PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and **separate** *standard $8\frac{1}{2}$"×11" or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 January 2000**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2446. Correction: *Proposed by Catherine Shevlin, Wallsend upon Tyne, England.*

A sequence of integers, $\{a_n\}$ with $a_1 > 0$, is defined by

$$a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } a(n) \equiv 0 \pmod 4, \\ 3a_n + 1 & \text{if } a(n) \equiv 1 \pmod 4, \\ 2a_n - 1 & \text{if } a(n) \equiv 2 \pmod 4, \\ \frac{a_n+1}{4} & \text{if } a(n) \equiv 3 \pmod 4. \end{cases}$$

Prove that there is an integer m such that $a_m = 1$.

(Compare **OQ**.117 in *OCTOGON*, vol 5, No. 2, p. 108.)

2451. *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Construct an infinite sequence, $\{A_n\}$, of infinite subsets of $\mathbb{N}$ with the following properties:

(a) the intersection of any two distinct sets A_n and A_m is a singleton;

(b) the singleton in (a) is a different one if at least one of the distinct sets A_n, A_m, is changed (so the new pair is again distinct);

(c) every natural number is the intersection of (exactly) one pair of distinct sets as in (a).

2452. *Proposed by Antal E. Fekete, Memorial University of Newfoundland, St. John's, Newfoundland.*

Establish the following equalities:

(a) $$\sum_{n=1}^{\infty} \frac{(2n+1)^2}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{(2n+2)^2}{(2n+2)!}.$$

(b) $$\sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^3}{(n+1)!} = \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^4}{(n+1)!}.$$

(c) $$\sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^6}{(n+1)!} = \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^7}{(n+1)!}.$$

2453. *Proposed by Antal E. Fekete, Memorial University of Newfoundland, St. John's, Newfoundland.*

Establish the following equalities:

(a) $$\sum_{n=1}^{\infty} (-1)^n \frac{(2n+1)^3}{(2n+1)!} = -3 \sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n+1)!}.$$

(b) $$\sum_{n=1}^{\infty} (-1)^n \frac{(2n)^3}{(2n)!} = -3 \sum_{n=1}^{\infty} (-1)^n \frac{1}{(n+1)!}.$$

(c) $$\left(\sum_{n=1}^{\infty} (-1)^n \frac{(2n+1)^2}{(2n+1)!} \right)^2 + \left(\sum_{n=1}^{\infty} (-1)^n \frac{(2n)^2}{(2n)!} \right)^2 = 9.$$

2454. *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Three circles intersect each other orthogonally at pairs of points A and A', B and B', and C and C'. Prove that the circumcircles of $\triangle ABC$ and $\triangle AB'C'$ touch at A.

2455. *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Three equal circles, centred at A, B and C intersect at a common point P. The other intersection points are L (not on circle centre A), M (not on circle centre B), and N (not on circle centre C). Suppose that Q is the centroid of $\triangle LMN$, that R is the centroid of $\triangle ABC$, and that S is the circumcentre of $\triangle LMN$.

(a) Show that P, Q, R and S are collinear.

(b) Establish how they are distributed on the line.

2456. *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Two circles intersect orthogonally at P. A third circle touches them at Q and R. Let X be any point on this third circle. Prove that the circumcircles of $\triangle XPQ$ and $\triangle XPR$ intersect at 45°.

2457. *Proposed by Gerry Leversha, St. Paul's School, London, England.*

In quadrilateral $ABCD$, we have $\angle A + \angle B = 2\alpha < 180^\circ$, and $BC = AD$. Construct isosceles triangles DCI, ACJ and DBK, where I, J and K are on the other side of CD from A, such that $\angle ICD = \angle IDC = \angle JAC = \angle JCA = \angle KDB = \angle KBD = \alpha$.

(a) Show that I, J and K are collinear.

(b) Establish how they are distributed on the line.

2458. *Proposed by Nikolaos Dergiades, Thessaloniki, Greece.*

Let $ABCD$ be a quadrilateral inscribed in the circle centre O, radius R, and let E be the point of intersection of the diagonals AC and BD. Let P be any point on the line segment OE and let K, L, M, N be the projections of P on AB, BC, CD, DA respectively.

Prove that the lines KL, MN, AC are either parallel or concurrent.

2459. *Proposed by Vedula N. Murty, Visakhapatnam, India, modified by the editors.*

Let P be a point on the curve whose equation is $y = x^2$. Suppose that the normal to the curve at P meets the curve again at Q. Determine the minimal length of the line segment PQ.

2460. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Let $y(x) = \sqrt[3]{x + \sqrt{x^2 - 1}} + \sqrt[3]{x - \sqrt{x^2 - 1}}$ for $0 \le x \le 1$.

(a) Show that $y(x)$ is real valued.

(b) Find an infinite sequence $\{x_n\}_{n=0}^{\infty}$ such that $y(x_n)$ can be expressed in terms of square roots only.

2461. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

Suppose that x_0, x_1, ..., x_n are integers which satisfy $x_0 > x_1 > \ldots > x_n$. Let

$$F(x) = \sum_{k=0}^{n} a_k x^{n-k}, \qquad a_k \in \mathbb{R}, \, a_0 = 1.$$

Prove that at least one of the numbers $|F(x_k)|$, $(k = 0, 1, \ldots, n)$ is greater than $\frac{n!}{2^n}$.

2462. *Proposed by Vedula N. Murty, Visakhapatnam, India.*

If the angles, A, B, C of $\triangle ABC$ satisfy

$$\cos A \sin\frac{A}{2} = \sin\frac{B}{2}\sin\frac{C}{2},$$

prove that $\triangle ABC$ is isosceles.