpendicular from D to BC, and that Q is the foot of the perpendicular from E to AB.

Prove that A is acute, right or obtuse according as $\overline{AH}-\overline{DP}-\overline{EQ}$ is positive, zero or negative.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

As noted by the proposer, this is a generalization of a problem he proposed in *College Mathematics Journal* **28**, no. 2, March 1997, 145-6.

Let $S=AH-DP-EQ$. Then, since $\angle DHA=\angle PDH=\angle B$, we have

$$DP \;=\; DH\cos B \;=\; AH\cos^2 B\,.$$

Similarly $EQ=AH\cos^2 C$, so,

$$\begin{aligned}S &= AH-AH\cos^2 B-AH\cos^2 C \;=\; AH(1-\cos^2 B-\cos^2 C)\\ &= AH(\sin^2 B-\cos^2 C) \;=\; AH\left(\frac{1-\cos 2B}{2}-\frac{1+\cos 2C}{2}\right)\\ &= -AH\cos(B+C)\cos(B-C)\,,\end{aligned}$$

or $S=AH\cos A\cos(B-C)$, where $\cos(B-C)>0$, and hence

$$A \text{ is acute if } S>0\,,\qquad A \text{ is right if } S=0\,,\qquad A \text{ is obtuse if } S<0\,.$$

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER

J. BRADLEY, Clifton College, Bristol, UK; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VJEKOSLAV KOVAČ, student, University of Zagreb, Croatia; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; TOSHIO SEIMIYA, Kawasaki, Japan; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

2348. [1998: 236] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Without the use of trigonometrical formulae, prove that

$$\sin(54^\circ) = \tfrac{1}{2} + \sin(18^\circ) \ .$$

I. *Solution by Richard I. Hess, Rancho Palos Verdes, California, USA.*
From Figure 1 below where we assume that $AC = 1$ we have $BC = \sin 54^\circ$, $AM = MC = MB = BN = \frac{1}{2}$, $\angle MBN = 36^\circ$, $\angle MBQ = \angle QBN = 18^\circ$. Thus $MQ = \frac{1}{2}\sin 18^\circ$.

Now $\angle CMN = 36^\circ$ which implies $CN = MN = 2MQ = \sin 18^\circ$. Since $BC = BN + CN$ we obtain $\sin 54^\circ = \frac{1}{2} + \sin 18^\circ$.

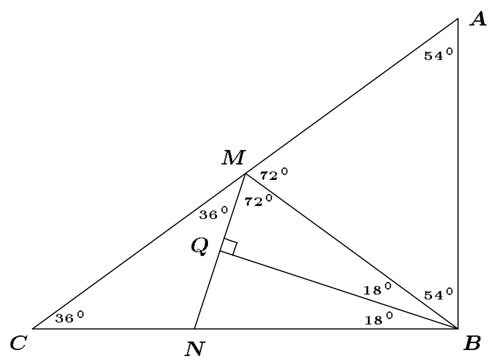

Figure 1.

II. *Solution by Nikolaos Dergiades, Thessaloniki, Greece.*
On the circle centred at O with radius r having a diameter AD we take the points B and C such that $\mathrm{arc}AB = \mathrm{arc}BC = 36^\circ$ (see Figure 2 below). Then $\mathrm{arc}CD = 108^\circ$, so that

$$AB \;=\; BC \;=\; 2r\sin 18^\circ\,, \qquad CD \;=\; 2r\sin 54^\circ\,.$$

Since $\angle BOA = \angle CDA$, we see that $BO \parallel CD$. Choosing E on CD such that $BE \parallel AD$, we see that the quadrilateral $BEDO$ is a rhombus; thus $ED = r$. Since $\angle CEB = \angle CDA = 36^\circ$, we have $\angle CBA = 144^\circ$, $\angle OBA = \angle OBC = 72^\circ$, and $\angle EBO = 36^\circ$. It then follows that $\angle CBE = 36^\circ$ and $CE = CB = 2r\sin 18^\circ$. Thus

$$2r\sin 54^\circ \;=\; CD \;=\; CE + ED \;=\; 2r\sin 18^\circ + r\,,$$

$$\text{or} \qquad \sin 54^\circ \;=\; \frac{1}{2} + \sin 18^\circ\,.$$

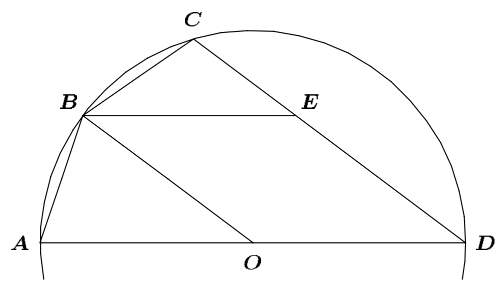

Figure 2.

III. *Solution by Jeremy Young, student, Nottingham High School, England.*

Set $x = \sin 18°$ and set $y = \sin 54°$. First consider the triangle in Figure 3 below, where we set $AC = 1$. Then $BC = x$ and $CD = 1$. Thus $AB = \sqrt{1-x^2}$ and $AD = \sqrt{1-x^2+(1+x)^2} = \sqrt{2(1+x)}$. Then

$$\sin 54° = y = \frac{1+x}{\sqrt{2(1+x)}} \implies 2y^2 - 1 = x\,.$$

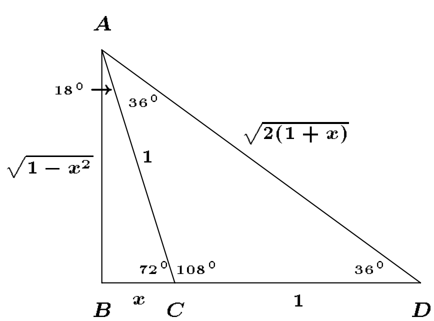

Figure 3.

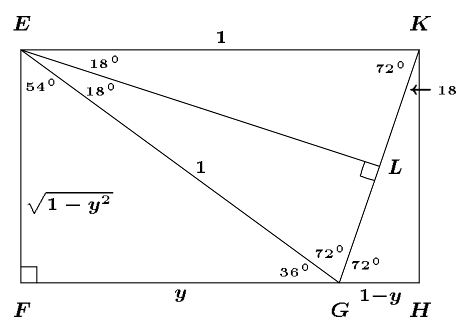

Figure 4.

In Figure 4 above where $EG = 1$, we have $FG = y$, $EK = 1$, $GH = 1-y$, and $EF = \sqrt{1-y^2} = KH$. Then

$$LG = x = \frac{1}{2}\sqrt{(1-y)^2+1-y^2} = \frac{1}{2}\sqrt{2-2y}\,.$$

which yields $1-2x^2 = y$. Adding this to the previous result ($2y^2-1=x$) gives

$$\begin{aligned} 2(y+x)(y-x) &= y+x\,, \\ 2(y-x) &= 1\,, \qquad \text{since } x, y > 0 \implies y+x \neq 0, \\ y &= x + \frac{1}{2}\,. \end{aligned}$$

Also solved by SAM BAETHGE, Nordheim, Texas, USA; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; M. BENITO and E. FERNÁNDEZ, Logroño, Spain; CHRISTOPHER J. BRADLEY, *Clifton College, Bristol, UK; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; DIANE and ROY DOWLING, University of Manitoba, Winnipeg, Manitoba; RICHARD EDEN, student, Ateneo de Manila University, Philippines; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ANGEL JOVAL ROQUET, Spain; GEOFFREY A. KANDALL, Hamden, CT; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; VJEKOSLAV KOVAČ, student, University of Zagreb, Croatia; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; VEDULA N. MURTY, Dover, PA, USA; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; BOB PRIELIPP, University of Wisconsin–Oshkosh, Wisconsin, USA; TOSHIO SEIMIYA, Kawasaki, Japan; J. SUCK, Essen, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposer.*

2351. [1998: 302] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.*

A triangle with integer sides is called ***Heronian*** if its area is an integer.

Does there exist a Heronian triangle whose sides are the arithmetic, geometric and harmonic means of two positive integers?

Solution by Michael Lambrou, University of Crete, Crete, Greece.

We show, slightly generalizing the given situation, that no triangle with sides a, $\sqrt{ac}$, c, with a, $c \in \mathbb{N}$ can have integral area. Indeed, if the area $\Delta \in \mathbb{N}$ we would have by Heron's formula:

$$\begin{aligned}\Delta^2 &= \frac{1}{16}(a+\sqrt{ac}+c)(a-\sqrt{ac}+c)(a+\sqrt{ac}-c)(-a+\sqrt{ac}+c) \\ &= \frac{1}{16}\left[4a^2c^2-(a^2+c^2-ac)^2\right] .\end{aligned}$$

If $(a,c)=t$ so that $a=pt$, $c=qt$ for some p, $q \in \mathbb{N}$ with $(p,q)=1$ we obtain

$$\frac{4\Delta}{t^2} = \sqrt{4p^2q^2-(p^2+q^2-pq)^2} .$$

But the left hand side is rational; so $4p^2q^2-(p^2+q^2-pq)^2$ must be an integer perfect square, say T^2. But this is impossible because p^2+q^2-pq is odd (as p, q are not both even) and so

$$T^2 = 4p^2q^2-(p^2+q^2-pq)^2 \equiv 0-1 \equiv 3 \pmod 4 ,$$

giving a contradiction, as no square is congruent to 3 modulo 4. This completes the proof that $\Delta \notin \mathbb{N}$.

Also solved by DUANE BROLINE, Eatsern Illinois University, Charleston, Illinois. USA; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

2352. [1998: 302] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Determine the shape of $\triangle ABC$ if

$$\cos A\cos B\cos(A-B)+\cos B\cos C\cos(B-C)$$
$$+\cos C\cos A\cos(C-A)+2\cos A\cos B\cos C \;=\; 1\,.$$

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

Since $\cos A\cos B-\sin A\sin B=\cos(A+B)=-\cos C$, we have $(\cos A\cos B+\cos C)^2=\left(1-\cos^2 A\right)\left(1-\cos^2 B\right)$, or

$$\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C \;=\; 1\,. \qquad (1)$$

Also,

$$\begin{aligned}\cos(A+B)\cos(A-B) &= \cos^2 A\cos^2 B-\sin^2 A\sin^2 B\\ &= \cos^2 A\cos^2 B-\left(1-\cos^2 A\right)\left(1-\cos^2 B\right)\\ &= \cos^2 A+\cos^2 B-1\,. \qquad (2)\end{aligned}$$

Using (2), we have

$$\begin{aligned}\cos A\cos B\cos(A-B) &= \tfrac{1}{2}\left(\cos(A-B)+\cos(A+B)\right)\cos(A-B)\\ &= \tfrac{1}{2}\left(\cos^2(A-B)+\cos^2 A+\cos^2 B-1\right)\\ &= \tfrac{1}{2}\left(\cos^2 A+\cos^2 B-\sin^2(A-B)\right)\,.\end{aligned}$$

Hence, the given equality is equivalent to

$$\begin{aligned}&\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C\\ &\quad -\tfrac{1}{2}\left(\sin^2(A-B)+\sin^2(B-C)+\sin^2(C-A)\right) \;=\; 1\,,\end{aligned}$$

which, in view of (1), becomes

$$\sin^2(A-B)+\sin^2(B-C)+\sin^2(C-A) \;=\; 0\,.$$

Hence, $A=B=C$, which means that $\triangle ABC$ is equilateral.

Also solved by MICHEL BATAILLE, Rouen, France; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; C. FESTRAETS-HAMOIR, Brussels, Belgium; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JUN-HUA HUANG, the Middle School Attached To Hunan Normal University, Changsha, China; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; VEDULA N. MURTY, Visakhapatnam, India; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; ARAM TANGBOONDOUANGJIT, Carnegie Mellon University, Pittsburgh, PA, USA; PARAYIOU THEOKLITOS, Limassol, Cyprus; PANOS E. TSAOUSSOGLOU, Athens, Greece; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer. There was also one incorrect solution.

Most solvers showed that the given equality is equivalent to either $\sum \cos^2(A-B) = 3$, *or to* $\sum \cos 2(A-B) = 3$, *both of which are clearly equivalent to* $\sum \sin^2(A-B) = 0$, *obtained in the solution above.*

Four solvers derived the equality $\prod \cos(A-B) = 1$, *from which the conclusion also follows immediately. The solution given above is self-contained, and does not use any known identity (for example,* $\sum \cos(2A) = -1 - 4\prod \cos A$, *which was used by a few solvers) beyond the elementary formula transforming product into sum. Lambrou obtained the slightly stronger result that*

$$\left(\sum \cos A \cos B \cos(A-B)\right) + 2\cos A \cos B \cos C \;\leq\; 1\,,$$

with equality holding if and only if $\triangle ABC$ *is equilateral.*

2353. [1998: 302] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Determine the shape of $\triangle ABC$ if

$$\sin A \sin B \sin(A-B) + \sin B \sin C \sin(B-C) + \sin C \sin A \sin(C-A) \;=\; 0\,.$$

Solution by Gerry Leversha, St. Paul's School, London, England (slightly modified by the editor).

Since

$$\begin{aligned} &\sin A \sin B \sin(A-B) \\ &\quad= \sin^2 A \sin B \cos B - \sin^2 B \sin A \cos A \\ &\quad= \tfrac{1}{4}\left((1-\cos(2A))\sin(2B) - (1-\cos(2B))\sin(2A)\right) \\ &\quad= \tfrac{1}{4}\left((\sin(2B) - \sin(2A)) + \sin(2(A-B))\right)\,, \end{aligned}$$

the given equality is equivalent to

$$\sum \sin(2(A-B)) \;=\; 0\,. \tag{1}$$

Using elementary formulae transforming sums into products, we have

$$\sin(2(A-B)) + \sin(2(B-C)) \;=\; 2\sin(A-C)\cos(A+C-2B)\,,$$

and thus

$$\begin{aligned} &\sin(2(A-B)) + \sin(2(B-C)) + \sin(2(C-A)) \\ &\quad= 2\sin(A-C)\cos(A+C-2B) + 2\sin(C-A)\cos(C-A) \\ &\quad= 2\sin(C-A)\left(\cos(C-A) - \cos(A+C-2B)\right) \\ &\quad= -4\sin(A-B)\sin(B-C)\sin(C-A)\,. \end{aligned} \tag{2}$$

From (1) and (2), we see that the given equality is equivalent to

$$\sin(A-B)\sin(B-C)\sin(C-A) \;=\; 0\,,$$

which holds if and only if at least two of A, B, C are equal to each other. Therefore, the given equality holds if and only if $\triangle ABC$ is isosceles.

Also solved by MICHEL BATAILLE, Rouen, France; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; NIKOLAOS DERGIADES, Thessaloniki, Greece; C. FESTRAETS-HAMOIR, Brussels, Belgium; PETER HURTHIG, Columbia College, Burnaby, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JUN-HUA HUANG, the Middle School Attached To Hunan Normal University, Changsha, China; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; VEDULA N. MURTY, Visakhapatnam, India; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; ARAM TANGBOONDOUANGJIT, Carnegie Mellon University, Pittsburgh, PA, USA; PARAYIOU THEOKLITOS, Limassol, Cyprus; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer. There were also two incorrect solutions.

The solution given above is interesting since it shows that the conclusion $A = B$ or $B = C$ or $C = A$ does not depend on the assumption that $A + B + C = \pi$; that is, that A, B and C are the three angles of a triangle.

2354. [1998: 302] *Proposed by Herbert Gülicher, Westfalische Wilhelms-Universität, Münster, Germany.*

In triangle $P_1P_2P_3$, the line joining $P_{i-1}P_{i+1}$ meets a line σ_j at the point $S_{i,j}$ ($i, j = 1, 2, 3$, all indices taken modulo 3), such that all the points $S_{i,j}$, P_k are distinct, and different from the vertices of the triangle.

1. Prove that if all the points $S_{i,i}$ [note the correction] are non-collinear, then any two of the following conditions imply the third condition:

   (a) $\dfrac{P_1S_{3,1}}{S_{3,2}P_2} \cdot \dfrac{P_2S_{1,2}}{S_{1,3}P_3} \cdot \dfrac{P_3S_{2,3}}{S_{2,1}P_1} = -1$;

   (b) $\dfrac{S_{1,2}S_{1,1}}{S_{1,1}S_{1,3}} \cdot \dfrac{S_{2,3}S_{2,2}}{S_{2,2}S_{2,1}} \cdot \dfrac{S_{3,1}S_{3,3}}{S_{3,3}S_{3,2}} = 1$;

   (c) σ_1, σ_2, σ_3 are either concurrent or parallel.

2. Prove further that (a) and (b) are equivalent if the $S_{i,i}$ are collinear.

Here, AB denotes the signed length of the directed line segment $[AB]$.

Solution by Günter Pickert, Giessen, Germany.

Define $S_i = S_{ii}$ and $Q_i = \sigma_i \bigcap S_{i-1}S_{i+1}$. If $\sigma_i || S_{i-1}S_{i+1}$ (so that Q_i is at infinity), then replace $\dfrac{S_{i+1}Q_i}{Q_iS_{i-1}}$ by -1.

(a) From Menelaus' Theorem applied to $\triangle S_{i+1}S_{i-1}P_i$ and the line σ_i $(1 = 1, 2, 3)$,

$$\frac{S_{i+1}Q_i}{Q_iS_{i-1}} \cdot \frac{S_{i-1}S_{i-1,i}}{S_{i-1,i}P_i} \cdot \frac{P_iS_{i+1,i}}{S_{i+1,i}S_{i+1}} = -1 . \qquad (1)$$

Define

$$a = \prod_{i=1}^{3} \frac{P_i S_{i-1,i}}{S_{i-1,i+1} P_{i+1}}, \quad b = \prod_{i=1}^{3} \frac{S_{i,i+1} S_i}{S_i S_{i,i-1}}, \quad c = \prod_{i=1}^{3} \frac{S_{i+1} Q_i}{Q_i S_{i-1}}.$$

The given conditions (a), (b), (c) say (respectively) that $a = -1$, $b = 1$, and $c = 1$. [The value $c = 1$ is just Ceva's Theorem applied to $\triangle S_1 S_2 S_3$, and extended to allow the possibility that one or two of the Q_i are at infinity.] From (1) we get

$$-1 = c \cdot \frac{\prod S_{i-1} S_{i-1,i}}{\prod S_{i+1,i} S_{i+1}} \cdot \frac{\prod P_i S_{i+1,i}}{\prod S_{i-1,i} P_i} = c \cdot b \cdot a^{-1},$$

so that $-a = b \cdot c$.

Two of c, b, $-a$ are equal to 1 if and only if two of (a), (b), (c) hold, in which case the third product is 1 and the third condition holds too.

(b) If the S_i are collinear then $Q_i = S_i$ and $c = 1$; therefore (a) is equivalent to (b).

Also solved by JUN-HUA HUANG, the Middle School Attached To Hunan Normal University, Changsha, China; and the proposer.

Huang's solution is much the same as our featured solution. Gülicher based his on his problem 64 in Math. Semesterber. **40** *(1992) 91–92.*

2355. [1998: 303] *Proposed by G.P. Henderson, Garden Hill, Campbellcroft, Ontario.*

For $j = 1, 2, \ldots, m$, let A_j be non-collinear points with $A_j \neq A_{j+1}$. Translate every even-numbered point by an equal amount to get new points A'_2, A'_4, $\ldots$, and consider the sequence B_j, where $B_{2i} = A'_{2i}$ and $B_{2i-1} = A_{2i-1}$. The last member of the new sequence is either A_{m+1} or A'_{m+1} according as m is even or odd.

Find a necessary and sufficient condition for the length of the path $B_1 B_2 B_3 \ldots B_m$ to be greater than the length of the path $A_1 A_2 A_3 \ldots A_m$ for all such non-zero translations.

CRUX 1985 [1994: 250; 1995: 280] provides an example of such a configuration. There, $m = 2n$, the A_i are the vertices of a regular $2n$–gon and $A_{2n+1} = A_1$.

Solution by the proposer.

(*Editor's note:* In the original submission the paths above were given as $B_1 B_2 \cdots B_{m+1}$ and $A_1 A_2 \cdots A_{m+1}$. Since this makes the upper bounds on summations simpler we have opted to present the solution with these endpoints, although logically it makes no difference.)

We use the same letter for a point P and a vector $\overrightarrow{P}$. Let the translation in the problem be $\overrightarrow{X}$. Then

$$\overrightarrow{B_{2k}} = \overrightarrow{A'_{2k}} = \overrightarrow{A_{2k}} + \overrightarrow{X}, \quad k = 1, 2, \ldots, \lfloor (m+1)/2 \rfloor ,$$

and the lengths of the new segments are

$$\begin{aligned} \left|\overrightarrow{B_2} - \overrightarrow{B_1}\right| &= \left|\overrightarrow{A'_2} - \overrightarrow{A_1}\right| &&= \left|\overrightarrow{X} + \overrightarrow{A_2} - \overrightarrow{A_1}\right| , \\ \left|\overrightarrow{B_3} - \overrightarrow{B_2}\right| &= \left|\overrightarrow{A_3} - \overrightarrow{A'_2}\right| &&= \left|\overrightarrow{X} - \left(\overrightarrow{A_3} - \overrightarrow{A_2}\right)\right| , \\ & &&\cdots . \end{aligned}$$

The lengths of the paths are

$$L' = \sum_{j=1}^{m} \left|\overrightarrow{B_{j+1}} - \overrightarrow{B_j}\right| = \sum_{j=1}^{m} \left|\overrightarrow{X} - (-1)^j \left(\overrightarrow{A_{j+1}} - \overrightarrow{A_j}\right)\right|$$

and $$L = \sum_{j=1}^{m} \left|\overrightarrow{A_{j+1}} - \overrightarrow{A_j}\right| .$$

Now L' is to be a minimum at $\overrightarrow{X} = \overrightarrow{0}$. If we form the partial derivatives of L' with respect to the components of $\overrightarrow{X}$ and set $\overrightarrow{X} = \overrightarrow{0}$, we get

$$\sum_{j=1}^{m} \frac{(-1)^j \left(\overrightarrow{A_{j+1}} - \overrightarrow{A_j}\right)}{\left|\overrightarrow{A_{j+1}} - \overrightarrow{A_j}\right|} = \overrightarrow{0} . \tag{1}$$

The minimum of a sum like L' does not always occur at a point where the derivatives are zero. However, in this case we will prove that (1) actually is the required condition.

That is, the sum of unit vectors along the odd segments $\overrightarrow{A_2} - \overrightarrow{A_1}$, $\overrightarrow{A_4} - \overrightarrow{A_3}$, ..., is equal to the sum of unit vectors along the even segments. The lengths of the segments are arbitrary provided they are greater than zero. It is only their directions that matter. In the case of the regular $2n$-gon, both sums are $\overrightarrow{0}$ because each consists of unit vectors parallel to the sides of a regular n-gon.

Set $\quad d_j = \left|\overrightarrow{A_{j+1}} - \overrightarrow{A_j}\right| > 0 \quad$ and $\quad \overrightarrow{C_j} = (-1)^j \left(\overrightarrow{A_{j+1}} - \overrightarrow{A_j}\right) / d_j ,$

a unit vector parallel to the jth segment. Equation (1) becomes

$$\sum_{j=1}^{m} \overrightarrow{C_j} = \overrightarrow{0} . \tag{2}$$

The lengths of the paths are $\displaystyle L' = \sum_{j=1}^{m} \left|\overrightarrow{X} - d_j \overrightarrow{C_j}\right|$ and $\displaystyle L = \sum_{j=1}^{m} d_j$.

To prove the necessity of (2), assume $L' \geq L$ for all $\overrightarrow{X}$.

Set $y_j = \left|\overrightarrow{X} - d_j\overrightarrow{C_j}\right| - d_j$. Then $\sum_{j=1}^{m} y_j \geq 0$. Squaring both sides of $\left|\overrightarrow{X} - d_j\overrightarrow{C_j}\right| = d_j + y_j$ and dividing by d_j, we get

$$\overrightarrow{X^2}/d_j - 2\overrightarrow{X}\cdot\overrightarrow{C_j} = 2y_j + y_j^2/d_j$$
$$\overrightarrow{X^2}\sum_{j=1}^{m}(1/d_j) - 2\overrightarrow{X}\cdot\sum_{j=1}^{m}\overrightarrow{C_j} = 2\sum_{j=1}^{m} y_j + \sum_{j=1}^{m}(y_j^2/d_j) \geq 0\,.$$

When we set $\overrightarrow{X} = \sum_{j=1}^{m}\overrightarrow{C_j}\Big/\sum_{j=1}^{m}(1/d_j)$, this becomes $\left(\sum_{j=1}^{m}\overrightarrow{C_j}\right)^2 \leq 0$, and we have (2).

Given (2) we are to prove that $L' > L$ for all $\overrightarrow{X} \neq \overrightarrow{0}$. We have

$$\left|\overrightarrow{X} - d_j\overrightarrow{C_j}\right| = \left|\overrightarrow{X} - d_j\overrightarrow{C_j}\right|\left|\overrightarrow{C_j}\right| \geq \left|\left(\overrightarrow{X} - d_j\overrightarrow{C_j}\right)\cdot\overrightarrow{C_j}\right|\,; \qquad (3)$$
$$L' \geq \sum_{j=1}^{m}\left|\left(\overrightarrow{X} - d_j\overrightarrow{C_j}\right)\cdot\overrightarrow{C_j}\right| \geq \left|\sum_{j=1}^{m}\left(\overrightarrow{X} - d_j\overrightarrow{C_j}\right)\cdot\overrightarrow{C_j}\right|$$
$$= \left|\overrightarrow{X}\cdot\sum_{j=1}^{m}\overrightarrow{C_j} - \sum_{j=1}^{m} d_j\right| = L\,.$$

If $L' = L$ for some $\overrightarrow{X} \neq \overrightarrow{0}$, there is equality in (3) for all j. Then $\overrightarrow{X} - d_j\overrightarrow{C_j} = c_j\overrightarrow{C_j}$, where $c_j \neq -d_j$; thus $\overrightarrow{C_j} = \overrightarrow{X}/(c_j + d_j)$, and the points are collinear.

All of the above is valid in a Euclidean space of any number of dimensions.

There were no other solutions submitted.