THE ACADEMY CORNER

No. 26

Bruce Shawyer

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Abstracts • Résumés

Canadian Undergraduate Mathematics Conference 1998 — Part 4

The Dirichlet-to-Neumann Map

Scott MacLachlan
University of British Columbia

There are two classic problems in potential theory - the Dirichlet and Neumann problems. In most courses on partial differential equations the problems are treated separately; however there is a very elegant link between these problems that can be explored via Fourier analysis. The Dirichlet-to-Neumann map takes data for a Dirichlet problem and translates it into data for a Neumann problem without calculating a solution first. This map is useful in many areas, particularly in potential theory.

Systèmes de racines

Maciej Mizerski
McGill University

Un des plus beaux résultats de la théorie des groupes et algèbres de Lie est la classification des algèbres de Lie semi-simples. Cette classification se ramène à l'étude de la structure des systèmes de racines. Dans ma présentation, je vais introduire les systèmes de racines et survoler leur classification en utilisant les diagrammes de Dynkin. Aussi je vais tenter de faire le lien avec les groupes et les algèbres de Lie.

Cardan's Formulas for Solving Cubic Equations Revisited

Afroze Naqvi
University of Regina

This presentation offers a very brief history of Algebra and then proceeds to derive Cardan's Formulas for solving cubic equations. These formulas will be used to solve some cubic equations.

Mathematics of the Ancient Jews and Ancient Israel

Peter Papez
University of Calgary

As mathematicians we are well acquainted with the mathematics of the ancient Babylonians, Greeks and Egyptians. Even Ancient Chinese mathematicians have gained great notoriety in recent years. But the mathematics of ancient Israel is certainly noteworthy, if not impressive, and has not received the attention it deserves. The purpose of this discussion will be to introduce these impressive achievements and discuss some problems and the solutions obtained by ancient Jewish scholars. In order to facilitate the discussion, a brief overview of ancient Jewish history will be given. As well, the Talmud and Talmudic Law will be presented. The discussion will encompass the very beginning construction of numerals and numeracy within the ancient Jewish culture, proceed through the development of arithmetic, touch on important notions concerning science and logic, and conclude with a discussion of some fairly advanced developments in sampling and statistics. Various problems and their solutions, as derived by the ancient Jews, will be used to illustrate these developments and the entire discussion will be placed in a historical and Talmudic context.

Computing in the Quantum World

Christian Paquin
Université de Montréal

The current computing model is based on the laws of classical physics. But the world is not classical; it follows the laws of quantum mechanics. A quantum computer is a model of computation based on quantum mechanics. It has been proved that such a model is more powerful than its classical counterpart, meaning that it can do the same computations as a classical computer (in approximately the same time) but there exist some problems for which the quantum computer is much faster. In this paper I will explain what is quantum information, why a quantum computer would be useful, what are the problems to build a quantum computer and how it will work.

Wavelet Compression on Fractal Tilings

Daniel Piché
University of Waterloo

Over the last decade, wavelets have become increasingly useful for studying the behaviour of functions, and for compression. Though much remains to be investigated in this field, certain types of wavelets are fairly easy to construct, namely Haar wavelets. This paper ties together the theory of these wavelets with that of complex bases. An algorithm is proposed for doing wavelet analysis, with wavelets arising in this fashion. This will enable further study of the properties of these wavelets.

Introduction to Representation Theory

Evelyne Robidoux
McGill University

This paper provides an introduction to representation theory, with emphasis on representations of finite groups. The background required will be only a bit of group theory (that is, being familiar with the concepts of groups, homomorphisms, conjugacy classes) and linear algebra (vector spaces, linear transformations, inner products, direct sums). A classical reference for this material is the book by Serre, *Linear Representations of Finite Groups*, which I really recommend.

The Effect of Impurities in One-Dimensional Antiferromagnets

Alistair Savage
University of British Columbia

A brief overview of a paper to be published shortly with Ian Affleck of the University of British Columbia Department of Physics and Astronomy.

Authentication Codes Without Secrecy

Nelly Simões
Simon Fraser University

Suppose Alice wants to say something to Bob and cares only about the *authentication* of her message. Authentication makes it possible for Bob to receive Alice's message and be certain it came from her. Basically an authentication code without secrecy is a process in which a mathematical function transforms what Alice wants to say to Bob, we call this the *source state*, into what is called an *authentication tag* and adds it to the source state to form the *message*. We also suppose that Alice and Bob mutually trust each other. Alice wants to use an *unconditionally secure* method of authentication. She does not want anyone to be able to modify her message, not even a person with lots of computer power. This is why Alice decides to use a combinatorial method. We are going to explore an unconditionally secure way of authenticating Alice's message.

Integer Triangles With a Side of Given Length

Jill Taylor
Mount Allison University

This presentation will give a glimpse into the history of *Heronian triangles* (triangles with integer sides and integer area) and one special case of such triangles which I have researched this summer. However, the main focus will be a particular problem involving Heronian triangles with a given perimeter. The solution of this problem requires both number theory and basic geometrical concepts.

Convergence and Transcendence in the Field of p–adic Numbers

Sarah Sumner
Queen's University

We will first explore convergence properties of series in $\mathbb{Q}_p$, and then study instances when the series

$$\sum_{n=0}^{\infty} a_n p^n, \ a_n \in \mathbb{Q}_p$$

is transcendental over $\mathbb{Q}_p$. We will give a general result describing a large class of series of this type which are transcendental over $\mathbb{Q}_p$. Our result unfortunately does not resolve the transcendence of $\sum_{n=0}^{\infty} n!$ in $\mathbb{Q}_p$. However, our theorem applies to show transcendence of

$$\sum_{n=0}^{\infty} \zeta_n n!$$

where ζ_n is a primitive n-th root of unity if $(p, n) = 1$ and is 1 otherwise.

Random Number Generation

Renée Touzin
Université de Montréal

Nowadays, in many scientific fields, random number distributions are needed. Those distributions such as a binomial, a normal, an exponential are built from iid uniform $(0, 1)$ distributions. But this distribution does not really exist. In fact, it consists of a mathematical and deterministic algorithm that tries to behave stochastically.

The building of an efficient generator requires a strong knowledge of mathematics and computer science. A priori, a generator must follow certain theoretical criteria. A posteriori, those same generators must pass many statistical tests to be sure they look random even though they are deterministic. In this presentation, we will talk about different kinds of existing generators and the qualities of a good generator. We will give examples of good and bad generators and finally we will present a simple implementation in $\mathbb{C}$.

The Probabilistic Method: Proving Existence by Chance

Alexander Yong
University of Waterloo

The Probabilistic Method is based on a simple concept: in order to prove the existence of some mathematical object, construct an appropriate probability space and show that the object occurs with positive probability. We will investigate this powerful proof technique via three examples.

THE OLYMPIAD CORNER

No. 199

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

The first Olympiad we give this issue is a Danish contest. My thanks go to Ravi Vakil for collecting a copy and forwarding it to me when he was Canadian Team Leader to the International Mathematical Olympiad at Mumbai.

GEORG MOHR KONKURRENCEN I MATEMATIK 1996
January 11, 9–13

Only writing and drawing materials are allowed.

1. $\angle C$ in $\triangle ABC$ is a right angle and the legs BC and AC are both of length 1. For an arbitrary point P on the leg BC, construct points Q, resp. R, on the hypotenuse, resp. on the other leg, such that PQ is parallel to AC and QR is parallel to BC. This divides the triangle into three parts.

Determine positions of the point P on BC such that the rectangular part has greater area than each of the other two parts.

2. Determine all triples (x, y, z), satisfying

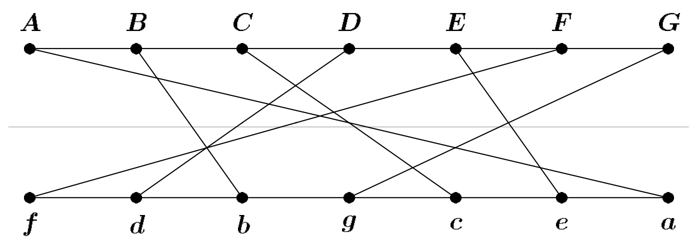
$$\begin{aligned} xy &= z \\ xz &= y \\ yz &= x\,. \end{aligned}$$

3. This year's idea for a gift is from "BabyMath", namely a series of 9 coloured plastic figures of decreasing sizes, alternating cube, sphere, cube, sphere, etc. Each figure may be opened and the succeeding one may be placed inside, fitting exactly. The largest and the smallest figures are both cubes. Determine the ratio between their side-lengths.

4. n is a positive integer. It is known that the last but one digit in the decimal expression of n^2 is 7. What is the last digit?

5. In a ballroom 7 gentlemen, A, B, C, D, E, F and G are sitting opposite 7 ladies a, b, c, d, e, f and g in arbitrary order. When the gentlemen walk across the dance floor to ask each of their ladies for a dance, one observes that at least two gentlemen walk distances of equal length. Is that always the case?

The figure shows an example. In this example $Bb = Ee$ and $Dd = Cc$.



The second two sets of problems come from the St. Petersburg City Mathematical Olympiad. Again my thanks go to Ravi Vakil, Canadian Team Leader to the International Mathematical Olympiad at Mumbai for collecting them for me.

ST. PETERSBURG CITY MATHEMATICAL OLYMPIAD

Third Round – February 25, 1996

11^{th} Grade (Time: 4 hours)

1. Serge was solving the equation $f(19x - 96/x) = 0$ and found 11 different solutions. Prove that if he tried hard he would be able to find at least one more solution.

2. The numbers $1, 2, \dots, 2n$ are divided into two groups of n numbers. Prove that pairwise sums of numbers in each group (sums of the form $a + a$ included) have identical sets of remainders on division by $2n$.

3. No three diagonals of a convex 1996–gon meet in one point. Prove that the number of the triangles lying in the interior of the 1996–gon and having sides on its diagonals is divisible by 11.

4. Points A' and C' are taken on the diagonal BD of a parallelogram $ABCD$ so that $AA' \| CC'$. Point K lies on the segment $A'C$, and the line AK meets the line $C'C$ at the point L. A line parallel to BC is drawn through K, and a line parallel to BD is drawn through C. These two lines meet at point M. Prove that the points D, M, L are collinear.

5. Find all quadruplets of polynomials $p_1(x)$, $p_2(x)$, $p_3(x)$, $p_4(x)$ with real coefficients possessing the following remarkable property: for all integers x, y, z, t satisfying the condition $xy - zt = 1$, the equality $p_1(x)p_2(y) - p_3(z)p_4(t) = 1$ holds.

6. In a convex pentagon $ABCDE$, $AB = BC$, $\angle ABE + \angle DBC = \angle EBD$, and $\angle AEB + \angle BDE = 180°$. Prove that the orthocentre of triangle BDE lies on diagonal AC.

7. Two players play the following game on a 100×100 board. The first player marks some free square of the board, then the second puts a domino figure (that is, a 2×1 rectangle) covering two free squares one of which is marked. The first player wins if all the board is covered; otherwise the second wins. Which of the players has a winning strategy?

Selective Round – March 10, 1996

11^{th} Grade (Time: 5 hours)

1. It is known about real numbers $a_1, \ldots, a_{n+1}$; $b_1, \ldots, b_n$ that $0 \leq b_k \leq 1$ $(k = 1, \ldots, n)$ and $a_1 \geq a_2 \geq \ldots \geq a_{n+1} = 0$. Prove the inequality:

$$\sum_{k=1}^{n} a_k b_k \leq \sum_{k=1}^{[\sum_{j=1}^{n} b_j]+1} a_k .$$

2. Segments AE and CF of equal length are taken on the sides AB and BC of a triangle ABC. The circle going through the points B, C, E and the circle going through the points A, B, F intersect at points B and D. Prove that the line BD is the bisector of angle ABC.

3. Prove that there are no positive integers a and b such that for all different primes p and q greater than 1000, the number $ap + bq$ is also prime.

4. A Young tableau is a figure obtained from an integral-sided rectangle by cutting out its cells covered by several integral-sided rectangles containing its right lower angle. We call a *hook* a part of the tableau consisting of some cell and all the cells lying either to the right of it (in the same row) or below it (in the same column). A Young tableau of n cells is given. Let s be the numbers of hooks containing exactly k cells. Prove that $s(k + s) \leq 2n$.

5. In a triangle ABC the angle A is $60°$. A point O is taken inside the triangle such that $\angle AOB = \angle BOC = 120°$. A point D is chosen on the half-line CO such that the triangle AOD is equilateral. The perpendicular bisector of the segment AO meets the line BC at point Q. Prove that the line OQ divides the segment BD into two equal parts.

6. There are 120 underground lines in a city; every station may be reached from each other with not more than 15 changes. We say that two stations are distant from each other if not less than 5 changes are needed to reach one of them from the other. What maximum number of pairwise distant stations can be in this city?

7. a, b, c are integers. It is known that the polynomial $x^3 + ax^2 + bx + c$ has three different pairwise coprime positive integral roots, and the polynomial $ax^2 + bx + c$ has a positive integral root. Prove that the number $|a|$ is composite.

8. Positive integers $1, 2, \ldots, n^2$ are being arranged in the squares of an $n \times n$ table. When the next number is put in a free square, the sum of the numbers already arranged in the row and column containing this square is written on the blackboard. When the table is full, the sum of the numbers on the blackboard is found. Show an example of this way of putting the numbers in squares such that the sum is minimum.

We next turn to readers' solutions to problems given earlier in the *Corner*. First some catching up. Over the summer I had some time to sort and file materials, and discovered solutions to problems of the 1994 Balkan Olympiad misfiled with 1999 material. So we begin with solutions to the 1994 Balkan Olympiad [1997: 198].

2. [*Greece*] Show that the polynomial

$$x^4 - 1994x^3 + (1993 + m)x^2 - 11x + m, \quad m \in \mathbb{Z}$$

has at most one integral root. (Success rate: $39.6\overline{6}\%$)

Solutions by Mansur Boase, student, Cambridge, England.

Consider

$$x^4 - 1994x^3 + (1993 + m)x^2 - 11x + m\,. \qquad (*)$$

Suppose the given polynomial has two integral roots. Then neither can be odd for otherwise

$$(x^4 + 1993x^2) - (1994x^3) + m(x^2 + 1) - 11x$$

will be odd (as each of the terms in brackets is even) and hence non-zero.

Suppose $x_1 = 2^{r_1}a$, $r_1 > 0$ and a odd is a solution.

Considering the polynomial $(\text{mod } 2^{2r_1})$, we then have that $m \equiv 11x \,(\text{mod } 2^{2r_1})$. Hence $m \equiv 2^{r_1}(11a)\,(\text{mod } 2^{2r_1})$, and since a is odd, m must be of the form $2^{r_1}l_1$, l_1 odd.

If $x_2 = 2^{r_2}b$, b odd, is also a solution, then $m = 2^{r_2}l_2$, l_2 odd, so we must have $r_1 = r_2$.

Thus, if there are two integral roots, both must be of the form $2^r k$, $r > 1$, k odd. The product of the two roots must be a multiple of 2^{2r}. The quadratic which has these two roots as zeros is $x^2 + px + 2^{2r}q$, where p, q are integers.

Now the given polynomial (*) can be factorized into two quadratics:

$$(x^2 + px + 2^{2r}q)(x^2 + sx + t)\,.$$

If s were not integral, then the coefficient of x^3 would not be integral in the quartic, and if t were not integral, the coefficient of x^2 would not be integral in the quartic. Thus s and t must be integers and $m = 2^{2r}qt$. But the highest power of 2 dividing m is 2^r so $r \geq 2r$ giving $r = 0$, a contradiction. Hence the given quartic cannot have more than one integral root.

4. [*Bulgaria*] Find the smallest number $n > 4$ such that there is a set of n people with the following properties:

(i) any two people who know each other have no common acquaintances;

(ii) any two people who do not know each other have exactly two common acquaintances.

Note: Acquaintance is a symmetric relation. (Success rate: 19%)

Solution by Mansur Boase, student, Cambridge, England.

Choose one of the people, A, and suppose A knows x_1, x_2, $\ldots$, x_r. Then by (i) no x_i knows an x_j for $i \neq j$. Therefore, by (ii), for each pair $\{x_i, x_j\}$ there must exist an X_{ij} who knows both x_i and x_j in order that x_i and x_j have two common acquaintances A and X_{ij}. Now A cannot know any of the X_{ij}. Thus by (ii) each X_{ij} can have only two acquaintances among x_1, x_2, $\ldots$, x_r, namely x_i and x_j, so all the X_{ij} are distinct.

Any person who is not A, nor an acquaintance of A must by (ii) be an X_{ij}. Thus the total number of people must be $\binom{r}{2} + r + 1$.

Now $r > 2$ and $n > 4$.

$$\begin{array}{llll} \text{If } r = 3, & \text{then } n = 7 & (A), \\ \text{If } r = 4, & \text{then } n = 11 & (B), \\ \text{If } r = 5, & \text{then } n = 16 & (C). \end{array}$$

Let us label the people $1, 2, \ldots, n$.

Case (A): Without loss of generality suppose 1 knows 2, 3 and 4 and that 5 knows 2 and 3, 6 knows 2 and 4 and 7 knows 3 and 4.

Now 5 must have three acquaintances, so he must know one of 6 and 7. But he has common acquaintances with both 6 and 7, contradicting (i).

Case (B): Without loss of generality, suppose 1 knows 2, 3, 4 and 5 and 6, 7, 8, 9, 10, 11 know pairs $\{2,3\}$, $\{2,4\}$, $\{2,5\}$, $\{3,4\}$, $\{3,5\}$ and $\{4,5\}$ respectively.

Then 6 cannot know 7, 8, 9 or 10 as he has common acquaintances with each of them. So 6 can only know 2, 3 and 11, while he must know $r = 4$ people, a contradiction.

Case (C): We claim that $n = 16$ is the smallest number of people required in such a set, by noting that cases (A) and (B) fail and that the following acquaintance table satisfies (i) and (ii):

Person	Acquaintances				
1	2	3	4	5	6
2	1	7	8	9	10
3	1	7	11	12	13
4	1	8	11	14	15
5	1	9	12	14	16
6	1	10	13	15	16
7	2	3	14	15	16
8	2	4	12	13	16
9	2	5	11	13	15
10	2	6	11	12	14
11	3	4	9	10	16
12	3	5	8	10	15
13	3	6	8	9	14
14	4	5	7	10	13
15	4	6	7	9	12
16	5	6	7	8	11

Now we turn to solutions to problems of the Fourth Grade of the 38^{th} Mathematics Competition of the Republic of Slovenia [1998: 132].

1. Prove that there does not exist a function $f : \mathbb{Z} \to \mathbb{Z}$, for which $f(f(x)) = x + 1$ for every $x \in \mathbb{Z}$.

Solutions by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec; by Pierre Bornsztein, Courdimanche, France; by Masoud Kamgarpour, Carson Graham Secondary School, North Vancouver, BC; by Pavlos Maragoudakis, Pireas, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution by Maragoudakis.

Suppose that there is such a function. Then $f(f(f(x))) = f(x) + 1$. Since $f(f(x)) = x + 1$, we get $f(x + 1) = f(x) + 1$.

By induction $f(x + n) = f(x) + n$ for every $n \in \mathbb{N}$. Also $f(x) = f(x - n + n) = f(x - n) + n$ so

$$f(x - n) = f(x) - n \quad \text{for every} \quad n \in \mathbb{N}\,.$$

Finally $f(x + y) = f(x) + y$ for $x, y \in \mathbb{Z}$.

For $x = 0$, $f(y) = f(0) + y$. For $y = f(0)$, $f(f(0)) = f(0) + f(0)$.

But $f(f(0)) = 1$; thus $2f(0) = 1$, a contradiction.

2. Put a natural number in every empty field of the table so that you get an arithmetic sequence in every row and every column.

	74			
				186
		103		
0				

Solutions by Pierre Bornsztein, Courdimanche, France; by Masoud Kamgarpour, Carson Graham Secondary School, North Vancouver, BC; and by Pavlos Maragoudakis, Pireas, Greece. We give the solution of Kamgarpour.

Let us say the numbers adjacent to 0 are a_0 and a_1, with a_0 in the row and a_1 the column. We know that in an arithmetic sequence every term is the arithmetic mean of the term before and after. Therefore, we can put numbers in the chart as follows:

$3a_1$	74			
$2a_1$	$a_1 - a_0 + 103$			186
a_1	$\frac{1}{2}(a_1 + 103)$	103	$\frac{1}{2}(309 - a_1)$	$206 - a_1$
0	a_0	$2a_0$	$3a_0$	$4a_0$

Now

$$\begin{aligned} \tfrac{1}{2}(186 + 4a_0) \;=\; 206 - a_1 \quad &\Longrightarrow \quad 93 + 2a_0 \;=\; 206 - a_1 \\ &\Longrightarrow \quad 2a_0 + a_1 \;=\; 113\,, \qquad (1) \end{aligned}$$

and

$$74 + \tfrac{1}{2}(a_1 + 103) \;=\; 2(103 + a_1 - a_0) \;\Longrightarrow\; 3a_1 - 4a_0 \;=\; -161\,. \quad (2)$$

Solving (1) and (2) gives $a_0 = 50$ and $a_1 = 13$, so we can easily put numbers in every field as shown:

52	82	112	142	172
39	74	109	144	179
26	66	106	146	186
13	58	103	148	193
0	50	100	150	200

3. Prove that every number of the sequence

$$49\,,\ 4489\,,\ 444889\,,\ 44448889\,,\ \ldots$$

is a perfect square (in every number there are n fours, $n-1$ eights and a nine).

Solutions by Pavlos Maragoudakis, Pireas, Greece; by Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Prielipp's solution.

Let $4_3 8_2 9$ denote the number 444889 and $6_2 7$ denote the number 667.

We shall show that

$$(6_{n-1}7)^2 \;=\; 4_n 8_{n-1} 9$$

for each positive integer n.

Because $(6_{n-1}7)^2 = \left(\frac{6(10^n-1)}{9}+1\right)^2$, it suffices to establish that

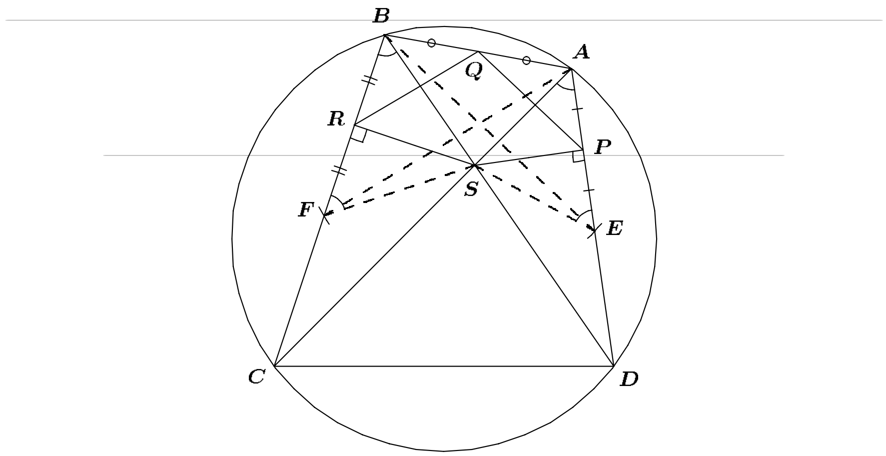
$$\left(\frac{6(10^n-1)}{9}+1\right)^2 \;=\; 4_n 8_{n-1} 9 \quad \text{for each positive integer } n\,. \qquad (*)$$

Let n be an arbitrary positive integer. Then

$$\begin{aligned}\left(\frac{6(10^n-1)}{9}+1\right)^2 &= \left(\frac{6\cdot 10^n+3}{9}\right)^2 \\ &= \frac{(2\cdot 10^n+1)^2}{9} \\ &= \frac{4\cdot 10^{2n}+4\cdot 10^n+1}{9} \\ &= \frac{40_{n-1}40_{n-1}1}{9} \;=\; 4_n 8_{n-1} 9\,.\end{aligned}$$

4. Let Q be the mid-point of the side AB of an inscribed quadrilateral $ABCD$ and S the intersection of its diagonals. Denote by P and R the orthogonal projections of S on AD and BC respectively. Prove that $|PQ| = |QR|$.

Solutions by Pierre Bornsztein, Courdimanche, France; and by Toshio Seimiya, Kawasaki, Japan. We give the solution of Seimiya.



Let E be the point symmetric to A with respect to P, and let F be the point symmetric to B with respect to R. Then we have

$$SE = SA, \quad \angle SEA = \angle SAE\,,$$

and

$$SF = SB, \quad \angle SFB = \angle SBF\,.$$

As A, B, C, D are concyclic we get $\angle CAD = \angle CBD$. Thus,

$$\angle SEA = \angle SAE = \angle SBF = \angle SFB\,.$$

Consequently we have $\angle ASE = \angle BSF$. Thus we get

$$\angle BSE = \angle BSA + \angle ASE = \angle BSA + \angle BSF = \angle FSA\,.$$

Since $SB = SF$ and $SE = SA$, we have

$$\triangle SEB \equiv \triangle SAF\,. \qquad (SAS)$$

Thus we get $EB = AF$. Since P, Q, R are mid-points of AE, AB, BF respectively, we have

$$PQ = \frac{1}{2}EB \quad \text{and} \quad QR = \frac{1}{2}AF\,.$$

Therefore we have $PQ = QR$.

That completes the Corner for this issue. Send me your nice solutions as well as contest materials.

BOOK REVIEWS

ALAN LAW

In Polya's Footsteps, by Ross Honsberger,
published by the Mathematical Association of America, 1997,
ISBN # 0-88385-326-4, softcover, 328+ pages, $28.95.
Reviewed by ***Murray S. Klamkin***.

This is another in a series of books by the author on problems and solutions taken from various national and international olympiad competitions and very many of which have appeared previously in the Olympiad Corner of ***Crux Mathematicorum***. Also as the author notes, the solutions are his own unless otherwise acknowledged. Otherwise how would it look if everything was copied from elsewhere. I find these solutions of the author to be a mixed bag, especially in light of the following two quotations,

A good proof is one that makes us wiser. **Yu. I. Manin**

Proofs really aren't there to convince you something is true — they're there to show you why it is true. **Andrew Gleason**

Some of his solutions are a welcome addition, but there are also quite a number for which the previously published ones are much better and certainly not as overblown.

I now give some specific comments on some of the problems and their solutions.

- **page 14**. Given any sequence of r digits, is there a perfect square k^2 with these digits immediately preceding the last digit of k^2? The nice solution of the impossibility is ascribed to Andy Liu, but not from the University of Calgary. As a related aside, it could be mentioned that in problem 4621, *School Science and Mathematics*, it is shown that the given sequence of digits can be the first r or middle r digits of an infinite number of squares k^2.
- **page 26**. Here it is shown in an elongated fashion that $\prod_{i=1}^{n}\left(1+\frac{1}{a_i}\right) \geq (n+1)^n$, where $\sum_{i=1}^{n} a_1 = 1$ and $a_i \geq 0$. This is a well-known inequality appearing in many places. For a more elegant proof, we can use Hölder's Inequality to immediately obtain the known inequality $\prod_{i=1}^{n}\left(1+\frac{1}{a_i}\right)^{\frac{1}{n}} \geq 1+\frac{1}{\sqrt[n]{a_1 a_2 \cdots a_n}}$. We can then finish off using the AM–GM Inequality $\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^n \geq a_1 a_2 \cdots a_n$. This proof not only leads to learning about the important Hölder's Inequality, but also leads to generalizations. For example, let $\sum a_i = A$ and $\sum b_i = B$. Then $\prod_{i=1}^{n} \sqrt[n]{1+\frac{1}{a_i}+\frac{1}{b_i}} \geq 1+\frac{n}{A}+\frac{n}{B}$.
- **page 60**. Not noted in this nice geometry problem, is that $PQRS$ also has the property of having the least perimeter for quadrilaterals inscribed in $ABCD$.

- **page 76**. Here the author generalizes a Chinese problem by determining the remainder when $F(x^{n+1})$ is divided by $F(x)$ where $F(x) = 1 + x + \cdots + x^{n-1} + x^n$, and again gives an elongated solution. At the end he noted: "An alternative solution is given in [1992: 102]; additional comment appears in [1994: 46]." Left out is that the additional comment gives a wider generalization and a solution which is simpler and more compact.

- **page 93**. Here we are to determine the area of a triangle ABC given three concurrent cevians AD, BE and CF intersecting at P, where $AD = 20$, $BP = 6 = PE$, $CF = 9$ and $PF = 3$. The solution here is almost four pages long and uses a guess in solving an irrational equation since it is known that the solution is an integer. For a much more compact solution with more understanding, consider the following. Let $[G]$ denote the area of a figure G. Then immediately, $\frac{[APB]}{[ABC]} = \frac{3}{3+9} = \frac{1}{4}$ and $\frac{[APC]}{[ABC]} = \frac{6}{6+6} = \frac{1}{2}$, so that $\frac{[BPC]}{[ABC]} = \frac{1}{4}$ and $AP = 15$. Now lettinq $\angle BPC = \pi - \alpha$, $\angle CPA = \pi - \beta$ and $\angle APB = \pi - \gamma$, we have $2[BPC] = 6 \cdot 9 \sin\alpha = \frac{1}{2}[ABC]$, $2[CPA] = 9 \cdot 15 \sin\beta = [ABC]$, and $2[APB] = 15 \cdot 6 \sin\gamma = \frac{1}{2}[ABC]$. All that is left is to determine one of the angles α, β and γ, whose sum is π, and so are angles of some triangle with sides proportional to 15, 12 and 9, respectively. Since this is a right triangle, $\alpha = \frac{\pi}{2}$ and $[ABC] = 2 \cdot 6 \cdot 9 = 108$. Note that even if the latter was not a right triangle, we still can determine the angles.

- **page 96**. Here we have an elementary but long solution to find the smallest positive integer n which makes $m^n - 1$ divisible by 2^{1989} no matter what odd integer greater than 1 might be substituted for m. For another way of showing that $n = 2^{1987}$ is the smallest n, we use the known extension of Euler's Theorem $a^{\phi(n)} \equiv 1 \pmod{n}$, where $(a, n) = 1$, to the λ function. Here,

$$\begin{aligned} \lambda(2^\alpha) &= \phi(2^\alpha) \quad \text{if } \alpha = 0, 1, 2; \\ \lambda(2^\alpha) &= \tfrac{1}{2}\phi(2^\alpha) \quad \text{if } \alpha > 2; \\ \lambda(p^\alpha) &= \phi(p^\alpha) \quad \text{if } p \text{ is an odd prime;} \end{aligned}$$

  and $\lambda(2^\alpha p_1^{\alpha_1} p_1^{\alpha_2} \cdots p_n^{\alpha_n})$ is the least common multiple of $\lambda(2^\alpha)$, $\lambda(p_1^{\alpha_1})$, $\lambda(p_2^{\alpha_2}), \ldots, \lambda(p_n^{\alpha_n})$, where $2, p_1, p_2, \ldots, p_n$ are different primes. Then there is no exponent μ less than $\lambda(n)$ for which the congruence $a^\mu \equiv 1 \pmod{n}$ is satisfied for every integer a relatively prime to n. Hence $\lambda(2^{1989}) = \frac{1}{2}\phi(2^{1989})$ $= 2^{1987}$. See R.D. Carmichael, *The Theory of Numbers* Wiley, NY 1914.

- **page 106**. Here the problem is to determine the acute angle A of a triangle if it is given that vertex A lies on the perpendicular bisector joining the circumcentre O and the orthocentre H. A simpler solution follows almost immediately by using vectors. Letting A, B and C be vectors from O to the vertices A, B and C, respectively. We have $\mathrm{H} = \mathrm{A} + \mathrm{B} + \mathrm{C}$ so that $|\mathrm{A}| = |\mathrm{B} + \mathrm{C}|$. Squaring the latter equation, we get $R^2 = R^2 + R^2 + 2\mathrm{B} \cdot \mathrm{C} = 4R^2 - a^2$. Hence $a = R\sqrt{3} = 2R \sin A$ so that $A = 60^\circ$. Note that many metric properties of a triangle can be determined easily using this vector representation and the following ones for the centroid $\mathrm{G} = \frac{\mathrm{A}+\mathrm{B}+\mathrm{C}}{3}$ and the incentre $\mathrm{I} = \frac{a\mathrm{A}+b\mathrm{B}+c\mathrm{C}}{a+b+c}$. For example, $OH^2 = (\mathrm{A} + \mathrm{B} + \mathrm{C})^2 = \mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2 + 2\mathrm{B} \cdot \mathrm{C} + 2\mathrm{C} \cdot \mathrm{A} + 2\mathrm{A} \cdot \mathrm{B} = 9R^2 - a^2 - b^2 - c^2$, so that we have a simple proof of the inequality $9R^2 \geq a^2 + b^2 + c^2$, or equivalently, $\frac{9}{4} \geq \sin^2 A + \sin^2 B + \sin^2 C$. It also follows immediately that O, H and G are collinear with $OH = 3OG$. As an exercise for the reader, show that $OI^2 = R^2 - 2Rr$.

- **page 118**. Here AB, CD and EF are chords of a circle that are concurrent at K and are inclined to each other at 60° angles. One has to show that $KA + KD + KE = KB + KC + KF$. Again the referred to solution is neater and shorter. Another easy solution is to use the polar equation of the circle with origin at K.

- **page 122**. Here E is a point on a diameter AC of a circle centre O and one has to determine the chord BD through E which yields the quadrilateral $ABCD$ of the greatest area. Here the author claims that BD should be perpendicular to AC. However, this is correct only if $OA \geq OE\sqrt{2}$. For a complete and simpler solution, let $OA = r$, $OE = a$ and $\angle BEC = \theta$. Then $BE = -a\cos\theta + \sqrt{r^2 - a^2\sin^2\theta}$ and $ED = a\cos\theta + \sqrt{r^2 - a^2\sin^2\theta}$, so that $[ABCD] = 2r\sin\theta\sqrt{r^2 - a^2\sin^2\theta}$. Letting $t = \sin^2\theta$, we wish to maximize $F \equiv t(r^2 - a^2t)$. Now F is increasing from $t = 0$ to $\frac{r^2}{2a^2}$. Consequently, if $r \leq a\sqrt{2}$, F is maximized for $t = \frac{r^2}{2a^2}$, and if $r > a\sqrt{2}$, F is maximized for $t = 1$.

- **page 133**. Here we have to maximize $abcd$ where a, b, c and d are integers with $a \leq b \leq c \leq d$ and $a + b + c + d = 30$. Again a previous solution in ***CRUX*** is simpler and more general and points out the use of the widely applicable Majorization Inequality.

- **page 134**. Here we are given tangents to a circle K from an external point P meeting K at A and B. We want to determine the position of C on the minor arc AB such that the tangent at C cuts from the figure a triangle PQR of maximum area. The author reduces the problem to minimizing the tangent chord QCR or, in his terms, $\tan\alpha + \tan(90^\circ - \phi - \alpha)$ where 2ϕ is the given angle APB. Then using calculus, he obtains the minimizing angle α as $\frac{90^\circ - \phi}{2}$, which places C at the mid-point of arc AB which is to be expected intuitively. He also justifies the minimum by examining the second derivative. My criticism here is one should eschew calculus methods in these problems. Since $\tan\alpha + \tan(90^\circ - \phi - \alpha) = \frac{\cos\phi}{\sin(\phi+\alpha)\cos\alpha} = \frac{2\cos\phi}{\sin\phi + \sin(\phi+2\alpha)}$, we have a more elementary and simpler solution for the minimizing angle.

- **page 152**. Here we have to show that if for every point P inside a convex quadrilateral $ABCD$, the sum of the perpendiculars to the four sides (or their extensions if necessary) is constant, then $ABCD$ is a parallelogram. The solution given in approximately three pages is attributed to me. However, I don't quite recognize it: my solution in ***CRUX*** was considerably shorter.

- **page 158**. Here one has equilateral triangles drawn outwardly on the sides of a triangle ABC, so that the three new vertices are D, E and F. Given D, E and F, one has to construct ABC. Here I have no criticisms of the solution, especially since the author first reviews some basic ideas of translations and rotations in the plane and then uses these to get a nice solution. I just like to point out that a solution using complex numbers, although not particularly elegant, is very direct. Let $(A, B, C) = (z_1, z_2, z_3)$ and $(D, E, F) = (w_1, w_2, w_3)$. Then $2w_1 = z_2 + z_3 + i\sqrt{3}(z_2 - z_3)$. The other two equations follow by a cyclic change in the subscripts. After some simple algebra, we find $2z_1 = w_1 + w_2\frac{1+i\sqrt{3}}{2} - w_3\frac{1-i\sqrt{3}}{2}$, so that z_1 is easy to construct given w_1, w_2

and w_3. We can then construct z_2 and z_3. It also follows, by summing the three equations, that the centroids of ABC and DEF coincide.

- **page 184**. Here we have a problem on numbering an infinite checkerboard with an approximate three page solution and with no attribution. Andy Liu informed me of a prior source of this problem. It occurs with an approximately one page solution in A. M. Yaglom and I. M. Yaglom, *Challenging Mathematical Problems with Elementary Solutions II*, Holden-Day, San Francisco, 1967, p.129.
- **page 203**. Here we have to find the minimum value of the function $f(x) = \sqrt{a^2 + x^2} + \sqrt{(b-x)^2 + c^2}$, where a, b and c are positive numbers. This is a classic minimum distance problem and is treated in many places. The author repeats the classic reflection solution given in ***CRUX*** and also refers to an alternate solution given there but does not indicate that this solution gives immediate generalizations via Minkowski's Inequality which is another useful tool for competition contestants: $\sqrt[p]{a^p + x^p + y^p} + \sqrt[p]{b^p + (c-x)^p + (d-y)^p} \geq \sqrt[p]{(a+b)^p + c^p + d^p}$ where $p > 1$. If $1 > p > 0$, the inequality is reversed.
- **page 216**. Here we have to show there are infinitely many non-zero solutions to the Diophantine equation $x^2 + y^5 = z^3$. The author, after a page of work, comes up with the solution $(x, y, z) = (2^{15a+10}, 2^{6a+4}, 2^{10a+7})$. More general equations of this type have appeared as problems very many times in the last 100 years with more general solutions. For example, consider the equation $x^r + y^s + z^t = w^u$ where r, s, t and u are given positive integers with u relatively prime to rst. We now just let $x = a(a^r + b^s + c^t)^{mst}, y = b(a^r + b^s + c^t)^{mtr}$ and $z = c(a^r + b^s + c^t)^{mrs}$ so that $x^r + y^s + z^t = (a^r + b^s + c^t)^{mrst+1}$. On setting $w = (a^r + b^s + c^t)^n$, we have to ensure that there are integers m and n such that $mrst + 1 = nu$. Since u is relatively prime to rst, there is an infinite set of such pairs of positive integers m and n. A much more difficult problem is to find relatively prime solutions.
- **page 247**. Here one has to show that $(1 + x_1)(1 + x_2) \cdots (1 + x_n) \leq 1 + S + \frac{S^2}{2!} + \cdots + \frac{S^n}{n!}$, where the x's are positive numbers and $S = \sum x_i$. In addition to his solution, the author refers to an alternate solution in ***CRUX*** in 1989. It should be pointed out that the following is a known stronger inequality: $(1+x_1x)(1+x_2x)\cdots(1+x_nx) \prec 1+xS+\frac{x^2S^2}{2!}+\cdots+\frac{x^nS^n}{n!} \quad (x > 0)$. Here the $\prec$ sign indicates majorization; that is, the coefficient of x^r on the left is less than or equal to the coefficient of x^r on the right for $r = 1, 2, \ldots, n$. See [1982: 296].

Aside from my criticisms above, the book is a nice collection of problems and some essays. In a review [1998: 78] of the author's previous book in the same vein, *From Erdős to Kiev*, Bill Sands was somewhat critical of the random order of the problems. While that is true here also, I would not be too critical of this provided the solutions were given in order; that is, having all the geometry solutions together, the algebra solutions together, and so on. For when one is writing a mathematics competition, the order of problems is essentially at random. But when one is learning to solve problems on one's own, it will be more effective to have solutions to like problems linked together. For a student competitor using this book, I strongly advise that she or he also look at the corresponding solutions in ***CRUX with MAYHEM***.

Finally, I do not think the book's title, "*In Pólya's Footsteps*", is appropriate.

APROPOS BELL AND STIRLING NUMBERS

Antal E. Fekete

Introduction

In 1877 Dobiñski stated [1] that there exist integers q_n such that

$$\frac{0^n}{0!}+\frac{1^n}{1!}+\frac{2^n}{2!}+\frac{3^n}{3!}+\cdots \;=\; q_n\left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots\right) \;=\; q_n e\,,$$

and he calculated their values for $n = 1$ through 5 (Table 1). Indeed, q_n are the Bell numbers, so named in honour of the American mathematician Eric Temple Bell (1883-1960), who was among the first to popularize these numbers; see [2] and [3], where further references can be found. It may be shown that q_n is just the sum of Stirling numbers of the second kind:

$$q_n \;=\; \left\{{n \atop 1}\right\}+\left\{{n \atop 2}\right\}+\cdots+\left\{{n \atop n}\right\}.$$

Since $\left\{{n \atop k}\right\}$ is the number of k-member quotient sets of an n-set, the Bell number q_n is the number of all quotient sets of an n-set. It can also be calculated via recursion in terms of the Stirling numbers of the first kind; the following formula is due to G.T. Williams [5]:

$$\left[{n \atop n}\right]q_n - \left[{n \atop n-1}\right]q_{n-1} + - \cdots + (-1)^{n-1}\left[{n \atop 1}\right]q_1 \;=\; 1\,,$$

where $\left[{n \atop k}\right]$ are the coefficients of the polynomial of degree n with roots 0 , 1 , 2 , $\ldots$, $(n-1)$:

$$\left[{n \atop n}\right]x^n - \left[{n \atop n-1}\right]x^{n-1} + - \cdots + (-1)^{n-1}\left[{n \atop 1}\right]x = x(x-1)(x-2)\cdots(x-n+1)\,.$$

Rényi numbers

Here we show that there exist integers r_n such that

$$\frac{0^n}{0!}-\frac{1^n}{1!}+\frac{2^n}{2!}-\frac{3^n}{3!}+-\cdots \;=\; r_n\left(\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+-\cdots\right) \;=\; \frac{r_n}{e}\,.$$

We calculate their values for $n = 1$ through 4 by bringing to the numerator the polynomial with coefficients $\left[{n \atop k}\right]$, and splitting it into linear factors which we can then cancel.

$r_1 = -1$:

$$\sum_0 (-1)^n \frac{n}{n!} \;=\; \sum_1 (-1)^n \frac{1}{(n-1)!} \;=\; -\frac{1}{e}\,.$$

$r_2 = 0$:

$$\sum_0 (-1)^n \frac{n^2}{n!} = \sum_0 (-1)^n \frac{n^2 - n}{n!} - \frac{1}{e} = \sum_0 (-1)^n \frac{(n-1)n}{n!} - \frac{1}{e}$$
$$= \sum_2 (-1)^n \frac{1}{(n-2)!} - \frac{1}{e} = \frac{1}{e} - \frac{1}{e} = 0\,.$$

$r_3 = 1$:

$$\sum_0 (-1)^n \frac{n^3}{n!} = \sum_0 (-1)^n \frac{n^3 - 3n^2 + 2n}{n!} + \frac{0}{e} + \frac{2}{e}$$
$$= \sum_0 (-1)^n \frac{(n-2)(n-1)n}{n!} + \frac{2}{e}$$
$$= \sum_2 (-1)^n \frac{1}{(n-3)!} + \frac{2}{e} = -\frac{1}{e} + \frac{2}{e} = \frac{1}{e}\,.$$

$r_4 = 1$:

$$\sum_0 (-1)^n \frac{n^4}{n!} = \sum_0 (-1)^n \frac{n^4 - 6n^3 + 11n^2 - 6n}{n!} + \frac{6}{e} - \frac{6}{e}$$
$$= \sum_0 (-1)^n \frac{(n-3)(n-2)(n-1)n}{n!}$$
$$= \sum_4 (-1)^n \frac{1}{(n-4)!} = \frac{1}{e}\,.$$

The recursion formula for r_n in terms of Stirling numbers of the first kind

$$\begin{bmatrix} n \\ n \end{bmatrix} r_n - \begin{bmatrix} n \\ n-1 \end{bmatrix} r_{n-1} + - \cdots + (-1)^{n-1} \begin{bmatrix} n \\ 1 \end{bmatrix} r_1 = (-1)^n$$

shows that r_n is an integer for all n (Table 1). To prove it we write the left–hand side as

$$e \sum_0 (-1)^k \frac{\left[{n \atop n}\right] k^n - \left[{n \atop n-1}\right] k^{n-1} + - \cdots + (-1)^{n-1} \left[{n \atop 1}\right] k}{k!}$$
$$= e \sum_0 (-1)^k \frac{(k-n+1)\cdots(k-1)k}{k!} = e \sum_n \frac{(-1)^k}{(k-n)!} = (-1)^n\,.$$

We call r_n Rényi numbers in honour of the Hungarian mathematician Alfréd Rényi (1921-1970) who first studied them [4]. They can be expressed as the alternating sum of the Stirling numbers of the second kind:

$$r_n = -\left\{ {n \atop 1} \right\} + \left\{ {n \atop 2} \right\} - + \cdots + (-1)^n \left\{ {n \atop n} \right\}.$$

Related numbers

We now introduce related numbers exhibiting properties similar to those of Bell and Rényi numbers. There exist integers a_n, b_n such that

$$\frac{0^n}{0!}+\frac{2^n}{2!}+\frac{4^n}{4!}+\cdots = a_n\cosh(1)+b_n\sinh(1)\,;$$

$$\text{and}\quad \frac{1^n}{1!}+\frac{3^n}{3!}+\frac{5^n}{5!}+\cdots = a_n\sinh(1)+b_n\cosh(1)$$

which can be calculated via recursion in terms of the Stirling numbers of the first kind:

$$\begin{bmatrix}n\\n\end{bmatrix}a_n-\begin{bmatrix}n\\n-1\end{bmatrix}a_{n-1}+-\cdots+(-1)^{n-1}\begin{bmatrix}n\\1\end{bmatrix}a_1 = \begin{cases}1 \text{ if } n \text{ is even}\\ 0 \text{ if } n \text{ is odd}\end{cases}$$

$$\begin{bmatrix}n\\n\end{bmatrix}b_n-\begin{bmatrix}n\\n-1\end{bmatrix}b_{n-1}+-\cdots+(-1)^{n-1}\begin{bmatrix}n\\1\end{bmatrix}b_1 = \begin{cases}0 \text{ if } n \text{ is even}\\ 1 \text{ if } n \text{ is odd}\end{cases}$$

Furthermore, there exist integers c_n, d_n such that

$$\frac{0^k}{0!}-\frac{2^k}{2!}+\frac{4^k}{4!}-+\cdots = c_k\cos(1)-d_k\sin(1)\,;$$

$$\text{and}\quad \frac{1^k}{1!}-\frac{3^k}{3!}+\frac{5^k}{5!}-+\cdots = c_k\sin(1)+d_k\cos(1)\,,$$

which can be calculated via recursion in terms of the Stirling numbers of the first kind:

$$\begin{bmatrix}n\\n\end{bmatrix}c_n-\begin{bmatrix}n\\n-1\end{bmatrix}c_{n-1}+-\cdots+(-1)^{n-1}\begin{bmatrix}n\\1\end{bmatrix}c_1 = \begin{cases}1 \text{ if } n=4k\\ -1 \text{ if } n=4k+2\\ 0 \text{ if } n \text{ is odd}\end{cases}$$

$$\begin{bmatrix}n\\n\end{bmatrix}d_n-\begin{bmatrix}n\\n-1\end{bmatrix}d_{n-1}+-\cdots+(-1)^{n-1}\begin{bmatrix}n\\1\end{bmatrix}d_1 = \begin{cases}0 \text{ if } n \text{ is even}\\ 1 \text{ if } n=4k+1\\ -1 \text{ if } n=4k+3\end{cases}$$

Table 1: Bell, Rényi and related numbers

n	0	1	2	3	4	5	6	7	8	9	10
q_n	1	1	2	5	15	52	203	877	4140	21147	115975
r_n	1	−1	0	1	1	−2	−9	−9	50	267	413
a_n	1	0	1	3	8	25	97	434	2095	10707	58194
b_n	0	1	1	2	7	27	106	443	2045	10440	57781
c_n	1	0	−1	−3	−6	−5	33	266	1309	4905	11516
d_n	0	1	1	0	−5	−23	−74	−161	57	3466	27361

These related numbers are useful in calculating the number of quotient sets of an n–set with an even or odd number of members:

$$a_n = \frac{q_n + r_n}{2} = \left\{ {n \atop 2} \right\} + \left\{ {n \atop 4} \right\} + \left\{ {n \atop 6} \right\} + \cdots ;$$

$$\text{and} \quad b_n = \frac{q_n - r_n}{2} = \left\{ {n \atop 1} \right\} + \left\{ {n \atop 3} \right\} + \left\{ {n \atop 5} \right\} + \cdots .$$

We also have

$$c_n = -\left\{ {n \atop 2} \right\} + \left\{ {n \atop 4} \right\} - \left\{ {n \atop 6} \right\} + - \cdots ;$$

$$\text{and} \quad d_n = \left\{ {n \atop 1} \right\} - \left\{ {n \atop 3} \right\} + \left\{ {n \atop 5} \right\} - + \cdots .$$

Extended Bell numbers

There exist integers q_{kn} such that

$$\frac{k^n}{0!} + \frac{(k+1)^n}{1!} + \frac{(k+2)^n}{2!} + \frac{(k+3)^n}{3!} + \cdots = e\, q_{k,n+1} .$$

In particular, $q_{1n} = q_n$; for this reason we call q_{kn} the extended Bell numbers. For each fixed k we have a recursion formula in terms of the Stirling numbers of the first kind

$$q_{kn} = (-1)^{k-1} \left(\left[{k \atop 1} \right] q_n - \left[{k \atop 2} \right] q_{n+1} + \left[{k \atop 3} \right] q_{n+2} - + \cdots + (-1)^{k-1} \left[{k \atop k} \right] q_{n+k-1} \right) ,$$

providing the *first* of three methods to calculate the extended Bell numbers (Table 2).

A *second* method is via the recursion

$$q_{kn} = k q_{k,n-1} + q_{k+1,n-1} .$$

For a fixed n, the extended Bell numbers $q_{0,n}$, $q_{1,n}$, $q_{2,n}$, ... are in arithmetic progression of order $n-1$. Therefore $q_{kn} \equiv Q_{n-1}(k+1)$ is a polynomial of degree $n-1$ in the variable k. We have the recursion

$$Q_n(k) \equiv (k-1)Q_{n-1}(k) + Q_{n-1}(k+1)$$

enabling us to calculate these polynomials:

$$Q_0(k) \equiv 1 \qquad \text{(constant)}$$
$$Q_1(k) \equiv k$$
$$Q_2(k) \equiv k^2 + 1$$
$$Q_3(k) \equiv k^3 + 3k + 1$$
$$Q_4(k) \equiv k^4 + 6k^2 + 4k + 4$$
$$Q_5(k) \equiv k^5 + 10k^3 + 10k^2 + 20k + 11$$

$$Q_6(k) \equiv k^6 + 15k^4 + 20k^3 + 60k^2 + 66k + 41$$
$$Q_7(k) \equiv k^7 + 21k^5 + 35k^4 + 140k^3 + 231k^2 + 287k + 162$$
$$Q_8(k) \equiv k^8 + 28k^6 + 56k^5 + 280k^4 + 616k^3 + 1148k^2 + 1296k + 715$$
$$\cdots\cdots\cdots\cdots$$

A *third* method of calculating q_{kn} is furnished by the difference equation $\Delta^n q_{k1} = q_{k-1,n}$ that may be verbalized as follows. In Table 2, the k–th *row can be obtained by calculating the first member of each of the higher order difference sequences of the* $(k+1)$–st *row*. This property is useful not only in calculating the k–th row from the $(k+1)$–st, quickly and efficiently, but also the other way around.

Table 2: Extended Bell Numbers

$k \backslash n$	1	2	3	4	5	6	7	8	n
·	·	·	·	·	·	·	·	·	·
−7	1	−6	37	−233	1492	−9685	63581	−421356	·
−6	1	−5	26	−139	759	−4214	23711	−134873	·
−5	1	−4	17	−75	340	−1573	7393	−35178	·
−4	1	−3	10	−35	127	−472	1787	−6855	·
−3	1	−2	5	−13	36	−101	293	−848	·
−2	1	−1	2	−3	7	−10	31	−21	·
−1	1	0	1	1	4	11	41	162	·
0	1	1	2	5	15	52	203	877	·
1	1	2	5	15	52	203	877	4140	·
2	1	3	10	37	151	674	3263	17007	·
3	1	4	17	77	372	1915	10481	60814	·
4	1	5	26	141	799	4736	29371	190497	·
5	1	6	37	235	1540	10427	73013	529032	·
6	1	7	50	365	2727	20878	163967	1322035	·
7	1	8	65	537	4516	38699	338233	3017562	·
k	·	·	·	·	·	·	·	·	q_{kn}

For example, suppose that row 3 is given and we wish to find the next, row 4. We enter row 3 as the slanting row indicated by the boxed numbers below, and calculate entries in successive slanting rows as the sum of the adjacent two entries in the previous slanting row.

$\boxed{1}$		5		26		141		799		4736 ...
	$\boxed{4}$		21		115		658		3937	...
		$\boxed{17}$		94		543		3279	...	
			$\boxed{77}$		449		2736	...		
				$\boxed{372}$		2287	...			
					$\boxed{1915}$	...				
						$\boxed{10481}$	...			

Then row 4 appears as the top line of the calculation. It is clear that, given the initial condition q_n, we can reconstruct the entire table for q_{kn} as the unique solution to the difference equation.

Problems

In passing we mention some further results, proposed here as problems that the reader may wish to solve using various ideas presented above.

(1) Show that

$$\frac{1^2}{1!}+\frac{3^2}{3!}+\frac{5^2}{5!}+\cdots = \frac{2^2}{2!}+\frac{4^2}{4!}+\frac{6^2}{6!}+\cdots .$$

Are there exponents other than $n = 2$ for which the equality holds?

(2) Show that

$$\frac{1^3}{1!}-\frac{2^3}{2!}+\frac{3^3}{3!}-+\cdots = \frac{1^4}{1!}-\frac{2^4}{2!}+\frac{3^4}{3!}-+\cdots .$$

Are there pairs of exponents other than 3, 4 for which an equality of this type holds?

(3) Show that

$$\frac{1^n}{1!}-\frac{3^n}{3!}+\frac{5^n}{5!}-+\cdots = -n\left(\frac{1}{1!}-\frac{1}{3!}+\frac{1}{5!}-+\cdots\right)$$

and

$$\frac{0^n}{0!}-\frac{2^n}{2!}+\frac{4^n}{4!}-+\cdots = -n\left(\frac{1}{0!}-\frac{1}{2!}+\frac{1}{4!}-+\cdots\right)$$

simultaneously hold for $n = 3$. Find all other n having the same property. If follows that

$$\left(\frac{0^n}{0!}-\frac{2^n}{2!}+\frac{4^n}{4!}-+\cdots\right)^2+\left(\frac{1^n}{1!}-\frac{3^n}{3!}+\frac{5^n}{5!}-+\cdots\right)^2$$

is an integer for $n = 3$. Clearly, this is also true for $n = 1$. Find all other n having the same property.

(4) Clearly,

$$\left(\frac{0^n}{0!}+\frac{2^n}{2!}+\frac{4^n}{4!}+\cdots\right)^2-\left(\frac{1^n}{1!}+\frac{3^n}{3!}+\frac{5^n}{5!}+\cdots\right)^2$$

is an integer for $n = 1$. Find all other n having the same property.

(5) Show that there are integers r_{kn} such that

$$\frac{k^n}{0!}-\frac{(k+1)^n}{1!}+\frac{(k+2)^n}{2!}-\frac{(k+3)^n}{3!}+-\cdots = \frac{r_{k,n+1}}{e} .$$

In particular $r_{1n} = r_n$, in consequence of which we may call r_{kn} the extended Rényi numbers.

(6) Show that there are integers a_{kn} , b_{kn} such that

$$\frac{(2k)^n}{0!} + \frac{(2k+2)^n}{2!} + \frac{(2k+4)^n}{4!} + \cdots = a_{k,n+1}\cosh(1) + b_{k,n+1}\sinh(1)$$

and
$$\frac{(2k+1)^n}{1!} + \frac{(2k+3)^n}{3!} + \frac{(2k+5)^n}{5!} + \cdots = a_{k,n+1}\sinh(1) + b_{k,n+1}\cosh(1)\,.$$

In particular, $a_{1n} = a_n$, $b_{1n} = b_n$.

(7) Show that there are integers c_{kn} , d_{kn} such that

$$\frac{(2k)^n}{0!} - \frac{(2k+2)^n}{2!} + \frac{(2k+4)^n}{4!} - + \cdots = c_{k,n+1}\cos(1) - d_{k,n+1}\sin(1)$$

and
$$\frac{(2k+1)^n}{1!} - \frac{(2k+3)^n}{3!} + \frac{(2k+5)^n}{5!} - + \cdots = c_{k,n+1}\sin(1) + d_{k,n+1}\cos(1)\,.$$

In particular, $c_{1n} = c_n$, $d_{1n} = d_n$.

(8) For a fixed k, write a recursion formula in terms of the Stirling numbers of the first kind for each of r_{kn} , a_{kn}, etc.

(9) Show that, for n fixed, each of the sequences of numbers r_{kn} , a_{kn}, etc., is in arithmetic progression of order $n-1$. Write polynomials of degree n, $R_n(k)$, $A_n(k)$, etc., defined such that $r_{kn} \equiv R_{n-1}(k+1)$, $a_{kn} \equiv A_{n-1}(k+1)$, etc.

(10) Write a recursion formula for each of the polynomials $R_n(k)$, $A_n(k)$. Write the polynomials in explicit form up to $n = 8$.

(11) Write a recursion formula for each of r_{kn} , a_{kn}, etc.

(12) Find and tabulate the values of each of r_{kn} , a_{kn}, etc.

(13) Recall that the difference equation $\Delta^n q_{k1} = q_{k-1,n}$ uniquely determines the extended Bell numbers q_{kn} under the initial condition $q_{1n} = q_n$. Write a difference equation for the extended Rényi numbers r_{kn} under the initial condition $r_{1n} = r_n$. Do the same for each of a_{kn}, b_{kn}, etc.

(14) Let n be fixed; continue the sequence of extended Bell numbers q_{kn} for negative k (there are at least three ways of doing that) in order to justify Table 2 for $k = -1, -2, -3, \ldots$. Do the same for each of r_{kn}, a_{kn}, etc.

(15) Show that for each fixed k we have

$$q_{k+1,n+1} = q_{k1} + \binom{n}{1} q_{k2} + \binom{n}{2} q_{k3} + \cdots + \binom{n}{n} q_{k,n+1}$$

and

$$q_{k+1,n-1} = q_{k1} - \binom{n}{1} q_{k2} + \binom{n}{2} q_{k3} - + \cdots + (-1)^{n-1} \binom{n}{n} q_{k,n+1} \,.$$

(16) Show that for each fixed k and m we have

$$q_{k+m,n+1} = q_{k1} m^n + \binom{n}{1} q_{k2} m^{n-1} + \binom{n}{2} q_{k3} m^{n-2} + \cdots + \binom{n}{n} q_{k,n+1} \,;$$

in particular,

$$q_{m,n+1} = q_0 m^n + \binom{n}{1} q_1 m^{n-1} + \binom{n}{2} q_2 m^{n-2} + \cdots + \binom{n}{n} q_n \,.$$

(17) Show that

$$\left\{ {k \atop 1} \right\} q_{1n} + \left\{ {k \atop 2} \right\} q_{2n} + \cdots + \left\{ {k \atop k} \right\} q_{kn} = q_{n+k-1} \,.$$

(18) Show that

$$q_n - q_{n-1} + - \cdots + (-1)^{n+1} q_1 = q_{n-1}(0) = q_{-1,n+2} \,.$$

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THE SKOLIAD CORNER

No. 39

R.E. Woodrow

This issue we give another example of a team competition with the problems of the 1998 Florida Mathematics Olympiad, written May 14, 1998. The contest was organized by Florida Atlantic University. My thanks go to John Grant McLoughlin, Memorial University of Newfoundland for sending me the problems.

FLORIDA MATHEMATICS OLYMPIAD
TEAM COMPETITION
May 14, 1998

1. Find all integers x, if any, such that $9 < x < 15$ and the sequence

$$1\,,\,2\,,\,6\,,\,7\,,\,9\,,\,x\,,\,15\,,\,18\,,\,20$$

does not have three terms in arithmetic progression. If there are no such integers, write "NONE."

2. A sequence a_1, a_2, $a_3, \ldots$ is said to satisfy a *linear recurrence relation of order two* if and only if there are numbers p and q such that, for all positive integers n,

$$a_{n+2} \;=\; pa_{n+1} + qa_n\,.$$

Find the next two terms of the sequence

$$2\,,\,5\,,\,14\,,\,41\,,\;\ldots\;,$$

assuming that this sequence satisfies a linear recurrence relation of order two.

3. Seven tests are given and on each test no ties are possible. Each person who is the top scorer on at least one of the tests or who is in the top six on at least four of these tests is given an award, but each person can receive at most one award. Find the maximum number of people who could be given awards, if 100 students take these tests.

4. Some primes can be written as a sum of two squares. We have, for example, that

$$5 \;=\; 1^2 + 2^2, \quad 13 \;=\; 2^2 + 3^2, \quad 17 \;=\; 1^2 + 4^2\,,$$

$$29 \;=\; 2^2 + 5^2, \quad 37 \;=\; 1^2 + 6^2, \quad \text{and} \quad 41 \;=\; 4^2 + 5^2\,.$$

The odd primes less than 108 are listed below; the ones that can be written as a sum of two squares are boxed in.

$$3\,,\ \boxed{5}\,,\ 7\,,\ 11\,,\boxed{13}\,,\ \boxed{17}\,,\ 19\,,\ 23\,,\ \boxed{29}\,,\ 31\,,$$

$$\boxed{37}\,,\ \boxed{41}\,,\ 43\,,\ 47\,,\ \boxed{53}\,,\ 59\,,\ \boxed{61}\,,\ 67\,,\ 71\,,$$

$$\boxed{73}\,,\ 79\,,\ 83\,,\ \boxed{89}\,,\ \boxed{97}\,,\ \boxed{101}\,,\ 103\,,\ 107\,.$$

The primes that can be written as a sum of two squares follow a simple pattern. See if you can correctly find this pattern. If you can, use this pattern to determine which of the primes between 1000 and 1050 can be written as a sum of two squares; there are five of them. The primes between 1000 and 1050 are

$$1009\,,\ 1013\,,\ 1019\,,\ 1021\,,\ 1031\,,\ 1033\,,\ 1039\,,\ 1049\,.$$

No credit unless the correct five primes are listed.

5. The sides of a triangle are 4, 13, and 15. Find the radius of the inscribed circle.

6. In Athenian criminal proceedings, ordinary citizens presented the charges, and the 500-man juries voted twice: first on guilt or innocence, and then (if the verdict was guilty) on the penalty. In 399BCE, Socrates (c469–399) was charged with dishonouring the gods and corrupting the youth of Athens. He was found guilty; the penalty was death. According to I.F. Stone's calculations on how the jurors voted:

(i) There were no abstentions;

(ii) There were 80 more votes for the death penalty than there were for the guilty verdict;

(iii) The sum of the number of votes for an innocent verdict and the number of votes against the death penalty equalled the number of votes in favour of the death penalty.

a) How many of the 500 jurors voted for an innocent verdict?

b) How many of the 500 jurors voted in favour of the death penalty?

7. Find all x such that $0 \le x \le \pi$ and

$$\tan^3 x - 1 + \frac{1}{\cos^2 x} - 3\cot\left(\frac{\pi}{2} - x\right) \;=\; 3\,.$$

Your answer should be in radian measure.

Last issue we gave the problems of the Newfoundland and Labrador Teachers' Association Mathematics League. Here are the answers.

NLTA MATH LEAGUE
GAME 1 — 1998–99

1. Find a two-digit number that equals twice the product of its digits.

Solution. Denote the number by ab; We get $10a + b = 2a \cdot b$. Trying $a = 1, 2, \ldots, 9$, the only integral solution is with $a = 3$, $b = 6$ and $36 = 2 \cdot 3 \cdot 6$.

2. The degree measures of the interior angles of a triangle are A, B, C where $A \le B \le C$. If A, B, and C are multiples of 15, how many possible values of (A, B, C) exist?

Solution. Let $A = 15m$, $B = 15n$, and $C = 15p$. Then $15m + 15n + 15p = 180$ so $m + n + p = 12$ and $m \le n \le p$. Since $m, n \ge 1$ we have $m + n \ge 2$ and $p \le 10$. Also $3p \ge 12$ so $p \ge 4$. For p fixed, $4 \le p \le 10$ we have $m + n = 12 - p$, or $m = 12 - p - n$. This leads to the solutions

p	4	5	5	6	6	6	7	7	8	8	9	10
n	4	4	5	3	4	5	3	4	2	3	2	1
m	4	3	2	3	2	1	2	1	2	1	1	1

There are twelve solutions.

3. Place an operation $(+, -, \times, \div)$ in each square so that the expression using 1, 2, 3, $\cdots$, 9 equals 100.

$$1 \square 2 \square 3 \square 4 \square 5 \square 6 \square 7 \square 8 \square 9 \ = \ 100\,.$$

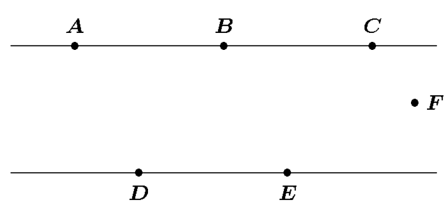
You may also freely place brackets before/after any digits in the expression. Note that the squares must be filled in with operational symbols only.

Solution. Here is one solution:

$$(((1 + 2 + 3 + 4 + 5) \times 6) - 7) + 8 + 9 \ = \ 100\,.$$

How many solutions are there?

4. A, B and C are points on a line that is parallel to another line containing points D and E, as shown. Point F does not lie on either of these lines.



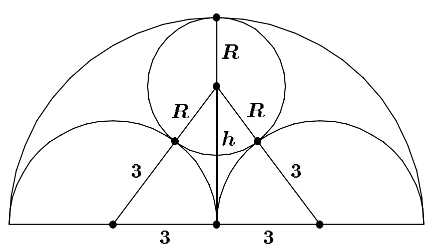
How many distinct triangles can be formed such that all three of its vertices are chosen from A, B, C, D, E, and F?

Solution. Any choice of three of the six vertices determines a triangle except when they lie on a line; that is, except for the one choice $\{A, B, C\}$. The total is thus $\binom{6}{3} - 1 = 20 - 1 = 19$.

5. Michael, Jane and Bert enjoyed a picnic lunch. The three of them were to contribute an equal amount of money toward the cost of the food. Michael spent twice as much money as Jane did buying food for lunch. Bert did not spend any money on food. Instead, Bert brought \$6 which exactly covered his share. How much (in dollars) of Bert's contribution should be given to Michael?

Solution. Bert brought \$6, which exactly covered his share, so the total cost of food is $3 \times \$6 = \$18$. Now Michael spent twice as much as Jane so he spent \$12 and she spent \$6. The total amount of \$6 brought by Bert should go to Michael.

6. Two semicircles of radius 3 are inscribed in a semicircle of radius 6. A circle of radius R is tangent to all three semicircles, as shown. Find R.



Solution. Join the centres of the two smaller semicircles and the centre of the circle. This forms an isosceles triangle with equal sides $3 + R$ and base 6 units. Call the altitude of this triangle h. The altitude extends to a radius of the large semicircle, so $h + R = 6$. By Pythagoras, $h^2 + 3^2 = (R + 3)^2$, so

$$\begin{aligned} (6 - R)^2 + 3^2 &= (R + 3)^2 , \\ 36 - 12R + R^2 + 9 &= R^2 + 6R + 9 , \\ 36 &= 18R , \\ 2 &= R . \end{aligned}$$

The radius of the small circle is 2.

7. If $5^A = 3$ and $9^B = 125$, find the value of AB.

Solution. Now $5^A = 3$, so $5^{2A} = 3^2 = 9$ and

$$5^{2AB} = (5^{2A})^B = 9^B = 125 = 5^3 ,$$

so $2AB = 3$ and $AB = \frac{3}{2}$.

8. The legs of a right angled triangle are 10 and 24 cm respectively.

$$\text{Let}\quad \begin{aligned} A &= \text{the length (cm) of the hypotenuse,}\\ B &= \text{the perimeter (cm) of the triangle,}\\ C &= \text{the area (cm}^2\text{) of the triangle.}\end{aligned}$$

Determine the lowest common multiple of A, B, and C.

Solution. Then

$$\begin{aligned} A &= \sqrt{10^2+24^2} &&= 26\\ B &= 10+24+26 &&= 60\\ C &= \tfrac{1}{2}\times 10\cdot 24 &&= 120\end{aligned}$$

$\text{lcm}(26,60,120) = 3\times 8\times 5\times 13 = 1560$.

9. A lattice point is a point (x, y) such that both x and y are integers. For example, $(2,-1)$ is a lattice point, whereas, $(3, \frac{1}{2})$ and $(-\frac{1}{3}, \frac{2}{3})$ are not. How many lattice points lie inside the circle defined by $x^2+y^2=20$? (Do NOT count lattice points that lie on the circumference of the circle.)

Solution. Since $4^2 < 20 < 5^2$ we have that $-4 \le x \le 4$. For a fixed x in the range we must have $-\sqrt{20-x^2} < y < \sqrt{20-x^2}$ for integer x, y solutions corresponding to interior points of the circle.

x	± 4	± 3	± 2	± 1	0
$\sqrt{20-x^2}$	2	$\sqrt{11}$	$\sqrt{16}$	$\sqrt{19}$	$\sqrt{20}$
y	$-1,0,1$	$-3,\ldots,3$	$-3,\ldots,3$	$-4,\ldots,4$	$-4,\ldots,4$
Lattice pts.	$2\cdot 3=6$	$2\cdot 7=14$	$2\cdot 7=14$	$2\cdot 9=18$	9

The total number is then $6+14+14+18+9=61$.

10. The quadratic equation $x^2+bx+c=0$ has roots r_1 and r_2 that have a sum which equals 3 times their product. Suppose that (r_1+5) and (r_2+5) are the roots of another quadratic equation $x^2+ex+f=0$. Given that the ratio of $e:f = 1:23$, determine the values of b and c in the original quadratic equation.

Solution. Now $r_1+r_2=-b$ and $r_1r_2=c$, so as $r_1+r_2=3r_1r_2$ we get $3c=-b$. Similarly we have

$$\begin{aligned} r_1+r_2+10 &= -e\,,\\ (r_1+5)(r_2+5) &= f\,.\end{aligned}$$

Thus

$$\begin{aligned} 10-b &= -e\\ \text{and } b-10 &= e\,,\\ r_1r_2+5(r_1+r_2)+25 &= f\,,\\ c-5b+25 &= f\,.\end{aligned}$$

From $\frac{e}{f} = \frac{1}{23}$, $23(b-10) = c - 5b + 25$. Using $b = -3c$

$$\begin{aligned} 23(-3c-10) &= c + 15c + 25\,, \\ \text{so that } -69c - 230 &= 16c + 25\,, \\ -255 &= 85c\,, \\ -3 &= c\,, \\ \text{and } 9 &= b\,. \end{aligned}$$

The quadratic is $x^2 + 9x - 3 = 0$.

RELAY

R1. Operations $*$ and $\diamond$ are defined as follows:

$$A * B \;=\; \frac{A^B + B^A}{A + B} \quad \text{and} \quad A \diamond B \;=\; \frac{A^B - B^A}{A - B}\,.$$

Simplify $N = (3 * 2) * (3 \diamond 2)$. Write the value of N in Box #1 of the relay answer sheet.

Solution.

$$\begin{aligned} 3 * 2 &= \frac{3^2 + 2^3}{3 + 2} = \frac{9 + 8}{5} = \frac{17}{5}\,, \\ 3 \diamond 2 &= \frac{3^2 - 2^3}{3 - 2} = \frac{9 - 8}{1} = 1\,, \end{aligned}$$

$$N \;=\; (3 * 2) * (3 \diamond 2) \;=\; \left(\frac{17}{5} * 1\right) = \frac{\left(\frac{17}{5}\right)^1 + (1)^{17/5}}{\frac{17}{5} + 1} \;=\; 1\,.$$

R2. A square has a perimeter of P cm and an area of Q sq.cm. Given that $3NP = 2Q$, determine the value of P. Write the value of P in Box #2 of the relay answer sheet.

Solution. From **R1**, $N = 1$, so $3P = 2Q$. Also the side length is $s = \frac{P}{4}$, so $Q = (\frac{P}{4})^2 = \frac{P^2}{16}$. So $3P = 2 \cdot \frac{P^2}{16}$ or $24P = P^2$. This gives $P = 0$ or $P = 24$. We use $P = 24$, assuming the square is not degenerate.

R3. List all two-digit numbers that have digits whose product is P. Call the sum of these two-digit numbers S. Write the value of S in Box #3 of the relay answer sheet.

Solution. $P = 1 \cdot 24 = 2 \cdot 12 = 4 \cdot 6 = 8 \cdot 3$. The two-digit numbers are 46, 64, 38 and 83; so

$$S \;=\; 231\,.$$

R4. How many integers between 6 and 24 share no common factors with S that are greater than 1?

Solution. Now $231 = 11 \times 21 = 11 \times 7 \times 3$. The numbers between 6 and 24 that share no common factor with 231 are

$$8\,,\ 10\,,\ 13\,,\ 16\,,\ 17\,,\ 19\,,\ 20\,,\ 23\,.$$

There are 8 of them.

TIE-BREAKER

Find the maximum value of

$$f(x) \;=\; 14 - \sqrt{x^2 - 6x + 25}\,.$$

Solution.

$$\begin{aligned} f(x) &= 14 - \sqrt{x^2 - 6x + 25} \\ &= 14\sqrt{x^2 - 6x + 9 - 9 + 25} \\ &= 14 - \sqrt{(x-3)^2 + 16}\,. \end{aligned}$$

For $f(x)$ to be maximum we want $(x-3)^2 + 16$ to be minimum. This occurs when $x = 3$, and

$$f(3) \;=\; 14 - \sqrt{16} \;=\; 14 - 4 \;=\; 10\,.$$

That completes the *Skoliad Corner* for this number. Send me your comments, suggestions, and especially suitable material for use in the Corner.

Announcement

The second volume in the **ATOM** series has just been published. *Algebra — Intermediate Methods* by Bruce Shawyer. Contents include:

Mathematical Induction, Series, Binomial Coefficients,
Solutions of Polynomial Equations, and Vectors and Matrices.

For more information, contact the CMS Office — address on outside back cover.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to Mathematical Mayhem, Department of Mathematics, University of Toronto, 100 St. George St., Toronto, Ontario, Canada. M5S 3G3. The electronic address is still

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Western Ontario). The rest of the staff consists of Adrian Chan (Upper Canada College), Jimmy Chui (University of Toronto), David Savitt (Harvard University) and Wai Ling Yee (University of Waterloo).

Shreds and Slices

Another Combinatorial Proof

Dear Cyrus:

In the most recent issue of *Crux Mathematicorum with Mathematical Mayhem*, Dave Arthur's two-step combinatorial proof of the identity

$$(n-r)\binom{n+r-1}{r}\binom{n}{r} = n\binom{n+r-1}{2r}\binom{2r}{r} \tag{1}$$

was presented, and a one-step combinatorial proof was sought. I have such a proof, but before I present the polished final version, let me explain how I was led to it.

Arthur's solution proceeded in two steps. He counted by two different methods:

(I) the number of ways of choosing two distinct sets A and B, each with r elements, from a set with $n+r-1$ elements, and

(II) the number of ways of choosing two distinct sets C and D, where C has one element and D has r elements, from a set with n elements.

It occurred to me that both (I) and (II) were ways to finish off the selection of three sets A, B, and C, with r, r, and 1 elements respectively, from a set with $n+r$ elements. We could first choose the small set C, and then choose sets A and B from the remaining $n+r-1$ elements as in (I), or we could choose one of the big sets A, and then choose sets C and $B=D$ from the remaining n elements as in (II). Counting both these methods yields the identity

$$(n+r)\left\{\binom{n+r-1}{2r}\binom{2r}{r}\right\} = \binom{n+r}{r}\left\{\binom{n}{r}(n-r)\right\}. \qquad (2)$$

We can easily see that this identity is equivalent to (1) by multiplying both sides of (2) by n and using the identity

$$n\binom{n+r}{r} = (n+r)\binom{n+r-1}{r} \qquad (3)$$

on the right-hand side.

There are two æsthetic problems with this: first, we derived the sought identity (1) with an extra factor of $n+r$ on both sides; second, we used the identity (3) which, though elementary, can be thought of as needing a combinatorial proof of its own (choosing disjoint subsets A and W of sizes r and 1 from a set with $n+r$ elements). The following proof of (1) is free of these defects.

Proof of (1). Suppose we have a circular arrangement of $n+r$ boxes, and we have r amber balls, r blue balls, 1 cyan ball, and 1 white ball. We want to arrange these balls in the boxes (arrangements that differ only by a rotation being regarded as the same), such that two balls are not allowed to be in the same box, except that the white ball may be in a box with a blue or cyan ball. We claim that both sides of (1) count the number of ways of doing this. The two counting methods are:

1. First place the white ball in a box; since rotated arrangements are considered the same, this can be done in only one way, but we have now used up our freedom to rotate the boxes. Next, place the r amber balls into r of the remaining $n+r-1$ empty boxes, which can be done in $\binom{n+r-1}{r}$ ways. Then, place the r blue balls into r of the n boxes that do not contain an amber ball (recall that a blue ball can coexist with a white ball), which can be done in $\binom{n}{r}$ ways. Finally, place the cyan ball in one of the $n-r$ boxes not containing an amber or blue ball (again, the white ball is allowed if it doesn't also contain a blue ball), which can be done in $n-r$ ways. The total number of such arrangements is thus

$$\binom{n+r-1}{r}\binom{n}{r}(n-r). \qquad (4)$$

2. This time, place the cyan ball first; as before, there is only one way to do this up to rotation. Next, select $2r$ of the remaining $n+r-1$ empty boxes (which we will momentarily fill with the $2r$ amber and blue balls), which can be done in $\binom{n+r-1}{2r}$ ways. Then, place the r amber balls into r of these selected $2r$ boxes, and place the r blue balls into the other r

selected boxes; this can be done in $\binom{2r}{r}$ ways. Finally, place the white ball into any of the n boxes that does not contain an amber ball, which can be done in n ways. The total number of such arrangements is thus

$$\binom{n+r-1}{2r}\binom{2r}{r}n\,. \tag{5}$$

Since (4) and (5) are the left- and right-hand sides of (1), respectively, this completes the proof. □

We remark that we could rephrase what we are counting as follows: we want to select three disjoint subsets A, B, and C, with r, r, and 1 elements respectively, from a set S with $n+r$ elements, and also label one element W of S that is not in A, except that selections that differ only by a cyclic permutation of S are to be considered equal.

Yours sincerely,

Greg Martin
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Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from this issue be submitted in time for issue 6 of 2000.

High School Problems

Editor: Adrian Chan, 229 Old Yonge Street, Toronto, Ontario, Canada. M2P 1R5 <a11238@sprint.com>

We correct problem H253, which first appeared in Issue 3.

H253. Find all real solutions to the equation

$$\sqrt{3x^2-12x+52}+\sqrt{2x^2-12x+162} \;=\; \sqrt{-x^2+6x+280}\,.$$

H257. Find all integers n such that $n^2 - 11n + 63$ is a perfect square.

H258. Solve in integers for x and y:

$$6(x! + 3) = y^2 + 5 .$$

H259. *Proposed by Alexandre Tritchtchenko, student, Carleton University, Ottawa, Ontario.*

Solve for x:

$$2^{m-n} \sin(2^n x) \prod_{i=1}^{m-n} \cos(2^{m-i} x) = 1 .$$

H260. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

Let x_1, x_2, ... , x_m be real numbers. Prove that

$$\sum_{1 \le i < j \le n} \cos(x_i - x_j) \ge -\frac{n}{2} .$$

Advanced Problems

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A233. *Proposed by Naoki Sato, Mayhem Editor.*

In C81, we defined the following sequence: $a_0 = 0$, $a_1 = 1$, and $a_{n+1} = 4a_n - a_{n-1}$ for $n = 1, 2, \ldots$. This sequence exhibits the following curious property: For $n \ge 1$, if we set $(a, b, c) = (a_{n-1}, 2a_n, a_{n+1})$, then $ab + 1$, $ac + 1$, and $bc + 1$ are always perfect squares. For example, for $n = 3$, $(a, b, c) = (a_2, 2a_3, a_4) = (4, 30, 56)$, and indeed, $4 \cdot 30 + 1 = 11^2$, $4 \cdot 56 + 1 = 15^2$, and $30 \cdot 56 + 1 = 41^2$. Show that this property holds. Generalize, using the sequence defined by $a_0 = 0$, $a_1 = 1$, and $a_{n+1} = Na_n - a_{n-1}$, and the triples $(a, b, c) = (a_{n-1}, (N-2)a_n, a_{n+1})$, where N is an arbitrary integer.

A234. In triangle ABC, AC^2 is the arithmetic mean of BC^2 and AB^2. Show that $\cot^2 B \ge \cot A \cot C$. (Note: $\cot\theta = \cos\theta / \sin\theta$.)

(1997 Baltic Way)

A235. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

The convex polygon $A_1 A_2 \cdots A_n$ is inscribed in a circle of radius R. Let A be some point on this circumcircle, different from the vertices. Set

$a_i = AA_i$, and let b_i denote the distance from A to the line A_iA_{i+1}, $i = 1$, $2, \ldots, n$, where $A_{n+1} = A_1$. Prove that

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq 2nR\,.$$

A236. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

For all positive integers n and positive reals x, prove the inequality

$$\frac{\binom{2n}{1}}{x+1} + \frac{\binom{2n}{3}}{x+3} + \cdots + \frac{\binom{2n}{2n-1}}{x+2n-1} < \frac{\binom{2n}{0}}{x} + \frac{\binom{2n}{2}}{x+2} + \cdots + \frac{\binom{2n}{2n}}{x+2n}\,.$$

Challenge Board Problems

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C87. *Proposed by Mark Krusemeyer, Carleton College.*

Find an example of three continuous functions $f(x)$, $g(x)$, and $h(x)$ from $\mathbb{R}$ to $\mathbb{R}$ with the property that exactly five of the six composite functions $f(g(h(x)))$, $f(h(g(x)))$, $g(f(h(x)))$, $g(h(f(x)))$, $h(f(g(x)))$, and $h(g(f(x)))$ are the same function and the sixth function is different.

C88.

(a) Let A be an $n \times n$ matrix whose entries are all either $+1$ or -1. Prove that $|\det A| \leq n^{n/2}$.

(b) It is conjectured that for there to exist an $n \times n$ matrix A whose entries are all either $+1$ or -1 and such that $|\det A| = n^{n/2}$, it is necessary and sufficient that $n = 1$, $n = 2$, or n is divisible by 4. Prove that this condition is necessary.

(c) Can you construct such a matrix, for n equal to a power of 2? For $n = 12$?

Problem of the Month

Jimmy Chui, student, University of Toronto

Problem. Let a, b, and c be positive real numbers. Prove that

$$a^a b^b c^c \;\geq\; (abc)^{(a+b+c)/3}\,.$$

(1995 CMO, Problem 2)

Solution I. Taking log of both sides, we see that we must prove

$$a\log a + b\log b + c\log c \;\geq\; \frac{a+b+c}{3}\log abc\,.$$

Let $f(x) = x\log x$. Then $f'(x) = \log x + 1$ and $f''(x) = 1/x$. We can see that $f''(x) > 0$ for $x > 0$. So, f is convex for $x > 0$. By Jensen's Inequality,

$$\begin{aligned} \frac{f(a)+f(b)+f(c)}{3} &\geq f\left(\frac{a+b+c}{3}\right) \\ \Longrightarrow \quad \frac{a\log a + b\log b + c\log c}{3} &\geq \frac{a+b+c}{3}\log\left(\frac{a+b+c}{3}\right)\,. \end{aligned}$$

By the AM-GM Inequality, $\frac{a+b+c}{3} \;\geq\; \sqrt[3]{abc}$. Thus, we have

$$\begin{aligned} a\log a + b\log b + c\log c &\geq (a+b+c)\log\left(\frac{a+b+c}{3}\right) \\ &\geq (a+b+c)\log(\sqrt[3]{abc}) \\ &= \frac{a+b+c}{3}\log abc\,. \end{aligned}$$

Solution II. Recall Chebychev's Inequality: Let x_1, x_2, $\ldots$, x_n and y_1, y_2, $\ldots$, y_n be two real sequences, either both increasing or both decreasing. Then

$$\frac{x_1y_1 + x_2y_2 + \cdots + x_ny_n}{n} \;\geq\; \frac{x_1 + x_2 + \cdots + x_n}{n}\cdot\frac{y_1 + y_2 + \cdots + y_n}{n}\,.$$

(In other words, the average of the products is greater than or equal to the product of the averages.)

Without loss of generality, let $a \geq b \geq c$. Then $\log a \geq \log b \geq \log c$. By Chebychev's Inequality,

$$\frac{a\log a + b\log b + c\log c}{3} \;\geq\; \frac{a+b+c}{3}\cdot\frac{\log a + \log b + \log c}{3}\,,$$

which implies that $a\log a + b\log b + c\log c \;\geq\; \frac{a+b+c}{3}\log abc\,.$

Solution III. Recall the Weighted AM-GM-HM Inequality: Let x_1, x_2, $\ldots$, x_n be positive real numbers, and let w_1, w_2, $\ldots$, w_n be non-negative real numbers which sum to 1. Then

$$w_1x_1 + w_2x_2 + \cdots + w_nx_n \;\geq\; x_1^{w_1}x_2^{w_2}\cdots x_n^{w_n} \;\geq\; \frac{1}{\frac{w_1}{x_2} + \frac{w_2}{x_2} + \cdots + \frac{w_n}{x_n}}\,.$$

Take $x_1 = a$, $x_2 = b$, $x_3 = c$, $w_1 = a/(a+b+c)$, $w_2 = b/(a+b+c)$, and $w_3 = c/(a+b+c)$. Then using the GM-HM portion of the above inequality, we obtain

$$\begin{aligned} a^{a/(a+b+c)}b^{b/(a+b+c)}c^{c/(a+b+c)} &= (a^ab^bc^c)^{1/(a+b+c)} \\ &\geq \frac{1}{\frac{1}{a+b+c} + \frac{1}{a+b+c} + \frac{1}{a+b+c}} \\ &= \frac{a+b+c}{3}\,. \end{aligned}$$

By the AM-GM Inequality,

$$a^ab^bc^c \;\geq\; \left(\frac{a+b+c}{3}\right)^{a+b+c} \;\geq\; (abc)^{(a+b+c)/3}\,.$$

J.I.R. McKnight Problems Contest 1986 Solutions

3. Prove that the sum of the squares of the first n even natural numbers exceeds the sum of the squares of the first n odd natural numbers by $n(2n+1)$. Hence, or otherwise, find the sum of the squares of the first n odd natural numbers.

Partial solution by Luyun Zhong-Qido, Columbia International College, Hamilton, Ontario, Canada.

The sum of the first n positive integers is given by

$$1^2 + 2^2 + \cdots + n^2 \;=\; \frac{n(n+1)(2n+1)}{6}\,. \qquad (1)$$

$$\text{Therefore,}\quad 1^2 + 2^2 + \cdots + (2n)^2 \;=\; \frac{2n(2n+1)(4n+1)}{6}\,, \qquad (2)$$

$$\text{and}\quad 2^2 + 4^2 + \cdots + (2n)^2 \;=\; \frac{4n(n+1)(2n+1)}{6}\,. \qquad (3)$$

From (2) and (3),

$$\begin{aligned}1^2+3^2+\cdots+(2n-1)^2 &= \frac{2n(2n+1)(4n+1)}{6}-\frac{4n(n+1)(2n+1)}{6}\\ &= \frac{2n(2n+1)[(4n+1)-2(n+1)]}{6}\\ &= \frac{2n(2n-1)(2n+1)}{6}\,. \qquad (4)\end{aligned}$$

From (3) and (4),

$$\begin{aligned}&2^2+4^2+\cdots+(2n)^2-[(1^2+3^2+\cdots+(2n-1)^2]\\ &= \frac{4n(n+1)(2n+1)}{6}-\frac{2n(2n-1)(2n+1)}{6}\\ &= \frac{2n(2n+1)[2(n+1)-(2n-1)]}{6}\\ &= \frac{3\cdot 2n(2n+1)}{6} = n(2n+1)\,.\end{aligned}$$

4. (b) Prove that in any acute triangle ABC,

$$\cot A+\cot B+\cot C = \frac{a^2+b^2+c^2}{4K}\,,$$

where K is the area of triangle ABC.

Solution by Luyun Zhong-Qido, Columbia International College, Hamilton, Ontario, Canada.

We have the following relations in a triangle:

$$\begin{aligned}2R &= \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}\,, \qquad (1)\\ K &= \frac{1}{2}\,ab\sin C = \frac{1}{2}\,ac\sin B = \frac{1}{2}\,bc\sin A\,. \qquad (2)\end{aligned}$$

From (1) and (2),

$$K = \frac{1}{2}\,bc\sin A = \frac{1}{2}\left(\frac{a\sin B}{\sin A}\right)\left(\frac{a\sin C}{\sin A}\right)\sin A = a^2\,\frac{\sin B\sin C}{2\sin A}\,,$$

so

$$a^2 = \frac{2K\sin A}{\sin B\sin C}\,.$$

Likewise,

$$b^2 = \frac{2K\sin B}{\sin A\sin C}\,, \qquad c^2 = \frac{2K\sin C}{\sin A\sin B}\,.$$

We note that

$$\begin{aligned}\cot A+\cot B &= \frac{\cos A}{\sin A}+\frac{\cos B}{\sin B} = \frac{\cos A\sin B+\cos B\sin A}{\sin A\sin B}\\ &= \frac{\sin(A+B)}{\sin A\sin B} = \frac{\sin C}{\sin A\sin B}\,.\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{a^2+b^2+c^2}{4K} &= \frac{\frac{2K\sin A}{\sin B\sin C}+\frac{2K\sin B}{\sin A\sin C}+\frac{2K\sin C}{\sin A\sin B}}{4K} \\
&= \frac{2K(\sin^2 A+\sin^2 B+\sin^2 C)}{4K\sin A\sin B\sin C} \\
&= \frac{1}{2}\left(\frac{\sin A}{\sin B\sin C}+\frac{\sin B}{\sin A\sin C}+\frac{\sin C}{\sin A\sin B}\right) \\
&= \frac{1}{2}(\cot B+\cot C+\cot A+\cot C+\cot A+\cot B) \\
&= \cot A+\cot B+\cot C\,.
\end{aligned}$$

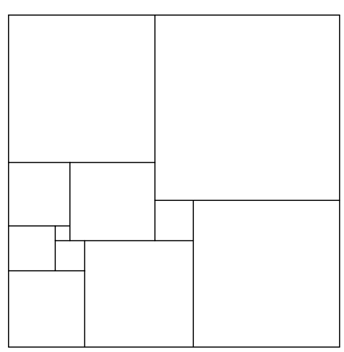
J.I.R. McKnight Problems Contest 1989

PART A

1. A curve has equation $y = x^3 - 3x$. A tangent is drawn to the curve at its relative minimum point. This tangent line also intersects the curve at P. Find the equation of the normal to the curve at P.

2. (a) In a triangle whose sides are 5, 12, and 13, find the length of the bisector of the larger acute angle.

   (b) This large rectangle has been cut into eleven squares of various sizes. The smallest square has an area of $81\ \mathrm{cm}^2$. Find the dimensions of the large rectangle.



3. Solve the following system of equations for x and y, where $x, y \in \mathbb{R}$:

$$x^3 - y^3 = 35\,, \qquad xy^2 - yx^2 = 30\,.$$

4. The combined volume of two cubes with integral sides is equal to the combined length of all their edges. Find the dimensions of all cubes satisfying these conditions.

5. A car at an intersection is heading west at 24 m/s. Simultaneously, a second car, 84 m north of the first car, is travelling directly south at 10 m/s. After 2 seconds:

   (a) Find the rate of change of the distance between the 2 cars.

   (b) Using vector methods, determine the velocity of the first car relative to the second.

   (c) Explain fully why the magnitude of the vector in (b) is not necessarily the same as the rate of change in (a).

PART B

1. Prove that

$$\sin 1^\circ + \sin 3^\circ + \sin 5^\circ + \cdots + \sin 97^\circ + \sin 99^\circ = \frac{\sin^2 50^\circ}{\sin 1^\circ}\,.$$

2. Determine the n^{th} term and the sum of n terms for the series

$$3 + 5 + 10 + 18 + 29 + \cdots\,.$$

3. A circle has equation $x^2+y^2=1$. Lines with slope $\frac{1}{2}$ and $-\frac{1}{2}$ are drawn through $C(0,0)$ to form a sector of the circle. Find the dimensions of the rectangle of maximal area which can be inscribed in this sector given that two sides of the rectangle are parallel to the y-axis and the rectangle is entirely below the x-axis.

4. In triangle ABC, angles A, B, and C are in the ratio $4:2:1$. Prove that the sides of the triangle are related by the equality $\frac{1}{a}+\frac{1}{b}=\frac{1}{c}$.

5. Prove that

$$\left(\frac{7+\sqrt{37}}{2}\right)^n + \left(\frac{7-\sqrt{37}}{2}\right)^n - 1$$

   is divisible by 3 for all $n \in \mathbb{N}$.

Derangements and Stirling Numbers

Naoki Sato
student, Yale University

In this article, we introduce two important classes of combinatorial numbers which appear frequently: derangements and Stirling numbers. We also hope to emphasize the importance and usefulness of basic counting principles.

We can think of a permutation on n objects as a 1-1 function π from the set $\{1, 2, \ldots, n\}$ to itself. For example, for $n = 3$, the map π given by $\pi(1) = 1$, $\pi(2) = 3$, and $\pi(3) = 2$ is a permutation on 3 objects, namely the elements of $\{1, 2, 3\}$; a permutation essentially re-arranges the elements. Note that there are $n!$ different permutations on n objects. Then, a *derangement* is a permutation π which has no fixed points; that is, $\pi(i) \neq i$ for all $i = 1, 2, \ldots, n$. Alternatively, if we think of a permutation on n objects as a distribution of n letters to n corresponding envelopes, then a derangement is a permutation where no letter is inserted into the correct corresponding envelope. Let D_n denote the number of derangements on n objects. The natural question to ask is, what is the formula for D_n?

Problems.

1. Write down all permutations on 2, 3, and 4 elements. How many of these are derangements? Is there a systematic way of writing down all permutations on n elements?

2. What is the total number of fixed points over all permutations on n elements? You should be able to guess the answer from small cases.

We immediately see that $D_1 = 0$ and $D_2 = 1$ (by convention, $D_0 = 0$). Assume that $n \geq 2$; we will derive a recurrence relation for the D_n, by dividing all derangements on n elements into two categories, and then counting the number in each category. Let π be a derangement on the n elements $1, 2, \ldots, n$. Let $k = \pi(1)$, so $k \neq 1$ since π is a derangement. There are two cases: $\pi(k) = 1$ or $\pi(k) \neq 1$. Let A_n be the number of derangements for which $\pi(k) = 1$, and let B_n be the number of derangements for which $\pi(k) \neq 1$.

If $\pi(k) = 1$, then π swaps the elements 1 and k, leaving what π does to the remaining elements $2, 3, \ldots, k-1, k+1, \ldots, n$ to be considered. Since π re-arranges these elements, it too acts as a derangement on these $n - 2$ elements, of which there are D_{n-2}. Going back, there are $n - 1$ possible values for k, as 1 is omitted, so $A_n = (n-1)D_{n-2}$.

If $\pi(k) \neq 1$, then let α be the permutation which swaps 1 and k, and leaves everything else fixed. Recall that for maps f and g, $f \circ g$ denotes the *composition* of the two maps; that is, $(f \circ g)(x) = f(g(x))$. In this case, if f and g are permutations, then $f \circ g$ is the permutation which arises from applying g, then applying f. Consider the permutation $\pi \circ \alpha$. We see that $(\pi \circ \alpha)(1) = \pi(\alpha(1)) = \pi(k) \neq 1$, $(\pi \circ \alpha)(k) = \pi(\alpha(k)) = \pi(1) = k$, and for all other elements i, $(\pi \circ \alpha)(i) = \pi(\alpha(i)) = \pi(i) \neq i$, since π is a derangement. Thus, $\pi \circ \alpha$ is a permutation which fixes the element k, and which deranges all $n-1$ others (this is why we composed π with α), of which there are D_{n-1}. Again, there are $n-1$ possible values of k (again, 1 is omitted), so $B_n = (n-1)D_{n-1}$. Therefore,

$$\begin{aligned} D_n &= A_n + B_n \\ &= (n-1)D_{n-2} + (n-1)D_{n-1} \\ &= (n-1)(D_{n-1} + D_{n-2}) \, . \end{aligned}$$

Problems.

3. Show that in general, for permutations α and β, $\alpha \circ \beta \neq \beta \circ \alpha$. Can you determine when $\alpha \circ \beta = \beta \circ \alpha$?

4. For a positive integer k, let π^k denote the permutation π composed with itself k times; that is,

$$\pi^k = \underbrace{\pi \circ \pi \circ \cdots \circ \pi}_{k} \, .$$

   Prove that for any permutation π, there exists a positive integer k such that $\pi^k = 1$. Moreover, if π is a permutation on n elements, then $\pi^{n!} = 1$. Here, 1 stands for the identity permutation, the permutation which takes every element to itself.

5. Classify all permutations π on n elements such that $\pi \circ \pi = \pi^2 = 1$.

6. Prove that for any permutation α, there exist unique permutations β and γ such that $\alpha \circ \beta = \gamma \circ \alpha = 1$. Problem 5 shows that $\beta = \gamma$ is possible; is this true in general? Hint: Apply Problem 4!

The next few terms in the sequence are then $D_3 = 2$, $D_4 = 9$, $D_5 = 44$, etc. We can use this recurrence to derive an explicit formula for D_n. By the relation,

$$\begin{aligned} D_n - nD_{n-1} &= -D_{n-1} + (n-1)D_{n-2} \\ &= -\left(D_{n-1} - (n-1)D_{n-2}\right) \\ &= (-1)^2\left(D_{n-2} - (n-2)D_{n-3}\right) \\ &= \cdots \\ &= (-1)^{n-2}(D_2 - 2D_1) = (-1)^n \, . \end{aligned}$$

Hence,

$$\begin{aligned}\frac{D_n}{n!} &= \frac{D_{n-1}}{(n-1)!} + \frac{(-1)^n}{n!} \\ &= \frac{D_{n-2}}{(n-2)!} + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!} \\ &= \cdots \\ &= \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!} \\ \Longrightarrow D_n &= n!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right).\end{aligned}$$

Remember this expression; as mentioned above, it comes up a lot! We note in passing that

$$\frac{1}{e} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots,$$

so that

$$D_n \approx \frac{n!}{e}$$

for large n. We also note that it is possible to derive the formula for D_n using the Principle of Inclusion-Exclusion. We finish off derangements with two quick problems.

Problem. Let n be a positive integer. Show that

$$\sum_{k=0}^{n} \binom{n}{k} D_k = n!.$$

Solution. It looks as if we might have to plug away at some algebra, but a combinatorial approach is much more natural and painless. Note that the RHS, $n!$, is simply the number of permutation on n objects, and this is a cue. What is the number of permutations which derange k elements, or alternatively, which fix $n-k$ elements? We first choose $n-k$ elements, and derange the rest, which there are D_k ways of doing, for a total of

$$\binom{n}{n-k} D_k = \binom{n}{k} D_k.$$

The result follows from summing over k (since every permutation deranges k elements for some k).

In general, if $x_0, x_1, x_2, \ldots$ and $y_0, y_1, y_2, \ldots$ are sequences satisfying

$$\sum_{k=0}^{n} \binom{n}{k} x_k = y_n,$$

then it is possible to recover $\{x_n\}$ from $\{y_n\}$, via

$$x_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} y_k .$$

Indeed, for $y_n = n!$, we obtain that $x_n = D_n$, since

$$\begin{aligned}
\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k! &= \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{(n-k)!} \\
&= n! \left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!} \right) \\
&= D_n .
\end{aligned}$$

Problem. Let n be a positive integer. Prove that

$$\sum_{k=0}^{n} k \binom{n}{k} D_k = (n-1) \cdot n! .$$

Solution. Left as an exercise for the reader. (Follow the same type of reasoning as the previous problem.) Hint: Combinatorially, what does the LHS represent?

We now draw our attention to the *Stirling numbers*, in particular those of the *second kind* (as opposed to the *first kind* for those in suspense). These arise in the following situation: Suppose that we have n distinguishable balls to distribute among k indistinguishable boxes, and no box can be empty. How many such distributions are there? First, consider the case where the boxes are distinguishable.

If the boxes were allowed to be empty, there would be k^n such distributions; assign a box to each ball. Following the Principle of Inclusion-Exclusion, we subtract the number of distributions with at least 1 box empty, which is

$$\binom{k}{k-1} (k-1)^n ,$$

since we must choose out of $k-1$ boxes for each ball. We then add the number of distributions with at least 2 boxes empty, which is

$$\binom{k}{k-2} (k-2)^n ,$$

and so on. Hence, the total number of distributions is

$$\sum_{i=0}^{n} (-1)^i \binom{k}{i} (k-i)^n .$$

Then, for the case where the boxes are indistinguishable, we may "remove the labels" of the boxes, and divide by a factor of $k!$, to obtain the number of distributions of the original problem:

$$S(n,k) := \frac{1}{k!}\sum_{i=0}^{n}(-1)^i\binom{k}{i}(k-i)^n .$$

These are the Stirling numbers of the second kind. We list the first few here:

n	$S(n,k),\ k=1,2,\ldots,n$				
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

An important property of these Stirling numbers, one that may be noticeable in the table, is the following:

$$S(n,k) \ = \ S(n-1,k-1) + kS(n-1,k) .$$

This rule easily allows us to generate further rows in the table. We stress that although we derived a formula for $S(n,k)$ above, it is in general better to consider the combinatorial significance of this number. So, we give a combinatorial proof here, and leave an algebraic proof for the reader.

Assume that the balls are labelled 1 through n (recall that they are distinguishable). Consider ball n. Either it is in a box all by itself, or with others. For the first case, the number of distributions is simply $S(n-1,k-1)$, as we can add one box containing ball n to a distribution of $n-1$ balls among $k-1$ boxes. For the second case, temporarily remove ball n. This leaves $n-1$ balls among k non-empty boxes, of which there are $S(n-1,k)$ distributions. We can then add ball n to any of the k boxes, giving rise to k distinct distributions. Hence,

$$S(n,k) \ = \ S(n-1,k-1) + kS(n-1,k) .$$

Problem. Show that

$$S(n,k) \ = \ \sum 1^{a_1}2^{a_2}\cdots k^{a_k} ,$$

where the sum is taken over all $\binom{n-1}{k-1}$ decompositions of $n-k$ into k non-negative integers $a_1, a_2, \ldots, a_k$: $a_1+a_2+\cdots+a_k = n-k$. (A decomposition of a non-negative integer N into m parts is a sequence of m non-negative integers, the sum of which is N. So, 3 has 4 different decompositions into 2 parts: $3 = 0+3 = 1+2 = 2+1 = 3+0$.)

Solution. The table above makes this virtually obvious, but we will flesh out the details. Draw in arrows in the table, joining entries in consecutive rows. An arrow from $S(n-1,k-1)$ has weight 1 and an arrow from $S(n-1,k)$ has weight k. By virtue of the identity proved above, an entry in the table is equal to the sum of the weights of the paths leading to it, where the weight of a path is simply the product of the weights of the arrows in it (think of this as a tweaked Pascal's Triangle).

There are $\binom{n-1}{k-1}$ paths from entry $S(1,1)$ to $S(n,k)$, and the weight of a path is precisely the term in the given sum. The result follows from summing over all paths, which clearly gives the given expression.

Problem. Show that

$$S(n+1,k) \;=\; \sum_{i=0}^{n} \binom{n+1}{i} S(i,k-1)\,.$$

Solution. In a distribution of $n+1$ balls among k boxes, select one box as "blue". Let i be the number of balls in the blue box, so $1 \le i \le n+1$. From the $n+1$ balls, we may choose i to be in the blue box, leaving $n-i+1$ to be distributed among $k-1$ boxes. There are

$$\binom{n+1}{i} S(n-i+1,k-1) = \binom{n+1}{n-i+1} S(n-i+1,k-1)$$

such distributions in this case. The result follows from summing over i.

Problems.

1. Let n be a positive integer. Prove that

$$\sum_{k=0}^{n} k\binom{n}{k} D_k \;=\; (n-1)\cdot n!\,.$$

2. Here is an example which illustrates the Principle of Inclusion-Exclusion: Let A, B, and C be subsets of a set S. Let $\overline{A}$ denote the complement of A. Prove that

$$|\overline{A}\cap\overline{B}\cap\overline{C}| \;=\; |S|-|A|-|B|-|C|+|A\cap B|+|A\cap C|+|B\cap C|-|A\cap B\cap C|\,.$$

   For example, let S be the set of all distributions of n distinguishable balls among 3 distinguishable boxes, let A be the subset of distributions with box 1 empty, and so on. Then $\overline{A}$ is the subset of distributions with box 1 containing at least one ball, and so on, so $\overline{A}\cap\overline{B}\cap\overline{C}$ is the subset of distributions with all three boxes containing at least one ball. Now determine what the formula above turns into. This is a useful formula, because the terms such as $|A|$ and $|A\cap B|$ are easy to compute. This formula also generalizes to any number of subsets.

3. By using the derived formula for $S(n,k)$, show that

$$S(n,k) \;=\; S(n-1,k-1) + kS(n-1,k)\,.$$

4. Let n and k be positive integers, with $k < n$. Verify that there are $\binom{n-1}{k-1}$ decompositions of $n-k$ into k parts. If we restrict the parts to be positive integers, then how many decompositions are there of n in total?

5. Define a sequence of polynomials $\{p_n(x)\}$ as follows: $p_1(x) = p(x)$ is given, and $p_{n+1}(x) = xp_n'(x)$. Find a closed formula for $p_n(x)$, in terms of $p(x)$ and n.

6. (a) Let n be a positive integer. Prove that the following is an identity in x:

$$x^n = \sum_{k=0}^{n} k!S(n,k)\binom{x}{k}.$$

   (b) Prove that

$$1^n + 2^n + \cdots + m^n = \sum_{k=0}^{n} k!S(n,k)\binom{m+1}{k+1}.$$

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PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and **separate** *standard $8\frac{1}{2}$"×11" or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 January 2000**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2446. Correction: *Proposed by Catherine Shevlin, Wallsend upon Tyne, England.*

A sequence of integers, $\{a_n\}$ with $a_1 > 0$, is defined by

$$a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } a(n) \equiv 0 \pmod 4, \\ 3a_n + 1 & \text{if } a(n) \equiv 1 \pmod 4, \\ 2a_n - 1 & \text{if } a(n) \equiv 2 \pmod 4, \\ \frac{a_n+1}{4} & \text{if } a(n) \equiv 3 \pmod 4. \end{cases}$$

Prove that there is an integer m such that $a_m = 1$.

(Compare **OQ**.117 in *OCTOGON*, vol 5, No. 2, p. 108.)

2451. *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Construct an infinite sequence, $\{A_n\}$, of infinite subsets of $\mathbb{N}$ with the following properties:

(a) the intersection of any two distinct sets A_n and A_m is a singleton;

(b) the singleton in (a) is a different one if at least one of the distinct sets A_n, A_m, is changed (so the new pair is again distinct);

(c) every natural number is the intersection of (exactly) one pair of distinct sets as in (a).

2452. *Proposed by Antal E. Fekete, Memorial University of Newfoundland, St. John's, Newfoundland.*

Establish the following equalities:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)^2}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{(2n+2)^2}{(2n+2)!}$.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^3}{(n+1)!} = \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^4}{(n+1)!}$.

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^6}{(n+1)!} = \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^7}{(n+1)!}$.

2453. *Proposed by Antal E. Fekete, Memorial University of Newfoundland, St. John's, Newfoundland.*

Establish the following equalities:

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(2n+1)^3}{(2n+1)!} = -3 \sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n+1)!}$.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(2n)^3}{(2n)!} = -3 \sum_{n=1}^{\infty} (-1)^n \frac{1}{(n+1)!}$.

(c) $\displaystyle\left(\sum_{n=1}^{\infty} (-1)^n \frac{(2n+1)^2}{(2n+1)!}\right)^2 + \left(\sum_{n=1}^{\infty} (-1)^n \frac{(2n)^2}{(2n)!}\right)^2 = 9$.

2454. *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Three circles intersect each other orthogonally at pairs of points A and A', B and B', and C and C'. Prove that the circumcircles of $\triangle ABC$ and $\triangle AB'C'$ touch at A.

2455. *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Three equal circles, centred at A, B and C intersect at a common point P. The other intersection points are L (not on circle centre A), M (not on circle centre B), and N (not on circle centre C). Suppose that Q is the centroid of $\triangle LMN$, that R is the centroid of $\triangle ABC$, and that S is the circumcentre of $\triangle LMN$.

(a) Show that P, Q, R and S are collinear.

(b) Establish how they are distributed on the line.

2456. *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Two circles intersect orthogonally at P. A third circle touches them at Q and R. Let X be any point on this third circle. Prove that the circumcircles of $\triangle XPQ$ and $\triangle XPR$ intersect at 45°.

2457. *Proposed by Gerry Leversha, St. Paul's School, London, England.*

In quadrilateral $ABCD$, we have $\angle A + \angle B = 2\alpha < 180^\circ$, and $BC = AD$. Construct isosceles triangles DCI, ACJ and DBK, where I, J and K are on the other side of CD from A, such that $\angle ICD = \angle IDC = \angle JAC = \angle JCA = \angle KDB = \angle KBD = \alpha$.

(a) Show that I, J and K are collinear.

(b) Establish how they are distributed on the line.

2458. *Proposed by Nikolaos Dergiades, Thessaloniki, Greece.*

Let $ABCD$ be a quadrilateral inscribed in the circle centre O, radius R, and let E be the point of intersection of the diagonals AC and BD. Let P be any point on the line segment OE and let K, L, M, N be the projections of P on AB, BC, CD, DA respectively.

Prove that the lines KL, MN, AC are either parallel or concurrent.

2459. *Proposed by Vedula N. Murty, Visakhapatnam, India, modified by the editors.*

Let P be a point on the curve whose equation is $y = x^2$. Suppose that the normal to the curve at P meets the curve again at Q. Determine the minimal length of the line segment PQ.

2460. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Let $y(x) = \sqrt[3]{x + \sqrt{x^2 - 1}} + \sqrt[3]{x - \sqrt{x^2 - 1}}$ for $0 \leq x \leq 1$.

(a) Show that $y(x)$ is real valued.

(b) Find an infinite sequence $\{x_n\}_{n=0}^{\infty}$ such that $y(x_n)$ can be expressed in terms of square roots only.

2461. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

Suppose that $x_0, x_1, \ldots, x_n$ are integers which satisfy $x_0 > x_1 > \ldots > x_n$. Let

$$F(x) = \sum_{k=0}^{n} a_k x^{n-k}, \qquad a_k \in \mathbb{R}, \, a_0 = 1.$$

Prove that at least one of the numbers $|F(x_k)|$, $(k = 0, 1, \ldots, n)$ is greater than $\frac{n!}{2^n}$.

2462. *Proposed by Vedula N. Murty, Visakhapatnam, India.*

If the angles, A, B, C of $\triangle ABC$ satisfy

$$\cos A \sin\frac{A}{2} = \sin\frac{B}{2}\sin\frac{C}{2},$$

prove that $\triangle ABC$ is isosceles.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2324. [1998: 109, 1999: 50] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Find the exact value of $\sum_{n=1}^{\infty}\frac{1}{u_n}$, where u_n is given by the recurrence

$$u_n = n! + \left(\frac{n-1}{n}\right) u_{n-1},$$

with the initial condition $u_1 = 2$.

Solution by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We first show that $u_n = n! + (n-1)!$. This is true for $n = 1$, since $u_1 = 2$. Suppose that $u_n = n! + (n-1)!$ for some $n \geq 1$. Then

$$\begin{aligned} u_{n+1} &= (n+1)! + \frac{n}{n+1}(n! + (n-1)!) \\ &= (n+1)! + \frac{n}{n+1}(n-1)!(n+1) \\ &= (n+1)! + n!, \end{aligned}$$

completing the induction.

Hence, for all $m \geq 1$, we have

$$\begin{aligned} \sum_{n=1}^{m}\frac{1}{u_n} &= \sum_{n=1}^{m}\frac{1}{n!+(n-1)!} = \sum_{n=1}^{m}\frac{1}{(n-1)!(n+1)} \\ &= \sum_{n=1}^{m}\frac{n+1-1}{(n+1)!} = \sum_{n=1}^{m}\left(\frac{1}{n!}-\frac{1}{(n+1)!}\right) = 1-\frac{1}{(m+1)!}, \end{aligned}$$

from which it follows that $\sum_{n=1}^{\infty}\frac{1}{u_n} = \lim_{m\to\infty}\sum_{n=1}^{m}\frac{1}{u_n} = 1$.

Also solved by the proposer.

2339. [1998: 234] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

A rhombus $ABCD$ has incircle Γ, and Γ touches AB at T. A tangent to Γ meets sides AB, AD at P, S respectively, and the line PS meets BC, CD at Q, R respectively. Prove that

(a) $\dfrac{1}{PQ} + \dfrac{1}{RS} = \dfrac{1}{BT}$,

and

(b) $\dfrac{1}{PS} - \dfrac{1}{QR} = \dfrac{1}{AT}$.

Solution by Michael Lambrou, University of Crete, Crete, Greece.

If PS meets Γ at U we show, slightly generalizing (and correcting) the situation, that, depending on the position of U, we have

$$\frac{1}{PQ} \pm \frac{1}{RS} \;=\; \pm\frac{1}{BT}\,,$$

$$\frac{1}{PS} \pm \frac{1}{QR} \;=\; \pm\frac{1}{AT}\,,$$

with an appropriate choice of $\pm$ each time.

Using as coordinate axes the two (perpendicular) diagonals, meeting at O say, we may assume that A, B, C, D have coordinates $A(a,0)$, $B(0,b)$, $C(-a,0)$, $D(0,-b)$ respectively, where a, $b > 0$. The radius of Γ is OT which, being perpendicular to AB is the altitude of OAB. Writing $OT = h$ we have $h \cdot AB = OA \cdot OB = ab$, and the coordinates of U are of the form $(h\cos\theta, h\sin\theta)$. As $PS \perp OU$, the equation of PS is clearly

$$y\sin\theta \;=\; -x\cos\theta + h\,.$$

Line AB is $\frac{x}{a} + \frac{y}{b} = 1$, so the coordinates of P, being on AB and PS, are easily seen to be

$$(x_P, y_P) \;=\; \frac{1}{b\sin\theta - a\cos\theta}\Big(a(b\sin\theta - h), b(h - a\cos\theta)\Big)\,. \tag{1}$$

Similarly (or quicker, by replacing a by $-a$) we find the coordinates of Q as

$$(x_Q, y_Q) \;=\; \frac{1}{b\sin\theta + a\cos\theta}\Big(-a(b\sin\theta - h), b(h + a\cos\theta)\Big)\,. \tag{2}$$

Assume now that the position of U is such that PS cuts, say, AB and BC internally (the rest of the cases follow by a trivial adaptation of our argument here) and so the coordinates of P are both positive [and Q has a negative x–coordinate and a positive y–coordinate]. By (1) we have $b\sin\theta > h > a\cos\theta$. Thus

$$PQ \;=\; \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2} \;=\; \frac{2ab(b\sin\theta - h)}{(b\sin\theta)^2 - (a\cos\theta)^2}\,.$$

Similarly

$$RS \;=\; \frac{2ab(b\sin\theta + h)}{(b\sin\theta)^2 - (a\cos\theta)^2}\,,$$

and so

$$\begin{aligned}\frac{1}{RS}-\frac{1}{PQ} &= \frac{(b\sin\theta)^2-(a\cos\theta)^2}{2ab}\left(\frac{1}{b\sin\theta+h}-\frac{1}{b\sin\theta-h}\right)\\ &= -\frac{(b\sin\theta)^2-(a\cos\theta)^2}{2ab}\cdot\frac{2h}{(b\sin\theta)^2-h^2}\,.\end{aligned}$$

Writing $h^2=\frac{a^2b^2}{(AB)^2}=\frac{a^2b^2}{a^2+b^2}$, this simplifies to $-\frac{(a^2+b^2)h}{ab^3}$ which equals $-\frac{1}{BT}$ as $BT\cdot AB=OB^2=b^2$, as required.

The proof of $\frac{1}{PS}\pm\frac{1}{QR}=\pm\frac{1}{AT}$ is similar and routine.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK.

Bradley and Lambrou were the only readers who recognized that the problem was incorrect as stated. There were eleven partial solutions.

2346. [1998: 236] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

The angles of $\triangle ABC$ satisfy $A>B\geq C$. Suppose that H is the foot of the perpendicular from A to BC, that D is the foot of the perpendicular from H to AB, that E is the foot of the perpendicular from H to AC, that P is the foot of the perpendicular from D to BC, and that Q is the foot of the perpendicular from E to AB.

Prove that A is acute, right or obtuse according as $\overline{AH}-\overline{DP}-\overline{EQ}$ is positive, zero or negative.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

As noted by the proposer, this is a generalization of a problem he proposed in *College Mathematics Journal* **28**, no. 2, March 1997, 145-6.

Let $S=AH-DP-EQ$. Then, since $\angle DHA=\angle PDH=\angle B$, we have

$$DP \;=\; DH\cos B \;=\; AH\cos^2 B\,.$$

Similarly $EQ=AH\cos^2 C$, so,

$$\begin{aligned}S &= AH-AH\cos^2 B-AH\cos^2 C \;=\; AH(1-\cos^2 B-\cos^2 C)\\ &= AH(\sin^2 B-\cos^2 C) \;=\; AH\left(\frac{1-\cos 2B}{2}-\frac{1+\cos 2C}{2}\right)\\ &= -AH\cos(B+C)\cos(B-C)\,,\end{aligned}$$

or $S=AH\cos A\cos(B-C)$, where $\cos(B-C)>0$, and hence

$$A \text{ is acute if } S>0\,,\qquad A \text{ is right if } S=0\,,\qquad A \text{ is obtuse if } S<0\,.$$

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER

J. BRADLEY, Clifton College, Bristol, UK; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VJEKOSLAV KOVAČ, student, University of Zagreb, Croatia; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; TOSHIO SEIMIYA, Kawasaki, Japan; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

2348. [1998: 236] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Without the use of trigonometrical formulae, prove that

$$\sin(54^\circ) = \tfrac{1}{2} + \sin(18^\circ) .$$

I. *Solution by Richard I. Hess, Rancho Palos Verdes, California, USA.* From Figure 1 below where we assume that $AC = 1$ we have $BC = \sin 54^\circ$, $AM = MC = MB = BN = \frac{1}{2}$, $\angle MBN = 36^\circ$, $\angle MBQ = \angle QBN = 18^\circ$. Thus $MQ = \frac{1}{2}\sin 18^\circ$.

Now $\angle CMN = 36^\circ$ which implies $CN = MN = 2MQ = \sin 18^\circ$. Since $BC = BN + CN$ we obtain $\sin 54^\circ = \frac{1}{2} + \sin 18^\circ$.

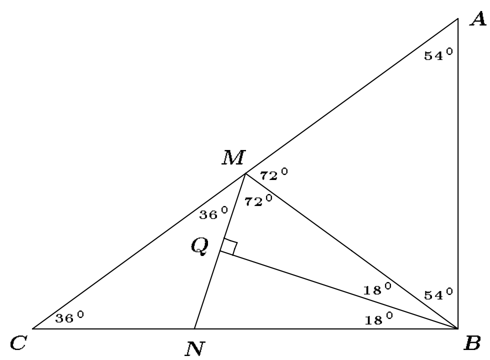

Figure 1.

II. *Solution by Nikolaos Dergiades, Thessaloniki, Greece.* On the circle centred at O with radius r having a diameter AD we take the points B and C such that $\text{arc}AB = \text{arc}BC = 36^\circ$ (see Figure 2 below). Then $\text{arc}CD = 108^\circ$, so that

$$AB = BC = 2r\sin 18^\circ , \qquad CD = 2r\sin 54^\circ .$$

Since $\angle BOA = \angle CDA$, we see that $BO \parallel CD$. Choosing E on CD such that $BE \parallel AD$, we see that the quadrilateral $BEDO$ is a rhombus; thus $ED = r$. Since $\angle CEB = \angle CDA = 36^\circ$, we have $\angle CBA = 144^\circ$, $\angle OBA = \angle OBC = 72^\circ$, and $\angle EBO = 36^\circ$. It then follows that $\angle CBE = 36^\circ$ and $CE = CB = 2r\sin 18^\circ$. Thus

$$2r\sin 54^\circ = CD = CE + ED = 2r\sin 18^\circ + r ,$$

$$\text{or} \qquad \sin 54^\circ = \frac{1}{2} + \sin 18^\circ .$$

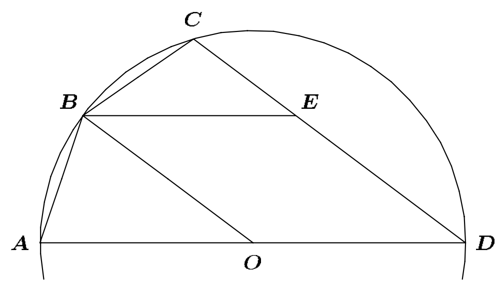

Figure 2.

III. *Solution by Jeremy Young, student, Nottingham High School, England.*

Set $x = \sin 18^\circ$ and set $y = \sin 54^\circ$. First consider the triangle in Figure 3 below, where we set $AC = 1$. Then $BC = x$ and $CD = 1$. Thus $AB = \sqrt{1-x^2}$ and $AD = \sqrt{1-x^2+(1+x)^2} = \sqrt{2(1+x)}$. Then

$$\sin 54^\circ \;=\; y \;=\; \frac{1+x}{\sqrt{2(1+x)}} \quad\Longrightarrow\quad 2y^2 - 1 \;=\; x\,.$$

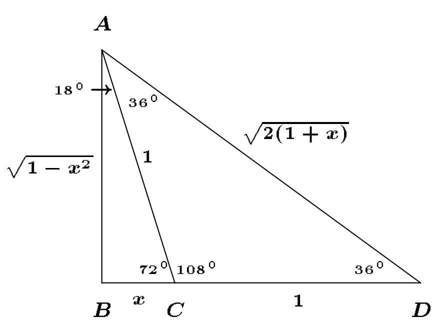

Figure 3.

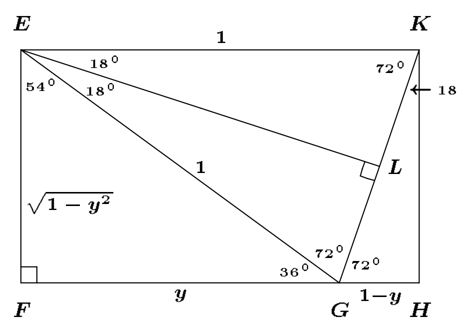

Figure 4.

In Figure 4 above where $EG = 1$, we have $FG = y$, $EK = 1$, $GH = 1 - y$, and $EF = \sqrt{1-y^2} = KH$. Then

$$LG \;=\; x \;=\; \frac{1}{2}\sqrt{(1-y)^2+1-y^2} \;=\; \frac{1}{2}\sqrt{2-2y}\,.$$

which yields $1 - 2x^2 = y$. Adding this to the previous result ($2y^2 - 1 = x$) gives

$$\begin{aligned} 2(y+x)(y-x) &= y+x\,, \\ 2(y-x) &= 1\,, \qquad \text{since } x,\,y>0 \implies y+x \neq 0, \\ y &= x+\frac{1}{2}\,. \end{aligned}$$

Also solved by SAM BAETHGE, Nordheim, Texas, USA; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; M. BENITO and E. FERNÁNDEZ, Logroño, Spain; CHRISTOPHER J. BRADLEY, *Clifton College, Bristol, UK; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; DIANE and ROY DOWLING, University of Manitoba, Winnipeg, Manitoba; RICHARD EDEN, student, Ateneo de Manila University, Philippines; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ANGEL JOVAL ROQUET, Spain; GEOFFREY A. KANDALL, Hamden, CT; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; VJEKOSLAV KOVAČ, student, University of Zagreb, Croatia; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; VEDULA N. MURTY, Dover, PA, USA; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; BOB PRIELIPP, University of Wisconsin–Oshkosh, Wisconsin, USA; TOSHIO SEIMIYA, Kawasaki, Japan; J. SUCK, Essen, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposer.*

2351. [1998: 302] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.*

A triangle with integer sides is called ***Heronian*** if its area is an integer.

Does there exist a Heronian triangle whose sides are the arithmetic, geometric and harmonic means of two positive integers?

Solution by Michael Lambrou, University of Crete, Crete, Greece.

We show, slightly generalizing the given situation, that no triangle with sides a, $\sqrt{ac}$, c, with a, $c \in \mathbb{N}$ can have integral area. Indeed, if the area $\Delta \in \mathbb{N}$ we would have by Heron's formula:

$$\begin{aligned}\Delta^2 &= \frac{1}{16}(a+\sqrt{ac}+c)(a-\sqrt{ac}+c)(a+\sqrt{ac}-c)(-a+\sqrt{ac}+c) \\ &= \frac{1}{16}\left[4a^2c^2-(a^2+c^2-ac)^2\right] .\end{aligned}$$

If $(a,c)=t$ so that $a=pt$, $c=qt$ for some p, $q \in \mathbb{N}$ with $(p,q)=1$ we obtain

$$\frac{4\Delta}{t^2} = \sqrt{4p^2q^2-(p^2+q^2-pq)^2}\,.$$

But the left hand side is rational; so $4p^2q^2-(p^2+q^2-pq)^2$ must be an integer perfect square, say T^2. But this is impossible because p^2+q^2-pq is odd (as p, q are not both even) and so

$$T^2 = 4p^2q^2-(p^2+q^2-pq)^2 \equiv 0-1 \equiv 3 \pmod 4,$$

giving a contradiction, as no square is congruent to 3 modulo 4. This completes the proof that $\Delta \notin \mathbb{N}$.

Also solved by DUANE BROLINE, Eatsern Illinois University, Charleston, Illinois. USA; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

2352. [1998: 302] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Determine the shape of $\triangle ABC$ if

$$\cos A\cos B\cos(A-B)+\cos B\cos C\cos(B-C)$$
$$+\cos C\cos A\cos(C-A)+2\cos A\cos B\cos C \;=\; 1\,.$$

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

Since $\cos A\cos B - \sin A\sin B = \cos(A+B) = -\cos C$, we have $(\cos A\cos B+\cos C)^2 = \left(1-\cos^2 A\right)\left(1-\cos^2 B\right)$, or

$$\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C \;=\; 1\,. \qquad (1)$$

Also,

$$\begin{aligned}\cos(A+B)\cos(A-B) &= \cos^2 A\cos^2 B-\sin^2 A\sin^2 B\\ &= \cos^2 A\cos^2 B-\left(1-\cos^2 A\right)\left(1-\cos^2 B\right)\\ &= \cos^2 A+\cos^2 B-1\,. \qquad (2)\end{aligned}$$

Using (2), we have

$$\begin{aligned}\cos A\cos B\cos(A-B) &= \tfrac{1}{2}\left(\cos(A-B)+\cos(A+B)\right)\cos(A-B)\\ &= \tfrac{1}{2}\left(\cos^2(A-B)+\cos^2 A+\cos^2 B-1\right)\\ &= \tfrac{1}{2}\left(\cos^2 A+\cos^2 B-\sin^2(A-B)\right)\,.\end{aligned}$$

Hence, the given equality is equivalent to

$$\begin{aligned}&\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C\\ &\quad -\tfrac{1}{2}\left(\sin^2(A-B)+\sin^2(B-C)+\sin^2(C-A)\right) \;=\; 1\,,\end{aligned}$$

which, in view of (1), becomes

$$\sin^2(A-B)+\sin^2(B-C)+\sin^2(C-A) \;=\; 0\,.$$

Hence, $A=B=C$, which means that $\triangle ABC$ is equilateral.

Also solved by MICHEL BATAILLE, Rouen, France; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; C. FESTRAETS-HAMOIR, Brussels, Belgium; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JUN-HUA HUANG, the Middle School Attached To Hunan Normal University, Changsha, China; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; VEDULA N. MURTY, Visakhapatnam, India; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; ARAM TANGBOONDOUANGJIT, Carnegie Mellon University, Pittsburgh, PA, USA; PARAYIOU THEOKLITOS, Limassol, Cyprus; PANOS E. TSAOUSSOGLOU, Athens, Greece; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer. There was also one incorrect solution.

Most solvers showed that the given equality is equivalent to either $\sum\cos^2(A-B) = 3$, *or to* $\sum\cos 2(A-B) = 3$, *both of which are clearly equivalent to* $\sum\sin^2(A-B) = 0$, *obtained in the solution above.*

Four solvers derived the equality $\prod\cos(A-B) = 1$, *from which the conclusion also follows immediately. The solution given above is self-contained, and does not use any known identity (for example,* $\sum\cos(2A) = -1-4\prod\cos A$, *which was used by a few solvers) beyond the elementary formula transforming product into sum. Lambrou obtained the slightly stronger result that*

$$\left(\sum\cos A\cos B\cos(A-B)\right) + 2\cos A\cos B\cos C \;\le\; 1\,,$$

with equality holding if and only if $\triangle ABC$ *is equilateral.*

2353. [1998: 302] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Determine the shape of $\triangle ABC$ if

$$\sin A\sin B\sin(A-B) + \sin B\sin C\sin(B-C) + \sin C\sin A\sin(C-A) \;=\; 0\,.$$

Solution by Gerry Leversha, St. Paul's School, London, England (slightly modified by the editor).

Since

$$\begin{aligned}
&\sin A\sin B\sin(A-B)\\
&\quad= \sin^2 A\sin B\cos B - \sin^2 B\sin A\cos A\\
&\quad= \tfrac{1}{4}\left((1-\cos(2A))\sin(2B) - (1-\cos(2B))\sin(2A)\right)\\
&\quad= \tfrac{1}{4}\left((\sin(2B)-\sin(2A)) + \sin(2(A-B))\right)\,,
\end{aligned}$$

the given equality is equivalent to

$$\sum\sin(2(A-B)) \;=\; 0\,. \tag{1}$$

Using elementary formulae transforming sums into products, we have

$$\sin(2(A-B)) + \sin(2(B-C)) \;=\; 2\sin(A-C)\cos(A+C-2B)\,,$$

and thus

$$\begin{aligned}
&\sin(2(A-B)) + \sin(2(B-C)) + \sin(2(C-A))\\
&\quad= 2\sin(A-C)\cos(A+C-2B) + 2\sin(C-A)\cos(C-A)\\
&\quad= 2\sin(C-A)\left(\cos(C-A) - \cos(A+C-2B)\right)\\
&\quad= -4\sin(A-B)\sin(B-C)\sin(C-A)\,.
\end{aligned} \tag{2}$$

From (1) and (2), we see that the given equality is equivalent to

$$\sin(A-B)\sin(B-C)\sin(C-A) \;=\; 0\,,$$

which holds if and only if at least two of A, B, C are equal to each other. Therefore, the given equality holds if and only if $\triangle ABC$ is isosceles.

Also solved by MICHEL BATAILLE, Rouen, France; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; NIKOLAOS DERGIADES, Thessaloniki, Greece; C. FESTRAETS-HAMOIR, Brussels, Belgium; PETER HURTHIG, Columbia College, Burnaby, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JUN-HUA HUANG, the Middle School Attached To Hunan Normal University, Changsha, China; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; VEDULA N. MURTY, Visakhapatnam, India; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; ARAM TANGBOONDOUANGJIT, Carnegie Mellon University, Pittsburgh, PA, USA; PARAYIOU THEOKLITOS, Limassol, Cyprus; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer. There were also two incorrect solutions.

The solution given above is interesting since it shows that the conclusion $A = B$ or $B = C$ or $C = A$ does not depend on the assumption that $A + B + C = \pi$; that is, that A, B and C are the three angles of a triangle.

2354. [1998: 302] *Proposed by Herbert Gülicher, Westfalische Wilhelms-Universität, Münster, Germany.*

In triangle $P_1P_2P_3$, the line joining $P_{i-1}P_{i+1}$ meets a line σ_j at the point $S_{i,j}$ ($i, j = 1, 2, 3$, all indices taken modulo 3), such that all the points $S_{i,j}$, P_k are distinct, and different from the vertices of the triangle.

1. Prove that if all the points $S_{i,i}$ [note the correction] are non-collinear, then any two of the following conditions imply the third condition:

   (a) $\dfrac{P_1S_{3,1}}{S_{3,2}P_2} \cdot \dfrac{P_2S_{1,2}}{S_{1,3}P_3} \cdot \dfrac{P_3S_{2,3}}{S_{2,1}P_1} = -1$;

   (b) $\dfrac{S_{1,2}S_{1,1}}{S_{1,1}S_{1,3}} \cdot \dfrac{S_{2,3}S_{2,2}}{S_{2,2}S_{2,1}} \cdot \dfrac{S_{3,1}S_{3,3}}{S_{3,3}S_{3,2}} = 1$;

   (c) σ_1, σ_2, σ_3 are either concurrent or parallel.

2. Prove further that (a) and (b) are equivalent if the $S_{i,i}$ are collinear.

Here, AB denotes the signed length of the directed line segment $[AB]$.

Solution by Günter Pickert, Giessen, Germany.

Define $S_i = S_{ii}$ and $Q_i = \sigma_i \bigcap S_{i-1}S_{i+1}$. If $\sigma_i || S_{i-1}S_{i+1}$ (so that Q_i is at infinity), then replace $\dfrac{S_{i+1}Q_i}{Q_iS_{i-1}}$ by -1.

(a) From Menelaus' Theorem applied to $\triangle S_{i+1}S_{i-1}P_i$ and the line σ_i $(1 = 1, 2, 3)$,

$$\frac{S_{i+1}Q_i}{Q_iS_{i-1}} \cdot \frac{S_{i-1}S_{i-1,i}}{S_{i-1,i}P_i} \cdot \frac{P_iS_{i+1,i}}{S_{i+1,i}S_{i+1}} = -1 . \qquad (1)$$

Define

$$a = \prod_{i=1}^{3} \frac{P_i S_{i-1,i}}{S_{i-1,i+1} P_{i+1}}, \quad b = \prod_{i=1}^{3} \frac{S_{i,i+1} S_i}{S_i S_{i,i-1}}, \quad c = \prod_{i=1}^{3} \frac{S_{i+1} Q_i}{Q_i S_{i-1}}.$$

The given conditions (a), (b), (c) say (respectively) that $a = -1$, $b = 1$, and $c = 1$. [The value $c = 1$ is just Ceva's Theorem applied to $\triangle S_1 S_2 S_3$, and extended to allow the possibility that one or two of the Q_i are at infinity.] From (1) we get

$$-1 = c \cdot \frac{\prod S_{i-1} S_{i-1,i}}{\prod S_{i+1,i} S_{i+1}} \cdot \frac{\prod P_i S_{i+1,i}}{\prod S_{i-1,i} P_i} = c \cdot b \cdot a^{-1},$$

so that $-a = b \cdot c$.

Two of c, b, $-a$ are equal to 1 if and only if two of (a), (b), (c) hold, in which case the third product is 1 and the third condition holds too.

(b) If the S_i are collinear then $Q_i = S_i$ and $c = 1$; therefore (a) is equivalent to (b).

Also solved by JUN-HUA HUANG, the Middle School Attached To Hunan Normal University, Changsha, China; and the proposer.

Huang's solution is much the same as our featured solution. Gülicher based his on his problem 64 in Math. Semesterber. **40** *(1992) 91–92.*

2355. [1998: 303] *Proposed by G.P. Henderson, Garden Hill, Campbellcroft, Ontario.*

For $j = 1, 2, \ldots, m$, let A_j be non-collinear points with $A_j \neq A_{j+1}$. Translate every even-numbered point by an equal amount to get new points A'_2, A'_4, $\ldots$, and consider the sequence B_j, where $B_{2i} = A'_{2i}$ and $B_{2i-1} = A_{2i-1}$. The last member of the new sequence is either A_{m+1} or A'_{m+1} according as m is even or odd.

Find a necessary and sufficient condition for the length of the path $B_1 B_2 B_3 \ldots B_m$ to be greater than the length of the path $A_1 A_2 A_3 \ldots A_m$ for all such non-zero translations.

CRUX 1985 [1994: 250; 1995: 280] provides an example of such a configuration. There, $m = 2n$, the A_i are the vertices of a regular $2n$–gon and $A_{2n+1} = A_1$.

Solution by the proposer.

(*Editor's note:* In the original submission the paths above were given as $B_1 B_2 \cdots B_{m+1}$ and $A_1 A_2 \cdots A_{m+1}$. Since this makes the upper bounds on summations simpler we have opted to present the solution with these endpoints, although logically it makes no difference.)

We use the same letter for a point P and a vector $\overrightarrow{P}$. Let the translation in the problem be $\overrightarrow{X}$. Then

$$\overrightarrow{B_{2k}} = \overrightarrow{A'_{2k}} = \overrightarrow{A_{2k}} + \overrightarrow{X}, \quad k = 1, 2, \ldots, \lfloor (m+1)/2 \rfloor,$$

and the lengths of the new segments are

$$\begin{aligned} \left|\overrightarrow{B_2} - \overrightarrow{B_1}\right| &= \left|\overrightarrow{A'_2} - \overrightarrow{A_1}\right| &= \left|\overrightarrow{X} + \overrightarrow{A_2} - \overrightarrow{A_1}\right|, \\ \left|\overrightarrow{B_3} - \overrightarrow{B_2}\right| &= \left|\overrightarrow{A_3} - \overrightarrow{A'_2}\right| &= \left|\overrightarrow{X} - \left(\overrightarrow{A_3} - \overrightarrow{A_2}\right)\right|, \\ & & \cdots\ . \end{aligned}$$

The lengths of the paths are

$$L' = \sum_{j=1}^{m} \left|\overrightarrow{B_{j+1}} - \overrightarrow{B_j}\right| = \sum_{j=1}^{m} \left|\overrightarrow{X} - (-1)^j \left(\overrightarrow{A_{j+1}} - \overrightarrow{A_j}\right)\right|$$

and $$L = \sum_{j=1}^{m} \left|\overrightarrow{A_{j+1}} - \overrightarrow{A_j}\right|.$$

Now L' is to be a minimum at $\overrightarrow{X} = \overrightarrow{0}$. If we form the partial derivatives of L' with respect to the components of $\overrightarrow{X}$ and set $\overrightarrow{X} = \overrightarrow{0}$, we get

$$\sum_{j=1}^{m} \frac{(-1)^j \left(\overrightarrow{A_{j+1}} - \overrightarrow{A_j}\right)}{\left|\overrightarrow{A_{j+1}} - \overrightarrow{A_j}\right|} = \overrightarrow{0}. \tag{1}$$

The minimum of a sum like L' does not always occur at a point where the derivatives are zero. However, in this case we will prove that (1) actually is the required condition.

That is, the sum of unit vectors along the odd segments $\overrightarrow{A_2} - \overrightarrow{A_1}$, $\overrightarrow{A_4} - \overrightarrow{A_3}$, $\ldots$, is equal to the sum of unit vectors along the even segments. The lengths of the segments are arbitrary provided they are greater than zero. It is only their directions that matter. In the case of the regular $2n$-gon, both sums are $\overrightarrow{0}$ because each consists of unit vectors parallel to the sides of a regular n-gon.

Set $d_j = \left|\overrightarrow{A_{j+1}} - \overrightarrow{A_j}\right| > 0$ and $\overrightarrow{C_j} = (-1)^j \left(\overrightarrow{A_{j+1}} - \overrightarrow{A_j}\right) / d_j$, a unit vector parallel to the jth segment. Equation (1) becomes

$$\sum_{j=1}^{m} \overrightarrow{C_j} = \overrightarrow{0}. \tag{2}$$

The lengths of the paths are $L' = \sum_{j=1}^{m} \left|\overrightarrow{X} - d_j \overrightarrow{C_j}\right|$ and $L = \sum_{j=1}^{m} d_j$.

To prove the necessity of (2), assume $L' \geq L$ for all $\overrightarrow{X}$.

Set $y_j = \left|\overrightarrow{X} - d_j\overrightarrow{C_j}\right| - d_j$. Then $\sum_{j=1}^{m} y_j \geq 0$. Squaring both sides of $\left|\overrightarrow{X} - d_j\overrightarrow{C_j}\right| = d_j + y_j$ and dividing by d_j, we get

$$\begin{aligned} \overrightarrow{X^2}/d_j - 2\overrightarrow{X}\cdot\overrightarrow{C_j} &= 2y_j + y_j^2/d_j \\ \overrightarrow{X^2}\sum_{j=1}^{m}(1/d_j) - 2\overrightarrow{X}\cdot\sum_{j=1}^{m}\overrightarrow{C_j} &= 2\sum_{j=1}^{m} y_j + \sum_{j=1}^{m}(y_j^2/d_j) \geq 0\,. \end{aligned}$$

When we set $\overrightarrow{X} = \sum_{j=1}^{m}\overrightarrow{C_j}\Big/\sum_{j=1}^{m}(1/d_j)$, this becomes $\left(\sum_{j=1}^{m}\overrightarrow{C_j}\right)^2 \leq 0$, and we have (2).

Given (2) we are to prove that $L' > L$ for all $\overrightarrow{X} \neq \overrightarrow{0}$. We have

$$\left|\overrightarrow{X} - d_j\overrightarrow{C_j}\right| = \left|\overrightarrow{X} - d_j\overrightarrow{C_j}\right|\left|\overrightarrow{C_j}\right| \geq \left|\left(\overrightarrow{X} - d_j\overrightarrow{C_j}\right)\cdot\overrightarrow{C_j}\right|\,; \qquad (3)$$

$$\begin{aligned} L' &\geq \sum_{j=1}^{m}\left|\left(\overrightarrow{X} - d_j\overrightarrow{C_j}\right)\cdot\overrightarrow{C_j}\right| \geq \left|\sum_{j=1}^{m}\left(\overrightarrow{X} - d_j\overrightarrow{C_j}\right)\cdot\overrightarrow{C_j}\right| \\ &= \left|\overrightarrow{X}\cdot\sum_{j=1}^{m}\overrightarrow{C_j} - \sum_{j=1}^{m} d_j\right| = L\,. \end{aligned}$$

If $L' = L$ for some $\overrightarrow{X} \neq \overrightarrow{0}$, there is equality in (3) for all j. Then $\overrightarrow{X} - d_j\overrightarrow{C_j} = c_j\overrightarrow{C_j}$, where $c_j \neq -d_j$; thus $\overrightarrow{C_j} = \overrightarrow{X}/(c_j + d_j)$, and the points are collinear.

All of the above is valid in a Euclidean space of any number of dimensions.

There were no other solutions submitted.