THE OLYMPIAD CORNER

No. 199

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The first Olympiad we give this issue is a Danish contest. My thanks go to Ravi Vakil for collecting a copy and forwarding it to me when he was Canadian Team Leader to the International Mathematical Olympiad at Mumbai.

GEORG MOHR KONKURRENCEN I MATEMATIK 1996
January 11, 9–13

Only writing and drawing materials are allowed.

1. $\angle C$ in $\triangle ABC$ is a right angle and the legs BC and AC are both of length 1. For an arbitrary point P on the leg BC, construct points Q, resp. R, on the hypotenuse, resp. on the other leg, such that PQ is parallel to AC and QR is parallel to BC. This divides the triangle into three parts.

Determine positions of the point P on BC such that the rectangular part has greater area than each of the other two parts.

2. Determine all triples (x, y, z), satisfying

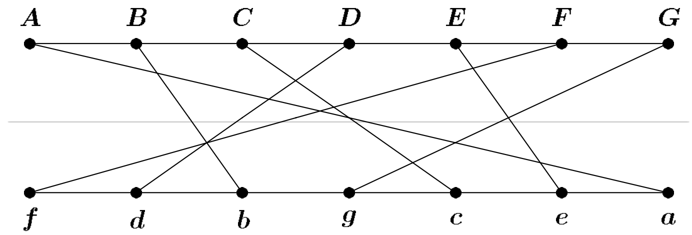
$$\begin{aligned} xy &= z \\ xz &= y \\ yz &= x\,. \end{aligned}$$

3. This year's idea for a gift is from "BabyMath", namely a series of 9 coloured plastic figures of decreasing sizes, alternating cube, sphere, cube, sphere, etc. Each figure may be opened and the succeeding one may be placed inside, fitting exactly. The largest and the smallest figures are both cubes. Determine the ratio between their side-lengths.

4. n is a positive integer. It is known that the last but one digit in the decimal expression of n^2 is 7. What is the last digit?

5. In a ballroom 7 gentlemen, A, B, C, D, E, F and G are sitting opposite 7 ladies a, b, c, d, e, f and g in arbitrary order. When the gentlemen walk across the dance floor to ask each of their ladies for a dance, one observes that at least two gentlemen walk distances of equal length. Is that always the case?

The figure shows an example. In this example $Bb = Ee$ and $Dd = Cc$.



The second two sets of problems come from the St. Petersburg City Mathematical Olympiad. Again my thanks go to Ravi Vakil, Canadian Team Leader to the International Mathematical Olympiad at Mumbai for collecting them for me.

ST. PETERSBURG CITY MATHEMATICAL OLYMPIAD

Third Round – February 25, 1996

11^{th} Grade (Time: 4 hours)

1. Serge was solving the equation $f(19x - 96/x) = 0$ and found 11 different solutions. Prove that if he tried hard he would be able to find at least one more solution.

2. The numbers $1, 2, \ldots, 2n$ are divided into two groups of n numbers. Prove that pairwise sums of numbers in each group (sums of the form $a + a$ included) have identical sets of remainders on division by $2n$.

3. No three diagonals of a convex 1996–gon meet in one point. Prove that the number of the triangles lying in the interior of the 1996–gon and having sides on its diagonals is divisible by 11.

4. Points A' and C' are taken on the diagonal BD of a parallelogram $ABCD$ so that $AA' \| CC'$. Point K lies on the segment $A'C$, and the line AK meets the line $C'C$ at the point L. A line parallel to BC is drawn through K, and a line parallel to BD is drawn through C. These two lines meet at point M. Prove that the points D, M, L are collinear.

5. Find all quadruplets of polynomials $p_1(x)$, $p_2(x)$, $p_3(x)$, $p_4(x)$ with real coefficients possessing the following remarkable property: for all integers x, y, z, t satisfying the condition $xy - zt = 1$, the equality $p_1(x)p_2(y) - p_3(z)p_4(t) = 1$ holds.

6. In a convex pentagon $ABCDE$, $AB = BC$, $\angle ABE + \angle DBC = \angle EBD$, and $\angle AEB + \angle BDE = 180°$. Prove that the orthocentre of triangle BDE lies on diagonal AC.

7. Two players play the following game on a 100×100 board. The first player marks some free square of the board, then the second puts a domino figure (that is, a 2×1 rectangle) covering two free squares one of which is marked. The first player wins if all the board is covered; otherwise the second wins. Which of the players has a winning strategy?

Selective Round – March 10, 1996

11^{th} Grade (Time: 5 hours)

1. It is known about real numbers $a_1, \ldots, a_{n+1}$; $b_1, \ldots, b_n$ that $0 \le b_k \le 1$ $(k = 1, \ldots, n)$ and $a_1 \ge a_2 \ge \ldots \ge a_{n+1} = 0$. Prove the inequality:

$$\sum_{k=1}^{n} a_k b_k \le \sum_{k=1}^{[\sum_{j=1}^{n} b_j]+1} a_k \,.$$

2. Segments AE and CF of equal length are taken on the sides AB and BC of a triangle ABC. The circle going through the points B, C, E and the circle going through the points A, B, F intersect at points B and D. Prove that the line BD is the bisector of angle ABC.

3. Prove that there are no positive integers a and b such that for all different primes p and q greater than 1000, the number $ap + bq$ is also prime.

4. A Young tableau is a figure obtained from an integral-sided rectangle by cutting out its cells covered by several integral-sided rectangles containing its right lower angle. We call a *hook* a part of the tableau consisting of some cell and all the cells lying either to the right of it (in the same row) or below it (in the same column). A Young tableau of n cells is given. Let s be the numbers of hooks containing exactly k cells. Prove that $s(k + s) \le 2n$.

5. In a triangle ABC the angle A is $60°$. A point O is taken inside the triangle such that $\angle AOB = \angle BOC = 120°$. A point D is chosen on the half-line CO such that the triangle AOD is equilateral. The perpendicular bisector of the segment AO meets the line BC at point Q. Prove that the line OQ divides the segment BD into two equal parts.

6. There are 120 underground lines in a city; every station may be reached from each other with not more than 15 changes. We say that two stations are distant from each other if not less than 5 changes are needed to reach one of them from the other. What maximum number of pairwise distant stations can be in this city?

7. a, b, c are integers. It is known that the polynomial $x^3 + ax^2 + bx + c$ has three different pairwise coprime positive integral roots, and the polynomial $ax^2 + bx + c$ has a positive integral root. Prove that the number $|a|$ is composite.

8. Positive integers $1, 2, \ldots, n^2$ are being arranged in the squares of an $n \times n$ table. When the next number is put in a free square, the sum of the numbers already arranged in the row and column containing this square is written on the blackboard. When the table is full, the sum of the numbers on the blackboard is found. Show an example of this way of putting the numbers in squares such that the sum is minimum.

We next turn to readers' solutions to problems given earlier in the *Corner*. First some catching up. Over the summer I had some time to sort and file materials, and discovered solutions to problems of the 1994 Balkan Olympiad misfiled with 1999 material. So we begin with solutions to the 1994 Balkan Olympiad [1997: 198].

2. [*Greece*] Show that the polynomial

$$x^4 - 1994x^3 + (1993+m)x^2 - 11x + m, \quad m \in \mathbb{Z}$$

has at most one integral root. (Success rate: $39.6\overline{6}\%$)

Solutions by Mansur Boase, student, Cambridge, England.

Consider

$$x^4 - 1994x^3 + (1993+m)x^2 - 11x + m\,. \qquad (*)$$

Suppose the given polynomial has two integral roots. Then neither can be odd for otherwise

$$(x^4 + 1993x^2) - (1994x^3) + m(x^2+1) - 11x$$

will be odd (as each of the terms in brackets is even) and hence non-zero.

Suppose $x_1 = 2^{r_1}a$, $r_1 > 0$ and a odd is a solution.

Considering the polynomial $(\bmod 2^{2r_1})$, we then have that $m \equiv 11x \pmod{2^{2r_1}}$. Hence $m \equiv 2^{r_1}(11a) \pmod{2^{2r_1}}$, and since a is odd, m must be of the form $2^{r_1}l_1$, l_1 odd.

If $x_2 = 2^{r_2}b$, b odd, is also a solution, then $m = 2^{r_2}l_2$, l_2 odd, so we must have $r_1 = r_2$.

Thus, if there are two integral roots, both must be of the form $2^r k$, $r > 1$, k odd. The product of the two roots must be a multiple of 2^{2r}. The quadratic which has these two roots as zeros is $x^2 + px + 2^{2r}q$, where p, q are integers.

Now the given polynomial (*) can be factorized into two quadratics:

$$(x^2 + px + 2^{2r}q)(x^2 + sx + t)\,.$$

If s were not integral, then the coefficient of x^3 would not be integral in the quartic, and if t were not integral, the coefficient of x^2 would not be integral in the quartic. Thus s and t must be integers and $m = 2^{2r}qt$. But the highest power of 2 dividing m is 2^r so $r \geq 2r$ giving $r = 0$, a contradiction. Hence the given quartic cannot have more than one integral root.

4. [*Bulgaria*] Find the smallest number $n > 4$ such that there is a set of n people with the following properties:

(i) any two people who know each other have no common acquaintances;

(ii) any two people who do not know each other have exactly two common acquaintances.

Note: Acquaintance is a symmetric relation. (Success rate: 19%)

Solution by Mansur Boase, student, Cambridge, England.

Choose one of the people, A, and suppose A knows x_1, x_2, $\ldots$, x_r. Then by (i) no x_i knows an x_j for $i \neq j$. Therefore, by (ii), for each pair $\{x_i, x_j\}$ there must exist an X_{ij} who knows both x_i and x_j in order that x_i and x_j have two common acquaintances A and X_{ij}. Now A cannot know any of the X_{ij}. Thus by (ii) each X_{ij} can have only two acquaintances among x_1, x_2, $\ldots$, x_r, namely x_i and x_j, so all the X_{ij} are distinct.

Any person who is not A, nor an acquaintance of A must by (ii) be an X_{ij}. Thus the total number of people must be $\binom{r}{2} + r + 1$.

Now $r > 2$ and $n > 4$.

$$\begin{aligned} \text{If } r &= 3, & \text{then } n &= 7 & (A), \\ \text{If } r &= 4, & \text{then } n &= 11 & (B), \\ \text{If } r &= 5, & \text{then } n &= 16 & (C). \end{aligned}$$

Let us label the people $1, 2, \ldots, n$.

Case (A): Without loss of generality suppose 1 knows 2, 3 and 4 and that 5 knows 2 and 3, 6 knows 2 and 4 and 7 knows 3 and 4.

Now 5 must have three acquaintances, so he must know one of 6 and 7. But he has common acquaintances with both 6 and 7, contradicting (i).

Case (B): Without loss of generality, suppose 1 knows 2, 3, 4 and 5 and 6, 7, 8, 9, 10, 11 know pairs $\{2, 3\}$, $\{2, 4\}$, $\{2, 5\}$, $\{3, 4\}$, $\{3, 5\}$ and $\{4, 5\}$ respectively.

Then 6 cannot know 7, 8, 9 or 10 as he has common acquaintances with each of them. So 6 can only know 2, 3 and 11, while he must know $r = 4$ people, a contradiction.

Case (C): We claim that $n = 16$ is the smallest number of people required in such a set, by noting that cases (A) and (B) fail and that the following acquaintance table satisfies (i) and (ii):

Person	Acquaintances				
1	2	3	4	5	6
2	1	7	8	9	10
3	1	7	11	12	13
4	1	8	11	14	15
5	1	9	12	14	16
6	1	10	13	15	16
7	2	3	14	15	16
8	2	4	12	13	16
9	2	5	11	13	15
10	2	6	11	12	14
11	3	4	9	10	16
12	3	5	8	10	15
13	3	6	8	9	14
14	4	5	7	10	13
15	4	6	7	9	12
16	5	6	7	8	11

Now we turn to solutions to problems of the Fourth Grade of the 38^{th} Mathematics Competition of the Republic of Slovenia [1998: 132].

1. Prove that there does not exist a function $f : \mathbb{Z} \to \mathbb{Z}$, for which $f(f(x)) = x + 1$ for every $x \in \mathbb{Z}$.

Solutions by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec; by Pierre Bornsztein, Courdimanche, France; by Masoud Kamgarpour, Carson Graham Secondary School, North Vancouver, BC; by Pavlos Maragoudakis, Pireas, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution by Maragoudakis.

Suppose that there is such a function. Then $f(f(f(x))) = f(x) + 1$. Since $f(f(x)) = x + 1$, we get $f(x + 1) = f(x) + 1$.

By induction $f(x + n) = f(x) + n$ for every $n \in \mathbb{N}$. Also $f(x) = f(x - n + n) = f(x - n) + n$ so

$$f(x - n) = f(x) - n \quad \text{for every} \quad n \in \mathbb{N}\,.$$

Finally $f(x + y) = f(x) + y$ for $x, y \in \mathbb{Z}$.

For $x = 0$, $f(y) = f(0) + y$. For $y = f(0)$, $f(f(0)) = f(0) + f(0)$.

But $f(f(0)) = 1$; thus $2f(0) = 1$, a contradiction.

2. Put a natural number in every empty field of the table so that you get an arithmetic sequence in every row and every column.

	74			
				186
		103		
0				

Solutions by Pierre Bornsztein, Courdimanche, France; by Masoud Kamgarpour, Carson Graham Secondary School, North Vancouver, BC; and by Pavlos Maragoudakis, Pireas, Greece. We give the solution of Kamgarpour.

Let us say the numbers adjacent to 0 are a_0 and a_1, with a_0 in the row and a_1 the column. We know that in an arithmetic sequence every term is the arithmetic mean of the term before and after. Therefore, we can put numbers in the chart as follows:

$3a_1$	74			
$2a_1$	$a_1 - a_0 + 103$			186
a_1	$\frac{1}{2}(a_1 + 103)$	103	$\frac{1}{2}(309 - a_1)$	$206 - a_1$
0	a_0	$2a_0$	$3a_0$	$4a_0$

Now

$$\tfrac{1}{2}(186 + 4a_0) \;=\; 206 - a_1 \;\Longrightarrow\; 93 + 2a_0 \;=\; 206 - a_1$$
$$\Longrightarrow\; 2a_0 + a_1 \;=\; 113\,, \qquad (1)$$

and

$$74 + \tfrac{1}{2}(a_1 + 103) \;=\; 2(103 + a_1 - a_0) \;\Longrightarrow\; 3a_1 - 4a_0 \;=\; -161\,. \quad (2)$$

Solving (1) and (2) gives $a_0 = 50$ and $a_1 = 13$, so we can easily put numbers in every field as shown:

52	82	112	142	172
39	74	109	144	179
26	66	106	146	186
13	58	103	148	193
0	50	100	150	200

3. Prove that every number of the sequence

$$49\,,\ 4489\,,\ 444889\,,\ 44448889\,,\ \ldots$$

is a perfect square (in every number there are n fours, $n-1$ eights and a nine).

Solutions by Pavlos Maragoudakis, Pireas, Greece; by Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Prielipp's solution.

Let $4_3 8_2 9$ denote the number 444889 and $6_2 7$ denote the number 667.

We shall show that

$$(6_{n-1}7)^2 \;=\; 4_n 8_{n-1} 9$$

for each positive integer n.

Because $(6_{n-1}7)^2 = \left(\frac{6(10^n-1)}{9}+1\right)^2$, it suffices to establish that

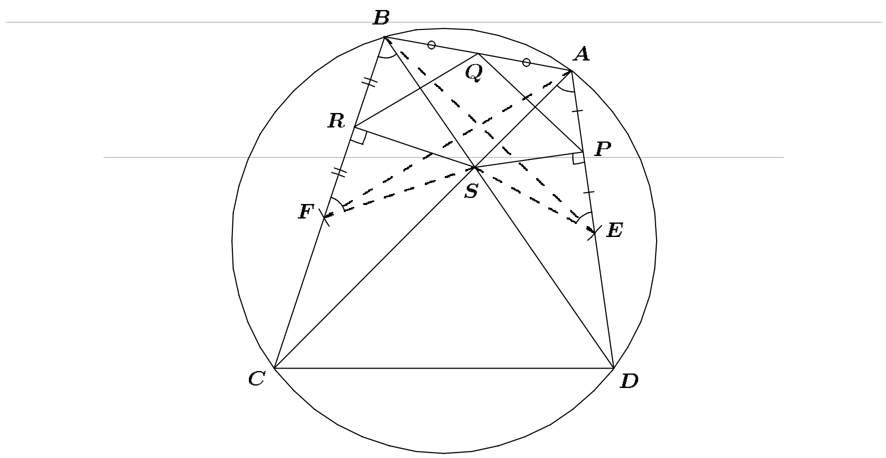
$$\left(\frac{6(10^n-1)}{9}+1\right)^2 \;=\; 4_n 8_{n-1} 9 \quad \text{for each positive integer } n\,. \qquad (*)$$

Let n be an arbitrary positive integer. Then

$$\begin{aligned}\left(\frac{6(10^n-1)}{9}+1\right)^2 &= \left(\frac{6\cdot 10^n+3}{9}\right)^2 \\ &= \frac{(2\cdot 10^n+1)^2}{9} \\ &= \frac{4\cdot 10^{2n}+4\cdot 10^n+1}{9} \\ &= \frac{40_{n-1}40_{n-1}1}{9} \;=\; 4_n 8_{n-1} 9\,.\end{aligned}$$

4. Let Q be the mid-point of the side AB of an inscribed quadrilateral $ABCD$ and S the intersection of its diagonals. Denote by P and R the orthogonal projections of S on AD and BC respectively. Prove that $|PQ| = |QR|$.

Solutions by Pierre Bornsztein, Courdimanche, France; and by Toshio Seimiya, Kawasaki, Japan. We give the solution of Seimiya.



Let E be the point symmetric to A with respect to P, and let F be the point symmetric to B with respect to R. Then we have

$$SE = SA, \quad \angle SEA = \angle SAE,$$

and

$$SF = SB, \quad \angle SFB = \angle SBF.$$

As A, B, C, D are concyclic we get $\angle CAD = \angle CBD$. Thus,

$$\angle SEA = \angle SAE = \angle SBF = \angle SFB.$$

Consequently we have $\angle ASE = \angle BSF$. Thus we get

$$\angle BSE = \angle BSA + \angle ASE = \angle BSA + \angle BSF = \angle FSA.$$

Since $SB = SF$ and $SE = SA$, we have

$$\triangle SEB \equiv \triangle SAF. \qquad (SAS)$$

Thus we get $EB = AF$. Since P, Q, R are mid-points of AE, AB, BF respectively, we have

$$PQ = \frac{1}{2}EB \quad \text{and} \quad QR = \frac{1}{2}AF.$$

Therefore we have $PQ = QR$.

That completes the Corner for this issue. Send me your nice solutions as well as contest materials.