APROPOS BELL AND STIRLING NUMBERS

Antal E. Fekete

Introduction

In 1877 Dobiñski stated [1] that there exist integers q_n such that

$$\frac{0^n}{0!} + \frac{1^n}{1!} + \frac{2^n}{2!} + \frac{3^n}{3!} + \cdots \;=\; q_n\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right) \;=\; q_n e\,,$$

and he calculated their values for $n = 1$ through 5 (Table 1). Indeed, q_n are the Bell numbers, so named in honour of the American mathematician Eric Temple Bell (1883-1960), who was among the first to popularize these numbers; see [2] and [3], where further references can be found. It may be shown that q_n is just the sum of Stirling numbers of the second kind:

$$q_n \;=\; \left\{{n \atop 1}\right\} + \left\{{n \atop 2}\right\} + \cdots + \left\{{n \atop n}\right\}.$$

Since $\left\{{n \atop k}\right\}$ is the number of k-member quotient sets of an n-set, the Bell number q_n is the number of all quotient sets of an n-set. It can also be calculated via recursion in terms of the Stirling numbers of the first kind; the following formula is due to G.T. Williams [5]:

$$\left[{n \atop n}\right] q_n - \left[{n \atop n-1}\right] q_{n-1} + - \cdots + (-1)^{n-1}\left[{n \atop 1}\right] q_1 \;=\; 1\,,$$

where $\left[{n \atop k}\right]$ are the coefficients of the polynomial of degree n with roots 0, 1, 2, ..., $(n-1)$:

$$\left[{n \atop n}\right] x^n - \left[{n \atop n-1}\right] x^{n-1} + - \cdots + (-1)^{n-1}\left[{n \atop 1}\right] x = x(x-1)(x-2)\cdots(x-n+1)\,.$$

Rényi numbers

Here we show that there exist integers r_n such that

$$\frac{0^n}{0!} - \frac{1^n}{1!} + \frac{2^n}{2!} - \frac{3^n}{3!} + - \cdots \;=\; r_n\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + - \cdots\right) \;=\; \frac{r_n}{e}\,.$$

We calculate their values for $n = 1$ through 4 by bringing to the numerator the polynomial with coefficients $\left[{n \atop k}\right]$, and splitting it into linear factors which we can then cancel.

$r_1 = -1$:

$$\sum_0 (-1)^n \frac{n}{n!} \;=\; \sum_1 (-1)^n \frac{1}{(n-1)!} \;=\; -\frac{1}{e}\,.$$

$r_2 = 0$:

$$\sum_0 (-1)^n \frac{n^2}{n!} = \sum_0 (-1)^n \frac{n^2 - n}{n!} - \frac{1}{e} = \sum_0 (-1)^n \frac{(n-1)n}{n!} - \frac{1}{e}$$
$$= \sum_2 (-1)^n \frac{1}{(n-2)!} - \frac{1}{e} = \frac{1}{e} - \frac{1}{e} = 0 .$$

$r_3 = 1$:

$$\sum_0 (-1)^n \frac{n^3}{n!} = \sum_0 (-1)^n \frac{n^3 - 3n^2 + 2n}{n!} + \frac{0}{e} + \frac{2}{e}$$
$$= \sum_0 (-1)^n \frac{(n-2)(n-1)n}{n!} + \frac{2}{e}$$
$$= \sum_2 (-1)^n \frac{1}{(n-3)!} + \frac{2}{e} = -\frac{1}{e} + \frac{2}{e} = \frac{1}{e} .$$

$r_4 = 1$:

$$\sum_0 (-1)^n \frac{n^4}{n!} = \sum_0 (-1)^n \frac{n^4 - 6n^3 + 11n^2 - 6n}{n!} + \frac{6}{e} - \frac{6}{e}$$
$$= \sum_0 (-1)^n \frac{(n-3)(n-2)(n-1)n}{n!}$$
$$= \sum_4 (-1)^n \frac{1}{(n-4)!} = \frac{1}{e} .$$

The recursion formula for r_n in terms of Stirling numbers of the first kind

$$\begin{bmatrix} n \\ n \end{bmatrix} r_n - \begin{bmatrix} n \\ n-1 \end{bmatrix} r_{n-1} + - \cdots + (-1)^{n-1} \begin{bmatrix} n \\ 1 \end{bmatrix} r_1 = (-1)^n$$

shows that r_n is an integer for all n (Table 1). To prove it we write the left–hand side as

$$e \sum_0 (-1)^k \frac{\left[{n \atop n}\right] k^n - \left[{n \atop n-1}\right] k^{n-1} + - \cdots + (-1)^{n-1} \left[{n \atop 1}\right] k}{k!}$$
$$= e \sum_0 (-1)^k \frac{(k-n+1)\cdots(k-1)k}{k!} = e \sum_n \frac{(-1)^k}{(k-n)!} = (-1)^n .$$

We call r_n Rényi numbers in honour of the Hungarian mathematician Alfréd Rényi (1921-1970) who first studied them [4]. They can be expressed as the alternating sum of the Stirling numbers of the second kind:

$$r_n = -\left\{ {n \atop 1} \right\} + \left\{ {n \atop 2} \right\} - + \cdots + (-1)^n \left\{ {n \atop n} \right\} .$$

Related numbers

We now introduce related numbers exhibiting properties similar to those of Bell and Rényi numbers. There exist integers a_n, b_n such that

$$\begin{aligned} \frac{0^n}{0!} + \frac{2^n}{2!} + \frac{4^n}{4!} + \cdots &= a_n \cosh(1) + b_n \sinh(1)\,; \\ \text{and} \quad \frac{1^n}{1!} + \frac{3^n}{3!} + \frac{5^n}{5!} + \cdots &= a_n \sinh(1) + b_n \cosh(1) \end{aligned}$$

which can be calculated via recursion in terms of the Stirling numbers of the first kind:

$$\begin{bmatrix} n \\ n \end{bmatrix} a_n - \begin{bmatrix} n \\ n-1 \end{bmatrix} a_{n-1} + - \cdots + (-1)^{n-1} \begin{bmatrix} n \\ 1 \end{bmatrix} a_1 = \begin{cases} 1 \text{ if } n \text{ is even} \\ 0 \text{ if } n \text{ is odd} \end{cases}$$

$$\begin{bmatrix} n \\ n \end{bmatrix} b_n - \begin{bmatrix} n \\ n-1 \end{bmatrix} b_{n-1} + - \cdots + (-1)^{n-1} \begin{bmatrix} n \\ 1 \end{bmatrix} b_1 = \begin{cases} 0 \text{ if } n \text{ is even} \\ 1 \text{ if } n \text{ is odd} \end{cases}$$

Furthermore, there exist integers c_n, d_n such that

$$\begin{aligned} \frac{0^k}{0!} - \frac{2^k}{2!} + \frac{4^k}{4!} - + \cdots &= c_k \cos(1) - d_k \sin(1)\,; \\ \text{and} \quad \frac{1^k}{1!} - \frac{3^k}{3!} + \frac{5^k}{5!} - + \cdots &= c_k \sin(1) + d_k \cos(1)\,, \end{aligned}$$

which can be calculated via recursion in terms of the Stirling numbers of the first kind:

$$\begin{bmatrix} n \\ n \end{bmatrix} c_n - \begin{bmatrix} n \\ n-1 \end{bmatrix} c_{n-1} + - \cdots + (-1)^{n-1} \begin{bmatrix} n \\ 1 \end{bmatrix} c_1 = \begin{cases} 1 \text{ if } n = 4k \\ -1 \text{ if } n = 4k+2 \\ 0 \text{ if } n \text{ is odd} \end{cases}$$

$$\begin{bmatrix} n \\ n \end{bmatrix} d_n - \begin{bmatrix} n \\ n-1 \end{bmatrix} d_{n-1} + - \cdots + (-1)^{n-1} \begin{bmatrix} n \\ 1 \end{bmatrix} d_1 = \begin{cases} 0 \text{ if } n \text{ is even} \\ 1 \text{ if } n = 4k+1 \\ -1 \text{ if } n = 4k+3 \end{cases}$$

Table 1: Bell, Rényi and related numbers

n	0	1	2	3	4	5	6	7	8	9	10
q_n	1	1	2	5	15	52	203	877	4140	21147	115975
r_n	1	−1	0	1	1	−2	−9	−9	50	267	413
a_n	1	0	1	3	8	25	97	434	2095	10707	58194
b_n	0	1	1	2	7	27	106	443	2045	10440	57781
c_n	1	0	−1	−3	−6	−5	33	266	1309	4905	11516
d_n	0	1	1	0	−5	−23	−74	−161	57	3466	27361

These related numbers are useful in calculating the number of quotient sets of an n–set with an even or odd number of members:

$$a_n = \frac{q_n + r_n}{2} = \left\{ {n \atop 2} \right\} + \left\{ {n \atop 4} \right\} + \left\{ {n \atop 6} \right\} + \cdots ;$$

$$\text{and} \quad b_n = \frac{q_n - r_n}{2} = \left\{ {n \atop 1} \right\} + \left\{ {n \atop 3} \right\} + \left\{ {n \atop 5} \right\} + \cdots .$$

We also have

$$c_n = -\left\{ {n \atop 2} \right\} + \left\{ {n \atop 4} \right\} - \left\{ {n \atop 6} \right\} + - \cdots ;$$

$$\text{and} \quad d_n = \left\{ {n \atop 1} \right\} - \left\{ {n \atop 3} \right\} + \left\{ {n \atop 5} \right\} - + \cdots .$$

Extended Bell numbers

There exist integers q_{kn} such that

$$\frac{k^n}{0!} + \frac{(k+1)^n}{1!} + \frac{(k+2)^n}{2!} + \frac{(k+3)^n}{3!} + \cdots = e\, q_{k,n+1} .$$

In particular, $q_{1n} = q_n$; for this reason we call q_{kn} the extended Bell numbers. For each fixed k we have a recursion formula in terms of the Stirling numbers of the first kind

$$q_{kn} =$$
$$(-1)^{k-1} \left(\left[{k \atop 1} \right] q_n - \left[{k \atop 2} \right] q_{n+1} + \left[{k \atop 3} \right] q_{n+2} - + \cdots + (-1)^{k-1} \left[{k \atop k} \right] q_{n+k-1} \right) ,$$

providing the *first* of three methods to calculate the extended Bell numbers (Table 2).

A *second* method is via the recursion

$$q_{kn} = k q_{k,n-1} + q_{k+1,n-1} .$$

For a fixed n, the extended Bell numbers $q_{0,n}$, $q_{1,n}$, $q_{2,n}$, ... are in arithmetic progression of order $n-1$. Therefore $q_{kn} \equiv Q_{n-1}(k+1)$ is a polynomial of degree $n-1$ in the variable k. We have the recursion

$$Q_n(k) \equiv (k-1)Q_{n-1}(k) + Q_{n-1}(k+1)$$

enabling us to calculate these polynomials:

$$Q_0(k) \equiv 1 \qquad \text{(constant)}$$
$$Q_1(k) \equiv k$$
$$Q_2(k) \equiv k^2 + 1$$
$$Q_3(k) \equiv k^3 + 3k + 1$$
$$Q_4(k) \equiv k^4 + 6k^2 + 4k + 4$$
$$Q_5(k) \equiv k^5 + 10k^3 + 10k^2 + 20k + 11$$

$$Q_6(k) \equiv k^6 + 15k^4 + 20k^3 + 60k^2 + 66k + 41$$
$$Q_7(k) \equiv k^7 + 21k^5 + 35k^4 + 140k^3 + 231k^2 + 287k + 162$$
$$Q_8(k) \equiv k^8 + 28k^6 + 56k^5 + 280k^4 + 616k^3 + 1148k^2 + 1296k + 715$$
$$\ldots\ldots\ldots\ldots\ldots$$

A *third* method of calculating q_{kn} is furnished by the difference equation $\Delta^n q_{k1} = q_{k-1,n}$ that may be verbalized as follows. In Table 2, the k–th *row can be obtained by calculating the first member of each of the higher order difference sequences of the* $(k+1)$–st *row*. This property is useful not only in calculating the k–th row from the $(k+1)$–st, quickly and efficiently, but also the other way around.

Table 2: Extended Bell Numbers

$k\backslash n$	1	2	3	4	5	6	7	8	n
·	·	·	·	·	·	·	·	·	·
−7	1	−6	37	−233	1492	−9685	63581	−421356	·
−6	1	−5	26	−139	759	−4214	23711	−134873	·
−5	1	−4	17	−75	340	−1573	7393	−35178	·
−4	1	−3	10	−35	127	−472	1787	−6855	·
−3	1	−2	5	−13	36	−101	293	−848	·
−2	1	−1	2	−3	7	−10	31	−21	·
−1	1	0	1	1	4	11	41	162	·
0	1	1	2	5	15	52	203	877	·
1	1	2	5	15	52	203	877	4140	·
2	1	3	10	37	151	674	3263	17007	·
3	1	4	17	77	372	1915	10481	60814	·
4	1	5	26	141	799	4736	29371	190497	·
5	1	6	37	235	1540	10427	73013	529032	·
6	1	7	50	365	2727	20878	163967	1322035	·
7	1	8	65	537	4516	38699	338233	3017562	·
k	·	·	·	·	·	·	·	·	q_{kn}

For example, suppose that row 3 is given and we wish to find the next, row 4. We enter row 3 as the slanting row indicated by the boxed numbers below, and calculate entries in successive slanting rows as the sum of the adjacent two entries in the previous slanting row.

$$\begin{array}{ccccccccccccc}
\boxed{1} & & 5 & & 26 & & 141 & & 799 & & 4736 & \ldots \\
& \boxed{4} & & 21 & & 115 & & 658 & & 3937 & \ldots & \\
& & \boxed{17} & & 94 & & 543 & & 3279 & \ldots & & \\
& & & \boxed{77} & & 449 & & 2736 & \ldots & & & \\
& & & & \boxed{372} & & 2287 & \ldots & & & & \\
& & & & & \boxed{1915} & \ldots & & & & & \\
& & & & & & \boxed{10481} & \ldots & & & &
\end{array}$$

Then row 4 appears as the top line of the calculation. It is clear that, given the initial condition q_n, we can reconstruct the entire table for q_{kn} as the unique solution to the difference equation.

Problems

In passing we mention some further results, proposed here as problems that the reader may wish to solve using various ideas presented above.

(1) Show that

$$\frac{1^2}{1!}+\frac{3^2}{3!}+\frac{5^2}{5!}+\cdots = \frac{2^2}{2!}+\frac{4^2}{4!}+\frac{6^2}{6!}+\cdots .$$

Are there exponents other than $n = 2$ for which the equality holds?

(2) Show that

$$\frac{1^3}{1!}-\frac{2^3}{2!}+\frac{3^3}{3!}-+\cdots = \frac{1^4}{1!}-\frac{2^4}{2!}+\frac{3^4}{3!}-+\cdots .$$

Are there pairs of exponents other than 3, 4 for which an equality of this type holds?

(3) Show that

$$\frac{1^n}{1!}-\frac{3^n}{3!}+\frac{5^n}{5!}-+\cdots = -n\left(\frac{1}{1!}-\frac{1}{3!}+\frac{1}{5!}-+\cdots\right)$$

and

$$\frac{0^n}{0!}-\frac{2^n}{2!}+\frac{4^n}{4!}-+\cdots = -n\left(\frac{1}{0!}-\frac{1}{2!}+\frac{1}{4!}-+\cdots\right)$$

simultaneously hold for $n = 3$. Find all other n having the same property. If follows that

$$\left(\frac{0^n}{0!}-\frac{2^n}{2!}+\frac{4^n}{4!}-+\cdots\right)^2+\left(\frac{1^n}{1!}-\frac{3^n}{3!}+\frac{5^n}{5!}-+\cdots\right)^2$$

is an integer for $n = 3$. Clearly, this is also true for $n = 1$. Find all other n having the same property.

(4) Clearly,

$$\left(\frac{0^n}{0!}+\frac{2^n}{2!}+\frac{4^n}{4!}+\cdots\right)^2-\left(\frac{1^n}{1!}+\frac{3^n}{3!}+\frac{5^n}{5!}+\cdots\right)^2$$

is an integer for $n = 1$. Find all other n having the same property.

(5) Show that there are integers r_{kn} such that

$$\frac{k^n}{0!}-\frac{(k+1)^n}{1!}+\frac{(k+2)^n}{2!}-\frac{(k+3)^n}{3!}+-\cdots = \frac{r_{k,n+1}}{e} .$$

In particular $r_{1n} = r_n$, in consequence of which we may call r_{kn} the extended Rényi numbers.

(6) Show that there are integers a_{kn} , b_{kn} such that

$$\begin{aligned}&\frac{(2k)^n}{0!}+\frac{(2k+2)^n}{2!}+\frac{(2k+4)^n}{4!}+\cdots\\&= a_{k,n+1}\cosh(1)+b_{k,n+1}\sinh(1)\end{aligned}$$

and

$$\begin{aligned}&\frac{(2k+1)^n}{1!}+\frac{(2k+3)^n}{3!}+\frac{(2k+5)^n}{5!}+\cdots\\&= a_{k,n+1}\sinh(1)+b_{k,n+1}\cosh(1)\,.\end{aligned}$$

In particular, $a_{1n}=a_n$, $b_{1n}=b_n$.

(7) Show that there are integers c_{kn} , d_{kn} such that

$$\begin{aligned}&\frac{(2k)^n}{0!}-\frac{(2k+2)^n}{2!}+\frac{(2k+4)^n}{4!}-+\cdots\\&= c_{k,n+1}\cos(1)-d_{k,n+1}\sin(1)\end{aligned}$$

and

$$\begin{aligned}&\frac{(2k+1)^n}{1!}-\frac{(2k+3)^n}{3!}+\frac{(2k+5)^n}{5!}-+\cdots\\&= c_{k,n+1}\sin(1)+d_{k,n+1}\cos(1)\,.\end{aligned}$$

In particular, $c_{1n}=c_n$, $d_{1n}=d_n$.

(8) For a fixed k, write a recursion formula in terms of the Stirling numbers of the first kind for each of r_{kn} , a_{kn}, etc.

(9) Show that, for n fixed, each of the sequences of numbers r_{kn} , a_{kn}, etc., is in arithmetic progression of order $n-1$. Write polynomials of degree n, $R_n(k)$, $A_n(k)$, etc., defined such that $r_{kn}\equiv R_{n-1}(k+1)$, $a_{kn}\equiv A_{n-1}(k+1)$, etc.

(10) Write a recursion formula for each of the polynomials $R_n(k)$, $A_n(k)$. Write the polynomials in explicit form up to $n=8$.

(11) Write a recursion formula for each of r_{kn} , a_{kn}, etc.

(12) Find and tabulate the values of each of r_{kn} , a_{kn}, etc.

(13) Recall that the difference equation $\Delta^n q_{k1}=q_{k-1,n}$ uniquely determines the extended Bell numbers q_{kn} under the initial condition $q_{1n}=q_n$. Write a difference equation for the extended Rényi numbers r_{kn} under the initial condition $r_{1n}=r_n$. Do the same for each of a_{kn}, b_{kn}, etc.

(14) Let n be fixed; continue the sequence of extended Bell numbers q_{kn} for negative k (there are at least three ways of doing that) in order to justify Table 2 for $k=-1,-2,-3,\ldots$. Do the same for each of r_{kn}, a_{kn}, etc.

(15) Show that for each fixed k we have

$$q_{k+1,n+1} \;=\; q_{k1} + \binom{n}{1} q_{k2} + \binom{n}{2} q_{k3} + \cdots + \binom{n}{n} q_{k,n+1}$$

and

$$q_{k+1,n-1} \;=\; q_{k1} - \binom{n}{1} q_{k2} + \binom{n}{2} q_{k3} - + \cdots + (-1)^{n-1}\binom{n}{n} q_{k,n+1}\,.$$

(16) Show that for each fixed k and m we have

$$q_{k+m,n+1} \;=\; q_{k1}m^n + \binom{n}{1} q_{k2}m^{n-1} + \binom{n}{2} q_{k3}m^{n-2} + \cdots + \binom{n}{n} q_{k,n+1}\,;$$

in particular,

$$q_{m,n+1} \;=\; q_0 m^n + \binom{n}{1} q_1 m^{n-1} + \binom{n}{2} q_2 m^{n-2} + \cdots + \binom{n}{n} q_n\,.$$

(17) Show that

$$\left\{ {k \atop 1} \right\} q_{1n} + \left\{ {k \atop 2} \right\} q_{2n} + \cdots + \left\{ {k \atop k} \right\} q_{kn} \;=\; q_{n+k-1}\,.$$

(18) Show that

$$q_n - q_{n-1} + - \cdots + (-1)^{n+1} q_1 = q_{n-1}(0) \;=\; q_{-1,n+2}\,.$$

References

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Memorial University of Newfoundland
St.John's, CANADA A1C 5S7
e-mail: fekete@math.mun.ca