THE SKOLIAD CORNER

No. 39

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This issue we give another example of a team competition with the problems of the 1998 Florida Mathematics Olympiad, written May 14, 1998. The contest was organized by Florida Atlantic University. My thanks go to John Grant McLoughlin, Memorial University of Newfoundland for sending me the problems.

FLORIDA MATHEMATICS OLYMPIAD
TEAM COMPETITION
May 14, 1998

1. Find all integers x, if any, such that $9 < x < 15$ and the sequence

$$1\,,\,2\,,\,6\,,\,7\,,\,9\,,\,x\,,\,15\,,\,18\,,\,20$$

does not have three terms in arithmetic progression. If there are no such integers, write "NONE."

2. A sequence a_1, a_2, $a_3, \ldots$ is said to satisfy a *linear recurrence relation of order two* if and only if there are numbers p and q such that, for all positive integers n,

$$a_{n+2} \;=\; pa_{n+1} + qa_n\,.$$

Find the next two terms of the sequence

$$2\,,\,5\,,\,14\,,\,41\,,\;\ldots\,,$$

assuming that this sequence satisfies a linear recurrence relation of order two.

3. Seven tests are given and on each test no ties are possible. Each person who is the top scorer on at least one of the tests or who is in the top six on at least four of these tests is given an award, but each person can receive at most one award. Find the maximum number of people who could be given awards, if 100 students take these tests.

4. Some primes can be written as a sum of two squares. We have, for example, that

$$5 \;=\; 1^2 + 2^2, \quad 13 \;=\; 2^2 + 3^2, \quad 17 \;=\; 1^2 + 4^2\,,$$

$$29 \;=\; 2^2 + 5^2, \quad 37 \;=\; 1^2 + 6^2, \quad \text{and} \quad 41 \;=\; 4^2 + 5^2\,.$$

The odd primes less than 108 are listed below; the ones that can be written as a sum of two squares are boxed in.

$$3\,,\ \boxed{5}\,,\ 7\,,\ 11\,,\boxed{13}\,,\ \boxed{17}\,,\ 19\,,\ 23\,,\ \boxed{29}\,,\ 31\,,$$

$$\boxed{37}\,,\ \boxed{41}\,,\ 43\,,\ 47\,,\ \boxed{53}\,,\ 59\,,\ \boxed{61}\,,\ 67\,,\ 71\,,$$

$$\boxed{73}\,,\ 79\,,\ 83\,,\ \boxed{89}\,,\ \boxed{97}\,,\ \boxed{101}\,,\ 103\,,\ 107\,.$$

The primes that can be written as a sum of two squares follow a simple pattern. See if you can correctly find this pattern. If you can, use this pattern to determine which of the primes between 1000 and 1050 can be written as a sum of two squares; there are five of them. The primes between 1000 and 1050 are

$$1009\,,\ 1013\,,\ 1019\,,\ 1021\,,\ 1031\,,\ 1033\,,\ 1039\,,\ 1049\,.$$

No credit unless the correct five primes are listed.

5. The sides of a triangle are 4, 13, and 15. Find the radius of the inscribed circle.

6. In Athenian criminal proceedings, ordinary citizens presented the charges, and the 500-man juries voted twice: first on guilt or innocence, and then (if the verdict was guilty) on the penalty. In 399BCE, Socrates (c469–399) was charged with dishonouring the gods and corrupting the youth of Athens. He was found guilty; the penalty was death. According to I.F. Stone's calculations on how the jurors voted:

(i) There were no abstentions;

(ii) There were 80 more votes for the death penalty than there were for the guilty verdict;

(iii) The sum of the number of votes for an innocent verdict and the number of votes against the death penalty equalled the number of votes in favour of the death penalty.

a) How many of the 500 jurors voted for an innocent verdict?

b) How many of the 500 jurors voted in favour of the death penalty?

7. Find all x such that $0 \leq x \leq \pi$ and

$$\tan^3 x - 1 + \frac{1}{\cos^2 x} - 3\cot\left(\frac{\pi}{2} - x\right) \;=\; 3\,.$$

Your answer should be in radian measure.

Last issue we gave the problems of the Newfoundland and Labrador Teachers' Association Mathematics League. Here are the answers.

NLTA MATH LEAGUE
GAME 1 — 1998–99

1. Find a two-digit number that equals twice the product of its digits.

Solution. Denote the number by ab; We get $10a + b = 2a \cdot b$. Trying $a = 1, 2, \ldots, 9$, the only integral solution is with $a = 3$, $b = 6$ and $36 = 2 \cdot 3 \cdot 6$.

2. The degree measures of the interior angles of a triangle are A, B, C where $A \leq B \leq C$. If A, B, and C are multiples of 15, how many possible values of (A, B, C) exist?

Solution. Let $A = 15m$, $B = 15n$, and $C = 15p$. Then $15m + 15n + 15p = 180$ so $m + n + p = 12$ and $m \leq n \leq p$. Since $m, n \geq 1$ we have $m + n \geq 2$ and $p \leq 10$. Also $3p \geq 12$ so $p \geq 4$. For p fixed, $4 \leq p \leq 10$ we have $m + n = 12 - p$, or $m = 12 - p - n$. This leads to the solutions

p	4	5	5	6	6	6	7	7	8	8	9	10
n	4	4	5	3	4	5	3	4	2	3	2	1
m	4	3	2	3	2	1	2	1	2	1	1	1

There are twelve solutions.

3. Place an operation $(+, -, \times, \div)$ in each square so that the expression using 1, 2, 3, $\cdots$, 9 equals 100.

$$1 \square 2 \square 3 \square 4 \square 5 \square 6 \square 7 \square 8 \square 9 \;=\; 100\,.$$

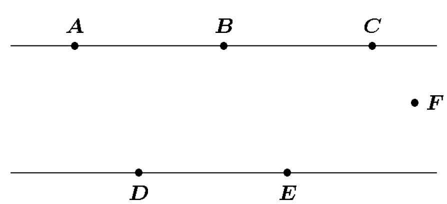
You may also freely place brackets before/after any digits in the expression. Note that the squares must be filled in with operational symbols only.

Solution. Here is one solution:

$$(((1 + 2 + 3 + 4 + 5) \times 6) - 7) + 8 + 9 \;=\; 100\,.$$

How many solutions are there?

4. A, B and C are points on a line that is parallel to another line containing points D and E, as shown. Point F does not lie on either of these lines.



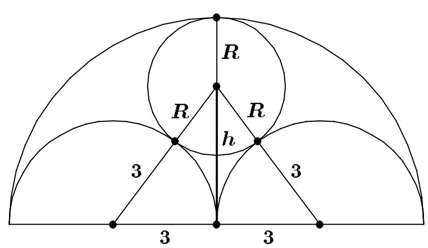
How many distinct triangles can be formed such that all three of its vertices are chosen from A, B, C, D, E, and F?

Solution. Any choice of three of the six vertices determines a triangle except when they lie on a line; that is, except for the one choice $\{A, B, C\}$. The total is thus $\binom{6}{3} - 1 = 20 - 1 = 19$.

5. Michael, Jane and Bert enjoyed a picnic lunch. The three of them were to contribute an equal amount of money toward the cost of the food. Michael spent twice as much money as Jane did buying food for lunch. Bert did not spend any money on food. Instead, Bert brought \$6 which exactly covered his share. How much (in dollars) of Bert's contribution should be given to Michael?

Solution. Bert brought \$6, which exactly covered his share, so the total cost of food is $3 \times \$6 = \$18$. Now Michael spent twice as much as Jane so he spent \$12 and she spent \$6. The total amount of \$6 brought by Bert should go to Michael.

6. Two semicircles of radius 3 are inscribed in a semicircle of radius 6. A circle of radius R is tangent to all three semicircles, as shown. Find R.



Solution. Join the centres of the two smaller semicircles and the centre of the circle. This forms an isosceles triangle with equal sides $3 + R$ and base 6 units. Call the altitude of this triangle h. The altitude extends to a radius of the large semicircle, so $h + R = 6$. By Pythagoras, $h^2 + 3^2 = (R + 3)^2$, so

$$\begin{aligned} (6 - R)^2 + 3^2 &= (R + 3)^2 \,, \\ 36 - 12R + R^2 + 9 &= R^2 + 6R + 9 \,, \\ 36 &= 18R \,, \\ 2 &= R \,. \end{aligned}$$

The radius of the small circle is 2.

7. If $5^A = 3$ and $9^B = 125$, find the value of AB.

Solution. Now $5^A = 3$, so $5^{2A} = 3^2 = 9$ and

$$5^{2AB} = (5^{2A})^B = 9^B = 125 = 5^3 \,,$$

so $2AB = 3$ and $AB = \frac{3}{2}$.

8. The legs of a right angled triangle are 10 and 24 cm respectively.

$$\begin{aligned} \text{Let } A &= \text{the length (cm) of the hypotenuse,} \\ B &= \text{the perimeter (cm) of the triangle,} \\ C &= \text{the area (cm}^2\text{) of the triangle.} \end{aligned}$$

Determine the lowest common multiple of A, B, and C.

Solution. Then

$$\begin{aligned} A &= \sqrt{10^2+24^2} &= 26 \\ B &= 10+24+26 &= 60 \\ C &= \tfrac{1}{2}\times 10\cdot 24 &= 120 \end{aligned}$$

$\mathrm{lcm}(26, 60, 120) = 3\times 8\times 5\times 13 = 1560$.

9. A lattice point is a point (x, y) such that both x and y are integers. For example, $(2, -1)$ is a lattice point, whereas, $(3, \frac{1}{2})$ and $(-\frac{1}{3}, \frac{2}{3})$ are not. How many lattice points lie inside the circle defined by $x^2+y^2=20$? (Do **NOT** count lattice points that lie on the circumference of the circle.)

Solution. Since $4^2 < 20 < 5^2$ we have that $-4 \le x \le 4$. For a fixed x in the range we must have $-\sqrt{20-x^2} < y < \sqrt{20-x^2}$ for integer x, y solutions corresponding to interior points of the circle.

x	± 4	± 3	± 2	± 1	0
$\sqrt{20-x^2}$	2	$\sqrt{11}$	$\sqrt{16}$	$\sqrt{19}$	$\sqrt{20}$
y	$-1, 0, 1$	$-3, \ldots, 3$	$-3, \ldots, 3$	$-4, \ldots, 4$	$-4, \ldots, 4$
Lattice pts.	$2\cdot 3=6$	$2\cdot 7=14$	$2\cdot 7=14$	$2\cdot 9=18$	9

The total number is then $6+14+14+18+9=61$.

10. The quadratic equation $x^2+bx+c=0$ has roots r_1 and r_2 that have a sum which equals 3 times their product. Suppose that (r_1+5) and (r_2+5) are the roots of another quadratic equation $x^2+ex+f=0$. Given that the ratio of $e : f = 1 : 23$, determine the values of b and c in the original quadratic equation.

Solution. Now $r_1+r_2=-b$ and $r_1r_2=c$, so as $r_1+r_2=3r_1r_2$ we get $3c=-b$. Similarly we have

$$\begin{aligned} r_1+r_2+10 &= -e\,, \\ (r_1+5)(r_2+5) &= f\,. \end{aligned}$$

Thus

$$\begin{aligned} 10-b &= -e \\ \text{and } b-10 &= e\,, \\ r_1r_2+5(r_1+r_2)+25 &= f\,, \\ c-5b+25 &= f\,. \end{aligned}$$

From $\frac{e}{f} = \frac{1}{23}$, $23(b-10) = c - 5b + 25$. Using $b = -3c$

$$\begin{aligned} 23(-3c-10) &= c + 15c + 25\,, \\ \text{so that } -69c - 230 &= 16c + 25\,, \\ -255 &= 85c\,, \\ -3 &= c\,, \\ \text{and } 9 &= b\,. \end{aligned}$$

The quadratic is $x^2 + 9x - 3 = 0$.

RELAY

R1. Operations $*$ and $\diamond$ are defined as follows:

$$A * B \;=\; \frac{A^B + B^A}{A+B} \quad \text{and} \quad A \diamond B \;=\; \frac{A^B - B^A}{A-B}\,.$$

Simplify $N = (3 * 2) * (3 \diamond 2)$. Write the value of N in Box #1 of the relay answer sheet.

Solution.

$$\begin{aligned} 3 * 2 &= \frac{3^2 + 2^3}{3+2} = \frac{9+8}{5} = \frac{17}{5}\,, \\ 3 \diamond 2 &= \frac{3^2 - 2^3}{3-2} = \frac{9-8}{1} = 1\,, \end{aligned}$$

$$N \;=\; (3 * 2) * (3 \diamond 2) \;=\; \left(\frac{17}{5} * 1\right) = \frac{\left(\frac{17}{5}\right)^1 + (1)^{17/5}}{\frac{17}{5} + 1} \;=\; 1\,.$$

R2. A square has a perimeter of P cm and an area of Q sq.cm. Given that $3NP = 2Q$, determine the value of P. Write the value of P in Box #2 of the relay answer sheet.

Solution. From **R1**, $N = 1$, so $3P = 2Q$. Also the side length is $s = \frac{P}{4}$, so $Q = (\frac{P}{4})^2 = \frac{P^2}{16}$. So $3P = 2 \cdot \frac{P^2}{16}$ or $24P = P^2$. This gives $P = 0$ or $P = 24$. We use $P = 24$, assuming the square is not degenerate.

R3. List all two-digit numbers that have digits whose product is P. Call the sum of these two-digit numbers S. Write the value of S in Box #3 of the relay answer sheet.

Solution. $P = 1 \cdot 24 = 2 \cdot 12 = 4 \cdot 6 = 8 \cdot 3$. The two-digit numbers are 46, 64, 38 and 83; so

$$S \;=\; 231\,.$$

R4. How many integers between 6 and 24 share no common factors with S that are greater than 1?

Solution. Now $231 = 11 \times 21 = 11 \times 7 \times 3$. The numbers between 6 and 24 that share no common factor with 231 are

$$8\,,\ 10\,,\ 13\,,\ 16\,,\ 17\,,\ 19\,,\ 20\,,\ 23\,.$$

There are 8 of them.

TIE-BREAKER

Find the maximum value of

$$f(x) \;=\; 14 - \sqrt{x^2 - 6x + 25}\,.$$

Solution.

$$\begin{aligned} f(x) &= 14 - \sqrt{x^2 - 6x + 25} \\ &= 14\sqrt{x^2 - 6x + 9 - 9 + 25} \\ &= 14 - \sqrt{(x-3)^2 + 16}\,. \end{aligned}$$

For $f(x)$ to be maximum we want $(x-3)^2 + 16$ to be minimum. This occurs when $x = 3$, and

$$f(3) \;=\; 14 - \sqrt{16} \;=\; 14 - 4 \;=\; 10\,.$$

That completes the *Skoliad Corner* for this number. Send me your comments, suggestions, and especially suitable material for use in the Corner.

Announcement

The second volume in the **ATOM** series has just been published. *Algebra — Intermediate Methods* by Bruce Shawyer. Contents include:

Mathematical Induction, Series, Binomial Coefficients,
Solutions of Polynomial Equations, and Vectors and Matrices.

For more information, contact the CMS Office — address on outside back cover.