BOOK REVIEWS

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In Polya's Footsteps, by Ross Honsberger,
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Reviewed by ***Murray S. Klamkin***.

This is another in a series of books by the author on problems and solutions taken from various national and international olympiad competitions and very many of which have appeared previously in the Olympiad Corner of ***Crux Mathematicorum***. Also as the author notes, the solutions are his own unless otherwise acknowledged. Otherwise how would it look if everything was copied from elsewhere. I find these solutions of the author to be a mixed bag, especially in light of the following two quotations,

A good proof is one that makes us wiser. **Yu. I. Manin**

Proofs really aren't there to convince you something is true — they're there to show you why it is true. **Andrew Gleason**

Some of his solutions are a welcome addition, but there are also quite a number for which the previously published ones are much better and certainly not as overblown.

I now give some specific comments on some of the problems and their solutions.

- **page 14**. Given any sequence of r digits, is there a perfect square k^2 with these digits immediately preceding the last digit of k^2? The nice solution of the impossibility is ascribed to Andy Liu, but not from the University of Calgary. As a related aside, it could be mentioned that in problem 4621, *School Science and Mathematics*, it is shown that the given sequence of digits can be the first r or middle r digits of an infinite number of squares k^2.
- **page 26**. Here it is shown in an elongated fashion that $\prod_{i=1}^{n}\left(1+\frac{1}{a_i}\right) \geq (n+1)^n$, where $\sum_{i=1}^{n} a_1 = 1$ and $a_i \geq 0$. This is a well-known inequality appearing in many places. For a more elegant proof, we can use Hölder's Inequality to immediately obtain the known inequality $\prod_{i=1}^{n}\left(1+\frac{1}{a_i}\right)^{\frac{1}{n}} \geq 1+\frac{1}{\sqrt[n]{a_1a_2\cdots a_n}}$. We can then finish off using the AM–GM Inequality $\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^n \geq a_1a_2\cdots a_n$. This proof not only leads to learning about the important Hölder's Inequality, but also leads to generalizations. For example, let $\sum a_i = A$ and $\sum b_i = B$. Then $\prod_{i=1}^{n} \sqrt[n]{1+\frac{1}{a_i}+\frac{1}{b_i}} \geq 1+\frac{n}{A}+\frac{n}{B}$.
- **page 60**. Not noted in this nice geometry problem, is that $PQRS$ also has the property of having the least perimeter for quadrilaterals inscribed in $ABCD$.

- **page 76**. Here the author generalizes a Chinese problem by determining the remainder when $F(x^{n+1})$ is divided by $F(x)$ where $F(x) = 1 + x + \cdots + x^{n-1} + x^n$, and again gives an elongated solution. At the end he noted: "An alternative solution is given in [1992: 102]; additional comment appears in [1994: 46]." Left out is that the additional comment gives a wider generalization and a solution which is simpler and more compact.

- **page 93**. Here we are to determine the area of a triangle ABC given three concurrent cevians AD, BE and CF intersecting at P, where $AD = 20$, $BP = 6 = PE$, $CF = 9$ and $PF = 3$. The solution here is almost four pages long and uses a guess in solving an irrational equation since it is known that the solution is an integer. For a much more compact solution with more understanding, consider the following. Let $[G]$ denote the area of a figure G. Then immediately, $\frac{[APB]}{[ABC]} = \frac{3}{3+9} = \frac{1}{4}$ and $\frac{[APC]}{[ABC]} = \frac{6}{6+6} = \frac{1}{2}$, so that $\frac{[BPC]}{[ABC]} = \frac{1}{4}$ and $AP = 15$. Now lettinq $\angle BPC = \pi - \alpha$, $\angle CPA = \pi - \beta$ and $\angle APB = \pi - \gamma$, we have $2[BPC] = 6 \cdot 9 \sin\alpha = \frac{1}{2}[ABC]$, $2[CPA] = 9 \cdot 15 \sin\beta = [ABC]$, and $2[APB] = 15 \cdot 6 \sin\gamma = \frac{1}{2}[ABC]$. All that is left is to determine one of the angles α, β and γ, whose sum is π, and so are angles of some triangle with sides proportional to 15, 12 and 9, respectively. Since this is a right triangle, $\alpha = \frac{\pi}{2}$ and $[ABC] = 2 \cdot 6 \cdot 9 = 108$. Note that even if the latter was not a right triangle, we still can determine the angles.

- **page 96**. Here we have an elementary but long solution to find the smallest positive integer n which makes $m^n - 1$ divisible by 2^{1989} no matter what odd integer greater than 1 might be substituted for m. For another way of showing that $n = 2^{1987}$ is the smallest n, we use the known extension of Euler's Theorem $a^{\phi(n)} \equiv 1 \pmod{n}$, where $(a, n) = 1$, to the λ function. Here,

$$\begin{aligned} \lambda(2^\alpha) &= \phi(2^\alpha) \quad \text{if } \alpha = 0, 1, 2; \\ \lambda(2^\alpha) &= \tfrac{1}{2}\phi(2^\alpha) \quad \text{if } \alpha > 2; \\ \lambda(p^\alpha) &= \phi(p^\alpha) \quad \text{if } p \text{ is an odd prime}; \end{aligned}$$

  and $\lambda(2^\alpha p_1^{\alpha_1} p_1^{\alpha_2} \cdots p_n^{\alpha_n})$ is the least common multiple of $\lambda(2^\alpha)$, $\lambda(p_1^{\alpha_1})$, $\lambda(p_2^{\alpha_2}), \ldots, \lambda(p_n^{\alpha_n})$, where $2, p_1, p_2, \ldots, p_n$ are different primes. Then there is no exponent μ less than $\lambda(n)$ for which the congruence $a^\mu \equiv 1 \pmod{n}$ is satisfied for every integer a relatively prime to n. Hence $\lambda(2^{1989}) = \frac{1}{2}\phi(2^{1989}) = 2^{1987}$. See R.D. Carmichael, *The Theory of Numbers* Wiley, NY 1914.

- **page 106**. Here the problem is to determine the acute angle A of a triangle if it is given that vertex A lies on the perpendicular bisector joining the circumcentre O and the orthocentre H. A simpler solution follows almost immediately by using vectors. Letting A, B and C be vectors from O to the vertices A, B and C, respectively. We have $\mathrm{H} = \mathrm{A} + \mathrm{B} + \mathrm{C}$ so that $|\mathrm{A}| = |\mathrm{B} + \mathrm{C}|$. Squaring the latter equation, we get $R^2 = R^2 + R^2 + 2\mathrm{B} \cdot \mathrm{C} = 4R^2 - a^2$. Hence $a = R\sqrt{3} = 2R \sin A$ so that $A = 60^\circ$. Note that many metric properties of a triangle can be determined easily using this vector representation and the following ones for the centroid $\mathrm{G} = \frac{\mathrm{A}+\mathrm{B}+\mathrm{C}}{3}$ and the incentre $\mathrm{I} = \frac{a\mathrm{A}+b\mathrm{B}+c\mathrm{C}}{a+b+c}$. For example, $OH^2 = (\mathrm{A} + \mathrm{B} + \mathrm{C})^2 = \mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2 + 2\mathrm{B} \cdot \mathrm{C} + 2\mathrm{C} \cdot \mathrm{A} + 2\mathrm{A} \cdot \mathrm{B} = 9R^2 - a^2 - b^2 - c^2$, so that we have a simple proof of the inequality $9R^2 \geq a^2 + b^2 + c^2$, or equivalently, $\frac{9}{4} \geq \sin^2 A + \sin^2 B + \sin^2 C$. It also follows immediately that O, H and G are collinear with $OH = 3OG$. As an exercise for the reader, show that $OI^2 = R^2 - 2Rr$.

- **page 118**. Here AB, CD and EF are chords of a circle that are concurrent at K and are inclined to each other at 60° angles. One has to show that $KA + KD + KE = KB + KC + KF$. Again the referred to solution is neater and shorter. Another easy solution is to use the polar equation of the circle with origin at K.

- **page 122**. Here E is a point on a diameter AC of a circle centre O and one has to determine the chord BD through E which yields the quadrilateral $ABCD$ of the greatest area. Here the author claims that BD should be perpendicular to AC. However, this is correct only if $OA \geq OE\sqrt{2}$. For a complete and simpler solution, let $OA = r$, $OE = a$ and $\angle BEC = \theta$. Then $BE = -a\cos\theta + \sqrt{r^2 - a^2\sin^2\theta}$ and $ED = a\cos\theta + \sqrt{r^2 - a^2\sin^2\theta}$, so that $[ABCD] = 2r\sin\theta\sqrt{r^2 - a^2\sin^2\theta}$. Letting $t = \sin^2\theta$, we wish to maximize $F \equiv t(r^2 - a^2 t)$. Now F is increasing from $t = 0$ to $\frac{r^2}{2a^2}$. Consequently, if $r \leq a\sqrt{2}$, F is maximized for $t = \frac{r^2}{2a^2}$, and if $r > a\sqrt{2}$, F is maximized for $t = 1$.

- **page 133**. Here we have to maximize $abcd$ where a, b, c and d are integers with $a \leq b \leq c \leq d$ and $a + b + c + d = 30$. Again a previous solution in ***CRUX*** is simpler and more general and points out the use of the widely applicable Majorization Inequality.

- **page 134**. Here we are given tangents to a circle K from an external point P meeting K at A and B. We want to determine the position of C on the minor arc AB such that the tangent at C cuts from the figure a triangle PQR of maximum area. The author reduces the problem to minimizing the tangent chord QCR or, in his terms, $\tan\alpha + \tan(90^\circ - \phi - \alpha)$ where 2ϕ is the given angle APB. Then using calculus, he obtains the minimizing angle α as $\frac{90^\circ - \phi}{2}$, which places C at the mid-point of arc AB which is to be expected intuitively. He also justifies the minimum by examining the second derivative. My criticism here is one should eschew calculus methods in these problems. Since $\tan\alpha + \tan(90^\circ - \phi - \alpha) = \frac{\cos\phi}{\sin(\phi+\alpha)\cos\alpha} = \frac{2\cos\phi}{\sin\phi + \sin(\phi + 2\alpha)}$, we have a more elementary and simpler solution for the minimizing angle.

- **page 152**. Here we have to show that if for every point P inside a convex quadrilateral $ABCD$, the sum of the perpendiculars to the four sides (or their extensions if necessary) is constant, then $ABCD$ is a parallelogram. The solution given in approximately three pages is attributed to me. However, I don't quite recognize it: my solution in ***CRUX*** was considerably shorter.

- **page 158**. Here one has equilateral triangles drawn outwardly on the sides of a triangle ABC, so that the three new vertices are D, E and F. Given D, E and F, one has to construct ABC. Here I have no criticisms of the solution, especially since the author first reviews some basic ideas of translations and rotations in the plane and then uses these to get a nice solution. I just like to point out that a solution using complex numbers, although not particularly elegant, is very direct. Let $(A, B, C) = (z_1, z_2, z_3)$ and $(D, E, F) = (w_1, w_2, w_3)$. Then $2w_1 = z_2 + z_3 + i\sqrt{3}(z_2 - z_3)$. The other two equations follow by a cyclic change in the subscripts. After some simple algebra, we find $2z_1 = w_1 + w_2\frac{1+i\sqrt{3}}{2} - w_3\frac{1-i\sqrt{3}}{2}$, so that z_1 is easy to construct given w_1, w_2

and w_3. We can then construct z_2 and z_3. It also follows, by summing the three equations, that the centroids of ABC and DEF coincide.

- **page 184**. Here we have a problem on numbering an infinite checkerboard with an approximate three page solution and with no attribution. Andy Liu informed me of a prior source of this problem. It occurs with an approximately one page solution in A. M. Yaglom and I. M. Yaglom, *Challenging Mathematical Problems with Elementary Solutions II*, Holden-Day, San Francisco, 1967, p.129.
- **page 203**. Here we have to find the minimum value of the function $f(x) = \sqrt{a^2 + x^2} + \sqrt{(b-x)^2 + c^2}$, where a, b and c are positive numbers. This is a classic minimum distance problem and is treated in many places. The author repeats the classic reflection solution given in ***CRUX*** and also refers to an alternate solution given there but does not indicate that this solution gives immediate generalizations via Minkowski's Inequality which is another useful tool for competition contestants: $\sqrt[p]{a^p + x^p + y^p} + \sqrt[p]{b^p + (c-x)^p + (d-y)^p} \geq \sqrt[p]{(a+b)^p + c^p + d^p}$ where $p > 1$. If $1 > p > 0$, the inequality is reversed.
- **page 216**. Here we have to show there are infinitely many non-zero solutions to the Diophantine equation $x^2 + y^5 = z^3$. The author, after a page of work, comes up with the solution $(x, y, z) = (2^{15a+10}, 2^{6a+4}, 2^{10a+7})$. More general equations of this type have appeared as problems very many times in the last 100 years with more general solutions. For example, consider the equation $x^r + y^s + z^t = w^u$ where r, s, t and u are given positive integers with u relatively prime to rst. We now just let $x = a(a^r + b^s + c^t)^{mst}, y = b(a^r + b^s + c^t)^{mtr}$ and $z = c(a^r + b^s + c^t)^{mrs}$ so that $x^r + y^s + z^t = (a^r + b^s + c^t)^{mrst+1}$. On setting $w = (a^r + b^s + c^t)^n$, we have to ensure that there are integers m and n such that $mrst + 1 = nu$. Since u is relatively prime to rst, there is an infinite set of such pairs of positive integers m and n. A much more difficult problem is to find relatively prime solutions.
- **page 247**. Here one has to show that $(1 + x_1)(1 + x_2) \cdots (1 + x_n) \leq 1 + S + \frac{S^2}{2!} + \cdots + \frac{S^n}{n!}$, where the x's are positive numbers and $S = \sum x_i$. In addition to his solution, the author refers to an alternate solution in ***CRUX*** in 1989. It should be pointed out that the following is a known stronger inequality: $(1+x_1x)(1+x_2x) \cdots (1+x_nx) \prec 1+xS+\frac{x^2S^2}{2!}+\cdots+\frac{x^nS^n}{n!} \quad (x > 0)$. Here the $\prec$ sign indicates majorization; that is, the coefficient of x^r on the left is less than or equal to the coefficient of x^r on the right for $r = 1, 2, \ldots, n$. See [1982: 296].

Aside from my criticisms above, the book is a nice collection of problems and some essays. In a review [1998: 78] of the author's previous book in the same vein, *From Erdős to Kiev*, Bill Sands was somewhat critical of the random order of the problems. While that is true here also, I would not be too critical of this provided the solutions were given in order; that is, having all the geometry solutions together, the algebra solutions together, and so on. For when one is writing a mathematics competition, the order of problems is essentially at random. But when one is learning to solve problems on one's own, it will be more effective to have solutions to like problems linked together. For a student competitor using this book, I strongly advise that she or he also look at the corresponding solutions in ***CRUX with MAYHEM***.

Finally, I do not think the book's title, "*In Pólya's Footsteps*", is appropriate.