MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to Mathematical Mayhem, Department of Mathematics, University of Toronto, 100 St. George St., Toronto, Ontario, Canada. M5S 3G3. The electronic address is still

mayhem@math.toronto.edu

Shreds and Slices

Another Combinatorial Proof

Dear Cyrus:

In the most recent issue of *Crux Mathematicorum with Mathematical Mayhem*, Dave Arthur's two-step combinatorial proof of the identity

$$(n-r)\binom{n+r-1}{r}\binom{n}{r} = n\binom{n+r-1}{2r}\binom{2r}{r} \qquad (1)$$

was presented, and a one-step combinatorial proof was sought. I have such a proof, but before I present the polished final version, let me explain how I was led to it.

Arthur's solution proceeded in two steps. He counted by two different methods:

(I) the number of ways of choosing two distinct sets A and B, each with r elements, from a set with $n+r-1$ elements, and

(II) the number of ways of choosing two distinct sets C and D, where C has one element and D has r elements, from a set with n elements.

It occurred to me that both (I) and (II) were ways to finish off the selection of three sets A, B, and C, with r, r, and 1 elements respectively, from a set with $n+r$ elements. We could first choose the small set C, and then choose sets A and B from the remaining $n+r-1$ elements as in (I), or we could choose one of the big sets A, and then choose sets C and $B=D$ from the remaining n elements as in (II). Counting both these methods yields the identity

$$(n+r)\left\{\binom{n+r-1}{2r}\binom{2r}{r}\right\} = \binom{n+r}{r}\left\{\binom{n}{r}(n-r)\right\}. \qquad (2)$$

We can easily see that this identity is equivalent to (1) by multiplying both sides of (2) by n and using the identity

$$n\binom{n+r}{r} = (n+r)\binom{n+r-1}{r} \qquad (3)$$

on the right-hand side.

There are two æsthetic problems with this: first, we derived the sought identity (1) with an extra factor of $n+r$ on both sides; second, we used the identity (3) which, though elementary, can be thought of as needing a combinatorial proof of its own (choosing disjoint subsets A and W of sizes r and 1 from a set with $n+r$ elements). The following proof of (1) is free of these defects.

Proof of (1). Suppose we have a circular arrangement of $n+r$ boxes, and we have r amber balls, r blue balls, 1 cyan ball, and 1 white ball. We want to arrange these balls in the boxes (arrangements that differ only by a rotation being regarded as the same), such that two balls are not allowed to be in the same box, except that the white ball may be in a box with a blue or cyan ball. We claim that both sides of (1) count the number of ways of doing this. The two counting methods are:

1. First place the white ball in a box; since rotated arrangements are considered the same, this can be done in only one way, but we have now used up our freedom to rotate the boxes. Next, place the r amber balls into r of the remaining $n+r-1$ empty boxes, which can be done in $\binom{n+r-1}{r}$ ways. Then, place the r blue balls into r of the n boxes that do not contain an amber ball (recall that a blue ball can coexist with a white ball), which can be done in $\binom{n}{r}$ ways. Finally, place the cyan ball in one of the $n-r$ boxes not containing an amber or blue ball (again, the white ball is allowed if it doesn't also contain a blue ball), which can be done in $n-r$ ways. The total number of such arrangements is thus

$$\binom{n+r-1}{r}\binom{n}{r}(n-r). \qquad (4)$$

2. This time, place the cyan ball first; as before, there is only one way to do this up to rotation. Next, select $2r$ of the remaining $n+r-1$ empty boxes (which we will momentarily fill with the $2r$ amber and blue balls), which can be done in $\binom{n+r-1}{2r}$ ways. Then, place the r amber balls into r of these selected $2r$ boxes, and place the r blue balls into the other r

selected boxes; this can be done in $\binom{2r}{r}$ ways. Finally, place the white ball into any of the n boxes that does not contain an amber ball, which can be done in n ways. The total number of such arrangements is thus

$$\binom{n+r-1}{2r}\binom{2r}{r}n\,. \tag{5}$$

Since (4) and (5) are the left- and right-hand sides of (1), respectively, this completes the proof. □

We remark that we could rephrase what we are counting as follows: we want to select three disjoint subsets A, B, and C, with r, r, and 1 elements respectively, from a set S with $n+r$ elements, and also label one element W of S that is not in A, except that selections that differ only by a cyclic permutation of S are to be considered equal.

Yours sincerely,

Greg Martin
Department of Mathematics, University of Toronto

Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from this issue be submitted in time for issue 6 of 2000.

High School Problems

Editor: Adrian Chan, 229 Old Yonge Street, Toronto, Ontario, Canada. M2P 1R5 <all238@sprint.com>

We correct problem H253, which first appeared in Issue 3.

H253. Find all real solutions to the equation

$$\sqrt{3x^2-12x+52}+\sqrt{2x^2-12x+162} \;=\; \sqrt{-x^2+6x+280}\,.$$

H257. Find all integers n such that $n^2 - 11n + 63$ is a perfect square.

H258. Solve in integers for x and y:

$$6(x! + 3) = y^2 + 5 .$$

H259. *Proposed by Alexandre Tritchtchenko, student, Carleton University, Ottawa, Ontario.*

Solve for x:

$$2^{m-n} \sin(2^n x) \prod_{i=1}^{m-n} \cos(2^{m-i} x) = 1 .$$

H260. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

Let $x_1, x_2, \ldots, x_m$ be real numbers. Prove that

$$\sum_{1 \le i < j \le n} \cos(x_i - x_j) \ge -\frac{n}{2} .$$

Advanced Problems

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A233. *Proposed by Naoki Sato, Mayhem Editor.*

In C81, we defined the following sequence: $a_0 = 0$, $a_1 = 1$, and $a_{n+1} = 4a_n - a_{n-1}$ for $n = 1, 2, \ldots$. This sequence exhibits the following curious property: For $n \ge 1$, if we set $(a, b, c) = (a_{n-1}, 2a_n, a_{n+1})$, then $ab + 1$, $ac + 1$, and $bc + 1$ are always perfect squares. For example, for $n = 3$, $(a, b, c) = (a_2, 2a_3, a_4) = (4, 30, 56)$, and indeed, $4 \cdot 30 + 1 = 11^2$, $4 \cdot 56 + 1 = 15^2$, and $30 \cdot 56 + 1 = 41^2$. Show that this property holds. Generalize, using the sequence defined by $a_0 = 0$, $a_1 = 1$, and $a_{n+1} = Na_n - a_{n-1}$, and the triples $(a, b, c) = (a_{n-1}, (N-2)a_n, a_{n+1})$, where N is an arbitrary integer.

A234. In triangle ABC, AC^2 is the arithmetic mean of BC^2 and AB^2. Show that $\cot^2 B \ge \cot A \cot C$. (Note: $\cot\theta = \cos\theta / \sin\theta$.)

(1997 Baltic Way)

A235. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

The convex polygon $A_1 A_2 \cdots A_n$ is inscribed in a circle of radius R. Let A be some point on this circumcircle, different from the vertices. Set

$a_i = AA_i$, and let b_i denote the distance from A to the line A_iA_{i+1}, $i = 1$, 2, ..., n, where $A_{n+1} = A_1$. Prove that

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq 2nR\,.$$

A236. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

For all positive integers n and positive reals x, prove the inequality

$$\frac{\binom{2n}{1}}{x+1} + \frac{\binom{2n}{3}}{x+3} + \cdots + \frac{\binom{2n}{2n-1}}{x+2n-1} < \frac{\binom{2n}{0}}{x} + \frac{\binom{2n}{2}}{x+2} + \cdots + \frac{\binom{2n}{2n}}{x+2n}\,.$$

Challenge Board Problems

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C87. *Proposed by Mark Krusemeyer, Carleton College.*

Find an example of three continuous functions $f(x)$, $g(x)$, and $h(x)$ from $\mathbb{R}$ to $\mathbb{R}$ with the property that exactly five of the six composite functions $f(g(h(x)))$, $f(h(g(x)))$, $g(f(h(x)))$, $g(h(f(x)))$, $h(f(g(x)))$, and $h(g(f(x)))$ are the same function and the sixth function is different.

C88.

(a) Let A be an $n \times n$ matrix whose entries are all either $+1$ or -1. Prove that $|\det A| \leq n^{n/2}$.

(b) It is conjectured that for there to exist an $n \times n$ matrix A whose entries are all either $+1$ or -1 and such that $|\det A| = n^{n/2}$, it is necessary and sufficient that $n = 1$, $n = 2$, or n is divisible by 4. Prove that this condition is necessary.

(c) Can you construct such a matrix, for n equal to a power of 2? For $n = 12$?

Problem of the Month

Jimmy Chui, student, University of Toronto

Problem. Let a, b, and c be positive real numbers. Prove that

$$a^a b^b c^c \;\geq\; (abc)^{(a+b+c)/3}\,.$$

(1995 CMO, Problem 2)

Solution I. Taking log of both sides, we see that we must prove

$$a\log a + b\log b + c\log c \;\geq\; \frac{a+b+c}{3}\log abc\,.$$

Let $f(x) = x\log x$. Then $f'(x) = \log x + 1$ and $f''(x) = 1/x$. We can see that $f''(x) > 0$ for $x > 0$. So, f is convex for $x > 0$. By Jensen's Inequality,

$$\begin{aligned} \frac{f(a)+f(b)+f(c)}{3} &\geq f\left(\frac{a+b+c}{3}\right) \\ \Longrightarrow\quad \frac{a\log a + b\log b + c\log c}{3} &\geq \frac{a+b+c}{3}\log\left(\frac{a+b+c}{3}\right)\,. \end{aligned}$$

By the AM-GM Inequality, $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$. Thus, we have

$$\begin{aligned} a\log a + b\log b + c\log c &\geq (a+b+c)\log\left(\frac{a+b+c}{3}\right) \\ &\geq (a+b+c)\log(\sqrt[3]{abc}) \\ &= \frac{a+b+c}{3}\log abc\,. \end{aligned}$$

Solution II. Recall Chebychev's Inequality: Let x_1, x_2, $\ldots$, x_n and y_1, y_2, $\ldots$, y_n be two real sequences, either both increasing or both decreasing. Then

$$\frac{x_1y_1 + x_2y_2 + \cdots + x_ny_n}{n} \;\geq\; \frac{x_1 + x_2 + \cdots + x_n}{n}\cdot\frac{y_1 + y_2 + \cdots + y_n}{n}\,.$$

(In other words, the average of the products is greater than or equal to the product of the averages.)

Without loss of generality, let $a \geq b \geq c$. Then $\log a \geq \log b \geq \log c$. By Chebychev's Inequality,

$$\frac{a\log a + b\log b + c\log c}{3} \;\geq\; \frac{a+b+c}{3}\cdot\frac{\log a + \log b + \log c}{3}\,,$$

which implies that $a\log a + b\log b + c\log c \;\geq\; \frac{a+b+c}{3}\log abc\,.$

Solution III. Recall the Weighted AM-GM-HM Inequality: Let x_1, x_2, ..., x_n be positive real numbers, and let w_1, w_2, ... , w_n be non-negative real numbers which sum to 1. Then

$$w_1x_1 + w_2x_2 + \cdots + w_nx_n \;\geq\; x_1^{w_1}x_2^{w_2}\cdots x_n^{w_n} \;\geq\; \frac{1}{\frac{w_1}{x_2} + \frac{w_2}{x_2} + \cdots + \frac{w_n}{x_n}}\,.$$

Take $x_1 = a$, $x_2 = b$, $x_3 = c$, $w_1 = a/(a+b+c)$, $w_2 = b/(a+b+c)$, and $w_3 = c/(a+b+c)$. Then using the GM-HM portion of the above inequality, we obtain

$$\begin{aligned} a^{a/(a+b+c)}b^{b/(a+b+c)}c^{c/(a+b+c)} &= (a^ab^bc^c)^{1/(a+b+c)} \\ &\geq \frac{1}{\frac{1}{a+b+c} + \frac{1}{a+b+c} + \frac{1}{a+b+c}} \\ &= \frac{a+b+c}{3}\,. \end{aligned}$$

By the AM-GM Inequality,

$$a^ab^bc^c \;\geq\; \left(\frac{a+b+c}{3}\right)^{a+b+c} \;\geq\; (abc)^{(a+b+c)/3}\,.$$

J.I.R. McKnight Problems Contest 1986 Solutions

3. Prove that the sum of the squares of the first n even natural numbers exceeds the sum of the squares of the first n odd natural numbers by $n(2n+1)$. Hence, or otherwise, find the sum of the squares of the first n odd natural numbers.

Partial solution by Luyun Zhong-Qido, Columbia International College, Hamilton, Ontario, Canada.

The sum of the first n positive integers is given by

$$1^2 + 2^2 + \cdots + n^2 \;=\; \frac{n(n+1)(2n+1)}{6}\,. \qquad (1)$$

$$\text{Therefore,}\quad 1^2 + 2^2 + \cdots + (2n)^2 \;=\; \frac{2n(2n+1)(4n+1)}{6}\,, \qquad (2)$$

$$\text{and}\quad 2^2 + 4^2 + \cdots + (2n)^2 \;=\; \frac{4n(n+1)(2n+1)}{6}\,. \qquad (3)$$

From (2) and (3),

$$\begin{aligned}1^2+3^2+\cdots+(2n-1)^2 &= \frac{2n(2n+1)(4n+1)}{6}-\frac{4n(n+1)(2n+1)}{6}\\ &= \frac{2n(2n+1)[(4n+1)-2(n+1)]}{6}\\ &= \frac{2n(2n-1)(2n+1)}{6}\,. \qquad (4)\end{aligned}$$

From (3) and (4),

$$\begin{aligned}&2^2+4^2+\cdots+(2n)^2-[(1^2+3^2+\cdots+(2n-1)^2]\\ &= \frac{4n(n+1)(2n+1)}{6}-\frac{2n(2n-1)(2n+1)}{6}\\ &= \frac{2n(2n+1)[2(n+1)-(2n-1)]}{6}\\ &= \frac{3\cdot 2n(2n+1)}{6} = n(2n+1)\,.\end{aligned}$$

4. (b) Prove that in any acute triangle ABC,

$$\cot A+\cot B+\cot C = \frac{a^2+b^2+c^2}{4K}\,,$$

where K is the area of triangle ABC.

Solution by Luyun Zhong-Qido, Columbia International College, Hamilton, Ontario, Canada.

We have the following relations in a triangle:

$$\begin{aligned}2R &= \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}\,, &(1)\\ K &= \frac{1}{2}\,ab\sin C = \frac{1}{2}\,ac\sin B = \frac{1}{2}\,bc\sin A\,. &(2)\end{aligned}$$

From (1) and (2),

$$K = \frac{1}{2}\,bc\sin A = \frac{1}{2}\left(\frac{a\sin B}{\sin A}\right)\left(\frac{a\sin C}{\sin A}\right)\sin A = a^2\,\frac{\sin B\sin C}{2\sin A}\,,$$

so

$$a^2 = \frac{2K\sin A}{\sin B\sin C}\,.$$

Likewise,

$$b^2 = \frac{2K\sin B}{\sin A\sin C}\,,\qquad c^2 = \frac{2K\sin C}{\sin A\sin B}\,.$$

We note that

$$\begin{aligned}\cot A+\cot B &= \frac{\cos A}{\sin A}+\frac{\cos B}{\sin B} = \frac{\cos A\sin B+\cos B\sin A}{\sin A\sin B}\\ &= \frac{\sin(A+B)}{\sin A\sin B} = \frac{\sin C}{\sin A\sin B}\,.\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{a^2+b^2+c^2}{4K} &= \frac{\frac{2K\sin A}{\sin B\sin C}+\frac{2K\sin B}{\sin A\sin C}+\frac{2K\sin C}{\sin A\sin B}}{4K} \\
&= \frac{2K(\sin^2 A+\sin^2 B+\sin^2 C)}{4K\sin A\sin B\sin C} \\
&= \frac{1}{2}\left(\frac{\sin A}{\sin B\sin C}+\frac{\sin B}{\sin A\sin C}+\frac{\sin C}{\sin A\sin B}\right) \\
&= \frac{1}{2}(\cot B+\cot C+\cot A+\cot C+\cot A+\cot B) \\
&= \cot A+\cot B+\cot C\,.
\end{aligned}$$

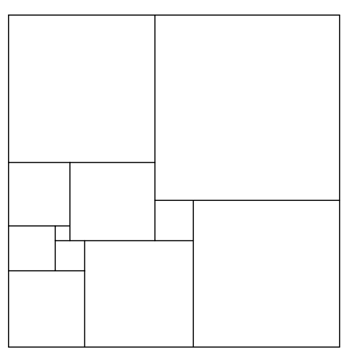
J.I.R. McKnight Problems Contest 1989

PART A

1. A curve has equation $y = x^3 - 3x$. A tangent is drawn to the curve at its relative minimum point. This tangent line also intersects the curve at P. Find the equation of the normal to the curve at P.

2. (a) In a triangle whose sides are 5, 12, and 13, find the length of the bisector of the larger acute angle.

   (b) This large rectangle has been cut into eleven squares of various sizes. The smallest square has an area of $81\ \mathrm{cm}^2$. Find the dimensions of the large rectangle.



3. Solve the following system of equations for x and y, where $x, y \in \mathbb{R}$:

$$x^3 - y^3 = 35\,, \qquad xy^2 - yx^2 = 30\,.$$

4. The combined volume of two cubes with integral sides is equal to the combined length of all their edges. Find the dimensions of all cubes satisfying these conditions.

5. A car at an intersection is heading west at 24 m/s. Simultaneously, a second car, 84 m north of the first car, is travelling directly south at 10 m/s. After 2 seconds:

   (a) Find the rate of change of the distance between the 2 cars.

   (b) Using vector methods, determine the velocity of the first car relative to the second.

   (c) Explain fully why the magnitude of the vector in (b) is not necessarily the same as the rate of change in (a).

PART B

1. Prove that

$$\sin 1^\circ + \sin 3^\circ + \sin 5^\circ + \cdots + \sin 97^\circ + \sin 99^\circ \;=\; \frac{\sin^2 50^\circ}{\sin 1^\circ}\,.$$

2. Determine the n^{th} term and the sum of n terms for the series

$$3 + 5 + 10 + 18 + 29 + \cdots\,.$$

3. A circle has equation $x^2 + y^2 = 1$. Lines with slope $\frac{1}{2}$ and $-\frac{1}{2}$ are drawn through $C(0,0)$ to form a sector of the circle. Find the dimensions of the rectangle of maximal area which can be inscribed in this sector given that two sides of the rectangle are parallel to the y-axis and the rectangle is entirely below the x-axis.

4. In triangle ABC, angles A, B, and C are in the ratio $4 : 2 : 1$. Prove that the sides of the triangle are related by the equality $\frac{1}{a} + \frac{1}{b} \;=\; \frac{1}{c}$.

5. Prove that

$$\left(\frac{7+\sqrt{37}}{2}\right)^n + \left(\frac{7-\sqrt{37}}{2}\right)^n - 1$$

is divisible by 3 for all $n \in \mathbb{N}$.

Derangements and Stirling Numbers

Naoki Sato
student, Yale University

In this article, we introduce two important classes of combinatorial numbers which appear frequently: derangements and Stirling numbers. We also hope to emphasize the importance and usefulness of basic counting principles.

We can think of a permutation on n objects as a 1-1 function π from the set $\{1, 2, \ldots, n\}$ to itself. For example, for $n = 3$, the map π given by $\pi(1) = 1$, $\pi(2) = 3$, and $\pi(3) = 2$ is a permutation on 3 objects, namely the elements of $\{1, 2, 3\}$; a permutation essentially re-arranges the elements. Note that there are $n!$ different permutations on n objects. Then, a *derangement* is a permutation π which has no fixed points; that is, $\pi(i) \neq i$ for all $i = 1, 2, \ldots, n$. Alternatively, if we think of a permutation on n objects as a distribution of n letters to n corresponding envelopes, then a derangement is a permutation where no letter is inserted into the correct corresponding envelope. Let D_n denote the number of derangements on n objects. The natural question to ask is, what is the formula for D_n?

Problems.

1. Write down all permutations on 2, 3, and 4 elements. How many of these are derangements? Is there a systematic way of writing down all permutations on n elements?
2. What is the total number of fixed points over all permutations on n elements? You should be able to guess the answer from small cases.

We immediately see that $D_1 = 0$ and $D_2 = 1$ (by convention, $D_0 = 0$). Assume that $n \geq 2$; we will derive a recurrence relation for the D_n, by dividing all derangements on n elements into two categories, and then counting the number in each category. Let π be a derangement on the n elements $1, 2, \ldots, n$. Let $k = \pi(1)$, so $k \neq 1$ since π is a derangement. There are two cases: $\pi(k) = 1$ or $\pi(k) \neq 1$. Let A_n be the number of derangements for which $\pi(k) = 1$, and let B_n be the number of derangements for which $\pi(k) \neq 1$.

If $\pi(k) = 1$, then π swaps the elements 1 and k, leaving what π does to the remaining elements $2, 3, \ldots, k-1, k+1, \ldots, n$ to be considered. Since π re-arranges these elements, it too acts as a derangement on these $n - 2$ elements, of which there are D_{n-2}. Going back, there are $n - 1$ possible values for k, as 1 is omitted, so $A_n = (n - 1)D_{n-2}$.

If $\pi(k) \neq 1$, then let α be the permutation which swaps 1 and k, and leaves everything else fixed. Recall that for maps f and g, $f \circ g$ denotes the *composition* of the two maps; that is, $(f \circ g)(x) = f(g(x))$. In this case, if f and g are permutations, then $f \circ g$ is the permutation which arises from applying g, then applying f. Consider the permutation $\pi \circ \alpha$. We see that $(\pi \circ \alpha)(1) = \pi(\alpha(1)) = \pi(k) \neq 1$, $(\pi \circ \alpha)(k) = \pi(\alpha(k)) = \pi(1) = k$, and for all other elements i, $(\pi \circ \alpha)(i) = \pi(\alpha(i)) = \pi(i) \neq i$, since π is a derangement. Thus, $\pi \circ \alpha$ is a permutation which fixes the element k, and which deranges all $n-1$ others (this is why we composed π with α), of which there are D_{n-1}. Again, there are $n-1$ possible values of k (again, 1 is omitted), so $B_n = (n-1)D_{n-1}$. Therefore,

$$\begin{aligned} D_n &= A_n + B_n \\ &= (n-1)D_{n-2} + (n-1)D_{n-1} \\ &= (n-1)(D_{n-1} + D_{n-2}) \,. \end{aligned}$$

Problems.

3. Show that in general, for permutations α and β, $\alpha \circ \beta \neq \beta \circ \alpha$. Can you determine when $\alpha \circ \beta = \beta \circ \alpha$?

4. For a positive integer k, let π^k denote the permutation π composed with itself k times; that is,

$$\pi^k = \underbrace{\pi \circ \pi \circ \cdots \circ \pi}_{k} \,.$$

Prove that for any permutation π, there exists a positive integer k such that $\pi^k = 1$. Moreover, if π is a permutation on n elements, then $\pi^{n!} = 1$. Here, 1 stands for the identity permutation, the permutation which takes every element to itself.

5. Classify all permutations π on n elements such that $\pi \circ \pi = \pi^2 = 1$.

6. Prove that for any permutation α, there exist unique permutations β and γ such that $\alpha \circ \beta = \gamma \circ \alpha = 1$. Problem 5 shows that $\beta = \gamma$ is possible; is this true in general? Hint: Apply Problem 4!

The next few terms in the sequence are then $D_3 = 2$, $D_4 = 9$, $D_5 = 44$, etc. We can use this recurrence to derive an explicit formula for D_n. By the relation,

$$\begin{aligned} D_n - nD_{n-1} &= -D_{n-1} + (n-1)D_{n-2} \\ &= -\left(D_{n-1} - (n-1)D_{n-2}\right) \\ &= (-1)^2\left(D_{n-2} - (n-2)D_{n-3}\right) \\ &= \cdots \\ &= (-1)^{n-2}(D_2 - 2D_1) = (-1)^n \,. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{D_n}{n!} &= \frac{D_{n-1}}{(n-1)!} + \frac{(-1)^n}{n!} \\ &= \frac{D_{n-2}}{(n-2)!} + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!} \\ &= \cdots \\ &= \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!} \\ \Longrightarrow D_n &= n! \left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!} \right) . \end{aligned}$$

Remember this expression; as mentioned above, it comes up a lot! We note in passing that

$$\frac{1}{e} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots ,$$

so that

$$D_n \approx \frac{n!}{e}$$

for large n. We also note that it is possible to derive the formula for D_n using the Principle of Inclusion-Exclusion. We finish off derangements with two quick problems.

Problem. Let n be a positive integer. Show that

$$\sum_{k=0}^{n} \binom{n}{k} D_k = n! .$$

Solution. It looks as if we might have to plug away at some algebra, but a combinatorial approach is much more natural and painless. Note that the RHS, $n!$, is simply the number of permutation on n objects, and this is a cue. What is the number of permutations which derange k elements, or alternatively, which fix $n-k$ elements? We first choose $n-k$ elements, and derange the rest, which there are D_k ways of doing, for a total of

$$\binom{n}{n-k} D_k = \binom{n}{k} D_k .$$

The result follows from summing over k (since every permutation deranges k elements for some k).

In general, if $x_0, x_1, x_2, \ldots$ and $y_0, y_1, y_2, \ldots$ are sequences satisfying

$$\sum_{k=0}^{n} \binom{n}{k} x_k = y_n ,$$

then it is possible to recover $\{x_n\}$ from $\{y_n\}$, via

$$x_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} y_k .$$

Indeed, for $y_n = n!$, we obtain that $x_n = D_n$, since

$$\begin{aligned} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k! &= \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{(n-k)!} \\ &= n! \left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!} \right) \\ &= D_n . \end{aligned}$$

Problem. Let n be a positive integer. Prove that

$$\sum_{k=0}^{n} k \binom{n}{k} D_k = (n-1) \cdot n! .$$

Solution. Left as an exercise for the reader. (Follow the same type of reasoning as the previous problem.) Hint: Combinatorially, what does the LHS represent?

We now draw our attention to the *Stirling numbers*, in particular those of the *second kind* (as opposed to the *first kind* for those in suspense). These arise in the following situation: Suppose that we have n distinguishable balls to distribute among k indistinguishable boxes, and no box can be empty. How many such distributions are there? First, consider the case where the boxes are distinguishable.

If the boxes were allowed to be empty, there would be k^n such distributions; assign a box to each ball. Following the Principle of Inclusion-Exclusion, we subtract the number of distributions with at least 1 box empty, which is

$$\binom{k}{k-1} (k-1)^n ,$$

since we must choose out of $k-1$ boxes for each ball. We then add the number of distributions with at least 2 boxes empty, which is

$$\binom{k}{k-2} (k-2)^n ,$$

and so on. Hence, the total number of distributions is

$$\sum_{i=0}^{n} (-1)^i \binom{k}{i} (k-i)^n .$$

Then, for the case where the boxes are indistinguishable, we may "remove the labels" of the boxes, and divide by a factor of $k!$, to obtain the number of distributions of the original problem:

$$S(n,k) := \frac{1}{k!}\sum_{i=0}^{n}(-1)^i\binom{k}{i}(k-i)^n .$$

These are the Stirling numbers of the second kind. We list the first few here:

n	$S(n,k),\ k=1,2,\dots,n$				
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

An important property of these Stirling numbers, one that may be noticeable in the table, is the following:

$$S(n,k) \ = \ S(n-1,k-1) + kS(n-1,k) .$$

This rule easily allows us to generate further rows in the table. We stress that although we derived a formula for $S(n,k)$ above, it is in general better to consider the combinatorial significance of this number. So, we give a combinatorial proof here, and leave an algebraic proof for the reader.

Assume that the balls are labelled 1 through n (recall that they are distinguishable). Consider ball n. Either it is in a box all by itself, or with others. For the first case, the number of distributions is simply $S(n-1,k-1)$, as we can add one box containing ball n to a distribution of $n-1$ balls among $k-1$ boxes. For the second case, temporarily remove ball n. This leaves $n-1$ balls among k non-empty boxes, of which there are $S(n-1,k)$ distributions. We can then add ball n to any of the k boxes, giving rise to k distinct distributions. Hence,

$$S(n,k) \ = \ S(n-1,k-1) + kS(n-1,k) .$$

Problem. Show that

$$S(n,k) \ = \ \sum 1^{a_1}2^{a_2}\cdots k^{a_k} ,$$

where the sum is taken over all $\binom{n-1}{k-1}$ decompositions of $n-k$ into k non-negative integers $a_1, a_2, \dots, a_k$: $a_1+a_2+\cdots+a_k = n-k$. (A decomposition of a non-negative integer N into m parts is a sequence of m non-negative integers, the sum of which is N. So, 3 has 4 different decompositions into 2 parts: $3 = 0+3 = 1+2 = 2+1 = 3+0$.)

Solution. The table above makes this virtually obvious, but we will flesh out the details. Draw in arrows in the table, joining entries in consecutive rows. An arrow from $S(n-1,k-1)$ has weight 1 and an arrow from $S(n-1,k)$ has weight k. By virtue of the identity proved above, an entry in the table is equal to the sum of the weights of the paths leading to it, where the weight of a path is simply the product of the weights of the arrows in it (think of this as a tweaked Pascal's Triangle).

There are $\binom{n-1}{k-1}$ paths from entry $S(1,1)$ to $S(n,k)$, and the weight of a path is precisely the term in the given sum. The result follows from summing over all paths, which clearly gives the given expression.

Problem. Show that

$$S(n+1,k) \;=\; \sum_{i=0}^{n}\binom{n+1}{i}S(i,k-1)\,.$$

Solution. In a distribution of $n+1$ balls among k boxes, select one box as "blue". Let i be the number of balls in the blue box, so $1 \le i \le n+1$. From the $n+1$ balls, we may choose i to be in the blue box, leaving $n-i+1$ to be distributed among $k-1$ boxes. There are

$$\binom{n+1}{i}S(n-i+1,k-1) = \binom{n+1}{n-i+1}S(n-i+1,k-1)$$

such distributions in this case. The result follows from summing over i.

Problems.

1. Let n be a positive integer. Prove that

$$\sum_{k=0}^{n} k\binom{n}{k}D_k \;=\; (n-1)\cdot n!\,.$$

2. Here is an example which illustrates the Principle of Inclusion-Exclusion: Let A, B, and C be subsets of a set S. Let $\overline{A}$ denote the complement of A. Prove that

$$|\overline{A}\cap\overline{B}\cap\overline{C}| \;=\; |S|-|A|-|B|-|C|+|A\cap B|+|A\cap C|+|B\cap C|-|A\cap B\cap C|\,.$$

   For example, let S be the set of all distributions of n distinguishable balls among 3 distinguishable boxes, let A be the subset of distributions with box 1 empty, and so on. Then $\overline{A}$ is the subset of distributions with box 1 containing at least one ball, and so on, so $\overline{A}\cap\overline{B}\cap\overline{C}$ is the subset of distributions with all three boxes containing at least one ball. Now determine what the formula above turns into. This is a useful formula, because the terms such as $|A|$ and $|A\cap B|$ are easy to compute. This formula also generalizes to any number of subsets.

3. By using the derived formula for $S(n,k)$, show that

$$S(n,k) \;=\; S(n-1,k-1)+kS(n-1,k)\,.$$

4. Let n and k be positive integers, with $k < n$. Verify that there are $\binom{n-1}{k-1}$ decompositions of $n - k$ into k parts. If we restrict the parts to be positive integers, then how many decompositions are there of n in total?

5. Define a sequence of polynomials $\{p_n(x)\}$ as follows: $p_1(x) = p(x)$ is given, and $p_{n+1}(x) = xp'_n(x)$. Find a closed formula for $p_n(x)$, in terms of $p(x)$ and n.

6. (a) Let n be a positive integer. Prove that the following is an identity in x:

$$x^n = \sum_{k=0}^{n} k!S(n,k)\binom{x}{k}.$$

(b) Prove that

$$1^n + 2^n + \cdots + m^n = \sum_{k=0}^{n} k!S(n,k)\binom{m+1}{k+1}.$$

Naoki Sato
Department of Mathematics
Yale University
New Haven
Connecticut
USA
sato-naoki@math.yale.edu