SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2321. [1998: 109] *Proposed by David Doster, Choate Rosemary Hall, Wallingford, Connecticut, USA.*

Suppose that $n \geq 2$. Prove that

$$\sum_{k=2}^{n} \left\lfloor \frac{n^2}{k} \right\rfloor = \sum_{k=n+1}^{n^2} \left\lfloor \frac{n^2}{k} \right\rfloor .$$

Here, as usual, $\lfloor x \rfloor$ means the greatest integer less than or equal to x.

Solution by Florian Herzig, student, Cambridge, UK.

We show more generally that

$$\sum_{k=2}^{n} \left\lfloor \frac{m}{k} \right\rfloor = \sum_{k=n+1}^{n^2} \left\lfloor \frac{m}{k} \right\rfloor \qquad (1)$$

for all $n^2 \leq m < (n+1)^2$ and $n \geq 2$. The proof is by induction on m. For $m = 4$ this is easily verified. For the induction step, first assume that $n^2 < m < (n+1)^2$ for some n and that (1) holds for $m - 1$. Then the left-hand side increases (from its value for $m - 1$) by the number of divisors d of m with $2 \leq d \leq n$. If $n^2 < m < (n+1)^2$, and d divides m, where $2 \leq d \leq n$, then $\frac{m}{d}$ also divides m, and

$$n < \frac{n^2+1}{n} \leq \frac{n^2+1}{d} \leq \frac{m}{d} \leq \frac{n^2+2n}{d} \leq \frac{n^2+2n}{2} \leq n^2 ;$$

that is,

$$n+1 \leq \frac{m}{d} \leq n^2 .$$

Conversely, if $n+1 \leq \frac{m}{d} \leq n^2$, where d divides m, then

$$1 < \frac{m}{n^2} \leq d \leq \frac{m}{n+1} < n + 1 ;$$

that is,

$$2 \leq d \leq n .$$

Consequently, there is a one-to-one correspondence between the divisors of m in the interval $[2, n]$ and those in $[n+1, n^2]$. Therefore both sides of (1) increase by the same amount as m increases by 1.

On the other hand, if $m = n^2 > 4$ for some n and (1) is true for $m - 1$, then the left-hand side increases by the number N of divisors d of n^2 with $2 \leq d < n$ and by $\left\lfloor \frac{n^2}{n} \right\rfloor = n$; that is, by $N + n$. The right-hand side increases by the number of divisors d of n^2 with $n \leq d \leq (n-1)^2$, which is $N+1$, and by

$$\sum_{k=(n-1)^2+1}^{n} \left\lfloor \frac{n^2}{k} \right\rfloor - \left\lfloor \frac{n^2}{n} \right\rfloor = (2n-1) - n = n-1\,.$$

Therefore both sides increase by the same value also in this case, and the result follows by induction.

Also solved by ZAVOSH AMIR-KHOSRAVI, student, North Toronto Collegiate Institute, Toronto; MICHEL BATAILLE, Rouen, France; MANSUR BOASE, student, Cambridge, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; PARAGIOU THEOKLITOS, Limassol, Cyprus; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer.

2322. [1998: 109] *Proposed by K.R.S. Sastry, Dodballapur, India.*

Suppose that the ellipse $\mathcal{E}$ has equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Suppose that Γ is any circle concentric with $\mathcal{E}$. Suppose that A is a point on $\mathcal{E}$ and B is a point of Γ such that AB is tangent to both $\mathcal{E}$ and Γ.

Find the maximum length of AB.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

Let the common tangent touch the ellipse at $A = (x_1, y_1)$ and the circle (of radius R) at $B = (x_2, y_2)$. Assume, without loss of generality, that $b < R < a$ and (since it is not perpendicular to an axis of the ellipse) that AB has the equation

$$y = px + q\,. \tag{1}$$

(x_1, y_1) satisfies (1) and also the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,. \tag{2}$$

Plugging (1) into (2) and setting the discriminant equal to zero, we find

$$x_1 = -\frac{pa^2}{q}\,, \qquad q^2 = b^2 + p^2a^2\,. \tag{3}$$

Similarly, (x_2, y_2) satisfies (1) and also the equation

$$x^2 + y^2 = R^2\,,$$

so that

$$x_2 = -\frac{pR^2}{q}\,, \qquad q^2 = R^2(1+p^2)\,. \tag{4}$$

From (3) and (4) we find that

$$p^2 \;=\; \frac{R^2 - b^2}{a^2 - R^2}\,.$$

Since $y_2 - y_1 \;=\; p(x_2 - x_1)$ and $x_2 - x_1 \;=\; \frac{p}{q}\,(a^2 - R^2)$, we have

$$\begin{aligned} AB^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (1 + p^2)(x_2 - x_1)^2 \\ &= \frac{(R^2 - b^2)(a^2 - R^2)}{R^2} \\ &= a^2 + b^2 - R^2 - \frac{a^2 b^2}{R^2} \\ &= (a - b)^2 - \left(R - \frac{ab}{R}\right)^2 \\ &\le (a - b)^2\,, \end{aligned}$$

with equality if and only if $R \;=\; \sqrt{ab}$, in which case the maximum value of AB is $a - b$.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; THEODORE CHRONIS, Athens, Greece; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; NICOLAS THÉRIAULT, étudiant, Université Laval, Québec; and the proposer.

2325*. [1998: 109] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that q is a prime and n is a positive integer. Suppose that $\{a_k\}$ $(0 \le k \le n)$ is given by

$$\sum_{k=0}^{n} a_k x^k \;=\; \frac{1}{q^n} \sum_{k=0}^{n} \binom{qn}{qk} (qx - 1)^k\,.$$

Prove that each a_k is an integer.

Solution by G.P. Henderson, Garden Hill, Campbellcroft, Ontario.

We are to prove that

$$F = \frac{1}{q^n} \sum_{k=0}^{n} \binom{qn}{qk} (qx - 1)^k$$

is an integer polynomial.

Let the (complex) q-th roots of the number $qx - 1$ be $y_1, y_2, \ldots, y_q$ so that $y_r^q = qx - 1$ for $r = 1, 2, \ldots, q$.

Lemma. If $P(y)$ is an integer polynomial, then

$$\frac{1}{q}\sum_{r=1}^{q} P(y_r)$$

is an integer polynomial in x.

Proof. Suppose

$$P(y) \;=\; \sum_{m=0}^{b} c_m y^m \,.$$

Then

$$\frac{1}{q}\sum_{r=1}^{q} P(y_r) \;=\; \frac{1}{q}\sum_{m=0}^{b} c_m \sum_{r=1}^{q} y_r^m \,.$$

If m is a multiple of q, say $m = qk$, the inner sum is

$$\sum_{r=1}^{q} y_r^{qk} \;=\; \sum_{r=1}^{q} (qx-1)^k \;=\; q(qx-1)^k \,,$$

and if m is not a multiple of q, the inner sum is zero. [*Editorial comment*: This follows because, with m and q relatively prime, $y_1^m, \ldots, y_q^m$ are the (distinct) q-th roots of $(qx-1)^m$ and thus sum to zero.] Therefore

$$\frac{1}{q}\sum_{r=1}^{q} P(y_r) \;=\; \sum_{k=0}^{\lfloor b/q \rfloor} c_{qk}(qx-1)^k \,, \tag{1}$$

which proves the lemma. □

Set $b = qn$ and $c_m = \binom{qn}{m}$. Then

$$P(y_r) \;=\; \sum_{m=0}^{qn} \binom{qn}{m} y_r^m \;=\; (1+y_r)^{qn}$$

for $r = 1, \ldots, q$. Thus, dividing (1) by q^n:

$$\frac{1}{q^{n+1}}\sum_{r=1}^{q}(1+y_r)^{qn} \;=\; \frac{1}{q^n}\sum_{k=0}^{n}\binom{qn}{qk}(qx-1)^k \;=\; F \,. \tag{2}$$

We have

$$(1+y_r)^q \;=\; 1 + \sum_{m=1}^{q-1}\binom{q}{m} y_r^m + y_r^q \;=\; q[x + Q(y_r)] \,,$$

where $Q(y)$ is the integer polynomial

$$Q(y) \;=\; \sum_{m=1}^{q-1} \frac{1}{q}\binom{q}{m} y^m \,.$$

Using this in (2),

$$F = \frac{1}{q}\sum_{r=1}^{q}[x+Q(y_r)]^n = \sum_{s=0}^{n}\binom{n}{s}x^{n-s}\frac{1}{q}\sum_{r=1}^{q}[Q(y_r)]^s . \qquad (3)$$

Applying the lemma to the integer polynomials $[Q(y)]^s$, we see that F is an integer polynomial.

Note. If $q = 2$, we can obtain an explicit expression for F. The roots y_1 and y_2 are $\pm\sqrt{2x-1}$, and $Q(y) = y$. From (3),

$$\begin{aligned} F &= \frac{1}{2}\left[(x+\sqrt{2x-1})^n + (x-\sqrt{2x-1})^n\right] \\ &= x^n + \binom{n}{2}x^{n-2}(2x-1) + \binom{n}{4}x^{n-4}(2x-1)^2 \\ &\qquad + \cdots + \binom{n}{2r}x^{n-2r}(2x-1)^r , \end{aligned}$$

where $r = \lfloor n/2 \rfloor$.

No other solutions were received.

2326*. [1998: 175, 301] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Prove or disprove that if A, B and C are the angles of a triangle, then

$$\frac{2}{\pi} < \sum_{\text{cyclic}} \frac{\left(1-\sin\frac{A}{2}\right)\left(1+2\sin\frac{A}{2}\right)}{\pi - A} \le \frac{9}{2\pi} .$$

Solution by Michael Lambrou, University of Crete, Crete, Greece.

Set $A' = (\pi - A)/2$ and similarly for B and C so that $A'+B'+C' = \pi$ and A', B', C' are the angles of an *acute*-angled triangle. Conversely, every acute-angled triangle arises in this way if we set $A = \pi - 2A'$, etc. So the substitution $A = \pi - 2A'$ transforms the inequality to be proved into

$$\frac{4}{\pi} < \sum \frac{\cos A' - \cos 2A'}{A'} \le \frac{9}{\pi} \qquad (1)$$

for all acute-angled triangles. For notational convenience drop the primes and write A, B, C in place of A', B', C' once again. We may further assume $\pi/2 > A \ge B \ge C > 0$. For $0 < x \le \pi/2$ set $f(x) = \dfrac{\cos x - \cos 2x}{x}$, and extend this, by continuity, to $x = 0$ [*Editorial note*: by defining $f(0) = 0$ — it is easy to show with calculus that $\lim_{x\to 0} f(x) = 0$].

For $x \in [0, \pi/2]$, we have $f''(x) = g(x)/x^3$, where

$$g(x) = (\cos x - \cos 2x)''x^2 - 2(\cos x - \cos 2x)'x + 2(\cos x - \cos 2x) ,$$

so that

$$g'(x) \;=\; (\cos x - \cos 2x)'''x^2 \;=\; x^2 \sin x(1 - 16\cos x)\,.$$

Hence $g'(x) = 0$ if and only if $x = 0$ or $x = \arccos(1/16) \approx 86^\circ$, and g decreases in $[0, \arccos(1/16)]$ and increases in $[\arccos(1/16), \pi/2]$ and so has absolute minimum at $x = \arccos(1/16)$. But $g(0) = 0$, $g(\pi/2) = -\pi^2 + \pi + 2 < 0$ so $g(x) \leq 0$ for all x in $[0, \pi/2]$. It follows that $f''(x) \leq 0$ for all x in $[0, \pi/2]$ and so f is concave down. By Jensen's inequality we have

$$f(A) + f(B) + f(C) \;\leq\; 3f\left(\frac{A+B+C}{3}\right) \;=\; 3f\left(\frac{\pi}{3}\right) \;=\; \frac{9}{\pi}\,,$$

showing the right hand side of (1).

The concavity of f together with $f(\pi/4) = 2\sqrt{2}/\pi > 2/\pi = f(\pi/2)$ shows that the least value of f in the restricted interval $[\pi/4, \pi/2]$ occurs at $x = \pi/2$. We also have $f(0) = 0 < f(\pi/2)$ so $f(x) \geq 0$ for all $x \in [0, \pi/2]$ with strict inequality $f(x) > 0$ if $x \neq 0$.

Finally, since $\pi/2 \geq A \geq B \geq C$, we have that $B \geq \pi/4$, so that $f(A)$, $f(B) \geq f(\pi/2) = 2/\pi$. Hence

$$f(A) + f(B) + f(C) \;>\; f(A) + f(B) \;\geq\; f\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = \frac{4}{\pi}\,,$$

which proves the left hand side of (1).

Editorial note. Both bounds in (1) are best possible: the lower bound $4/\pi$ is attained by the degenerate $\pi/2$, $\pi/2$, 0 triangle, and the upper bound $9/\pi$ by the equilateral triangle.

Also solved by HAYO AHLBURG, Benidorm, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; and HEINZ-JÜRGEN SEIFFERT, Berlin, Germany.

Konečný, Seiffert and the proposer rewrote the sum in the problem as

$$\sum \frac{\sin(A/2) + \cos A}{\pi - A}\,,$$

which may be a more attractive form, though not perhaps quite as attractive as Lambrou's equivalent form (which was also found by Ahlburg).

The problem arose from Janous's solution to his ***CRUX with MAYHEM*** *problem* **2190**.

2327. [1998: 175] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

The sequence $\{a_n\}$ is defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and

$$a_{n+1} = a_n - a_{n-1} + \frac{a_n^2}{a_{n-2}}, \qquad n \geq 3\,.$$

Prove that each $a_n \in \mathbb{N}$, and that no a_n is divisible by 4.

Composite solutions by Edward J. Barbeau, University of Toronto, Toronto, Ontario and Florian Herzig, student, Cambridge, UK.

Note first that $\{a_n\}$ is increasing and so has no zero terms.
[Ed.: A simple induction suffices.]

Rearranging, we have $\dfrac{a_{n+1} + a_{n-1}}{a_n + a_{n-2}} = \dfrac{a_n}{a_{n-2}}$.

Setting $n = 3, 4, \ldots, m$, and taking the product, we get, for all $m \geq 2$,

$$\frac{a_{m+1} + a_{m-1}}{a_3 + a_1} = \frac{a_m a_{m-1}}{a_2 a_1}\,.$$

Since $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, we get $a_{m+1} + a_{m-1} = 2a_m a_{m-1}$, or $a_{m+1} = a_{m-1}(2a_m - 1)$; whence $a_n \in \mathbb{N}$ for all n, by induction. Since $2a_m - 1$ is odd, another induction shows that a_{m+1} is not divisible by 4, since a_{m-1} is not.

Also solved by ZAVOSH AMIR-KHOSRAVI, student, North-Toronto Collegiate Institute, Toronto, Ontario; MICHEL BATAILLE, Rouen, France; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; ANTHONY FONG, student, Eric Hamber Secondary School, Vancouver, BC; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; HADI SALMASIAN, student, Sharif University of Technology, Tehran, Iran; JOEL SCHLOSBERG, student, Bayside, NY, USA; NICOLAS THÉRIAULT, student, Université Laval, Montréal, Québec; TODD THOMPSON, student, University of Arizona, Tucson, AZ, USA; and the proposer. There was also a partially incorrect solution submitted.

The recurrence relation shown in the solution above was also obtained by nine other solvers. However, only Herzig, Lambrou and the proposer actually derived the relation, while the others all used induction. Strictly speaking, a complete proof should include a statement and proof for the fact that $a_n \neq 0$ for all n. However, only a few solvers pointed this out explicitly.

Janous considered more generally the sequence defined by

$$a_{n+1} = ka_n - ka_{n-1} + \frac{a_n^2}{a_{n-2}}, \qquad (n \geq 3)\,,$$

where k, a_1, a_2, $a_3 \in \mathbb{N}$, and showed that if $\dfrac{ka_1 + a_3}{a_1 a_2} \in \mathbb{N}$ and $\dfrac{ka_1 + a_3}{a_1 a_2} > k$, then $a_n \in \mathbb{N}$ for all n.

2328★. [1998: 176] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

It is known from Wilson's Theorem that the sequence $\{y_n : n \geq 0\}$, with $y_n = \dfrac{n!+1}{n+1}$, contains infinitely many integers; namely, $y_n \in \mathbb{N}$ if and only if $n+1$ is prime.

(a) Determine all integer members of the sequences $\{y_n(a) : n \geq 0\}$, with $y_n = \dfrac{n!+a}{n+a}$, in the cases $a = 2, 3, 4$.

(b) Determine all integer members of the sequences $\{y_n(a) : n \geq 0\}$, with $y_n = \dfrac{n!+a}{n+a}$, in the cases $a \geq 5$.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

For all $a \in \mathbb{N}$, we have that $y_n(a) = \dfrac{n!+a}{n+a}$ is an integer when $n = 1$ and $n = 2$. If $n = a > 2$, then $y_n(a) = \dfrac{(a-1)!+1}{2}$ is not an integer.

Case 1: $n + a$ is not prime.

Suppose $n > a$. There are two possibilities:

(i) $n + a = pq$ where $\gcd(p, q) = 1$. Then $2n > n + a \geq 2q$ which implies $q \leq n-1$. Similarly $p \leq n-1$. This means $p|(n-1)!$ and $q|(n-1)!$, which further implies $pq|n!$. Thus $n!+a \equiv a \pmod{n+a}$, whence the number $\dfrac{n!+a}{n+a}$ is not an integer.

(ii) $n + a = p^k$, where $k > 1$ and p is prime. Arguing as above we get $p^{k-1}|(n-1)!$. If the number $\dfrac{n!+a}{n+a}$ is an integer, then $p^{k-1}|(n!+a)$. [*Ed.* note: we actually have $p^k|(n!+a)$, but that is not needed in the proof.] From $p^{k-1}|n!$ we conclude that $p^{k-1}|a$; from this and $p^{k-1}|(n+a)$ we get $p^{k-1}|n$ and finally

$$p^{k-1}|(n-1)! \text{ and } p^{k-1}|n \implies p^{k+k-2}|n! \implies n+a = p^k|n!$$

or $n! + a \equiv a \pmod{n+a}$. So the number $\dfrac{n!+a}{n+a}$ is not an integer.

Therefore, in case 1 if the number $\dfrac{n!+a}{n+a}$ is an integer we must have $n < a$.

Case 2: $n + a$ is prime.

From Wilson's Theorem we have $(n+a-1)! + 1 \equiv 0 \pmod{n+a}$.

If $n! + a \equiv 0 \pmod{n+a}$, then

$$\begin{aligned} n!(n+1)\cdots(n+a-1) + a(n+1)\cdots(n+a-1) &\equiv 0 \pmod{n+a}\,, \\ -1 + a(n+1)\cdots(n+a-1) &\equiv 0 \pmod{n+a}\,, \\ -1 + a(-a+1)(-a+2)\cdots(-2)(-1) &\equiv 0 \pmod{n+a}\,, \\ -1 - (-1)^a a! &\equiv 0 \pmod{n+a}\,. \end{aligned}$$

In this case the number $\dfrac{n!+a}{n+a}$ is an integer if $n+a$ is a prime divisor of $s = -1 - (-1)^a a!$. So there are a finite number of terms of $y_n(a)$ such that $\dfrac{n!+a}{n+a}$ is an integer, since when $a \neq 1$ we have $s \neq 0$.

Examples:

Let $a = 2$. Then $s = -3$ with 3 the only prime divisor, so $n + 2 = 3$ gives $n = 1$. Thus the number $\dfrac{n!+2}{n+2}$ is an integer only when $n = 1, 2$.

Let $a = 3$. Then $s = 5$. So $n + 3 = 5$ gives $n = 2$ and the number $\dfrac{n!+3}{n+3}$ is an integer only when $n = 1, 2$.

Let $a = 4$. Then $s = -25$ with 5 the only prime divisor, so $n + 4 = 5$ gives $n = 1$. We also need to check $n < a$: if $n = 3$, then $\dfrac{n!+4}{n+4} = \dfrac{10}{7}$, which is not an integer. Thus the number $\dfrac{n!+4}{n+4}$ is an integer only when $n = 1, 2$.

Let $a = 5$. Then $s = 119$ with prime divisors 7 and 17, so $n + 5 = 7$ or 17 implies $n = 2$ or 12, respectively. We also need to check $n < a$: if $n = 3, 4$, then $\dfrac{n!+5}{n+5} = \dfrac{11}{8}, \dfrac{29}{9}$, respectively, neither of which is an integer. So the number $\dfrac{n!+5}{n+5}$ is an integer only when $n = 1, 2, 12$.

As a last example, if $a = 22$, then the number $\dfrac{n!+22}{n+22}$ is an integer only when $n = 1, 2, 12, 499, 93\,799\,610\,095\,769\,625$.

Also solved by MANSUR BOASE, student, Cambridge, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, California, USA; and the proposer.

Hess was the only solver who specifically showed that $y_n(a)$ is never an integer when $n = 0$ and $a \geq 2$. He also gave a complete list of the values of n which yield integers for $y_n(a)$ for values of $6 \leq a \leq 17$ (the list below gives only the values of $n > 2$):

$a = 6$	:	4 , 97
$a = 7$	:	5 032
$a = 8$	:	6 , 53 , 653
$a = 9$	:	2 990
$a = 10$	:	329 881
$a = 11$	:	6 , 12 , 7 842
$a = 12$	:	10 , 2 834 317
$a = 13$	:	1 720 , 3 593 190
$a = 14$	:	9 , 12 , 3 790 360 473
$a = 15$	:	6 , 16 , 38 , 1 510 244
$a = 16$	:	4 , 45 , 121 , 123 , 1 059 495
$a = 17$	:	56 , 256 443 711 660

Janous observes that there are quite a few new questions coming from this problem, such as:

What is $\ell = \limsup_{a\to\infty} i(a)$, *where* $i(a)$ *is the number of integer members of the set* $\left\{\dfrac{n!+a}{n+a} : n = 1, 2, 3 \ldots\right\}$?

If $n(k) = \#\{a \geq 2 : i(a) = k\}$, *what can be said about* $n(k)$? *Especially, in the event of* $\ell = \infty$, *is it true that, for all* $k \geq 2$, *we always have* $n(k) \geq 1$? *If not, for what values of* k *is* $n(k) = 0$?

2330. [1998: 176] *Proposed by Florian Herzig, student, Perchtoldsdorf, Austria.*

Prove that

$$e = 3 - \frac{1!}{1\cdot 3} + \frac{2!}{3\cdot 11} - \frac{3!}{11\cdot 53} + \frac{4!}{53\cdot 309} - \frac{5!}{309\cdot 2119} + \cdots,$$

where

$$\begin{aligned} 11 &= 3\cdot 3 + 2\cdot 1,\\ 53 &= 4\cdot 11 + 3\cdot 3,\\ 309 &= 5\cdot 53 + 4\cdot 11,\\ 2119 &= 6\cdot 309 + 5\cdot 53,\\ &\vdots \end{aligned}$$

Solution by Con Amore Problem Group, Royal Danish School of Educational Studies, Copenhagen, Denmark.

We have to prove that

$$e = 3 - \frac{1!}{a_1a_2} + \frac{2!}{a_2a_3} - \frac{3!}{a_3a_4} + \cdots + (-1)^n\cdot\frac{n!}{a_na_{n+1}} + \cdots, \qquad (1)$$

where a_1, a_2, $\ldots$, are such that $a_1 = 1$, $a_2 = 3$, and for $n \geq 3$:

$$a_n = na_{n-1} + (n-1)a_{n-2}\,. \qquad (2)$$

Along with $a_1, a_2, \ldots$, we consider $d_1, d_2, \ldots$, defined by

$$d_n = n!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{(-1)^n}{n!}\right) \tag{3}$$

for $n = 1, 2, \ldots$. These are well-known as the derangement (or displacement) numbers, so called because d_n is the number of permutations of $(1, 2, \ldots, n)$ where all the elements are displaced, but this need not concern us here. We just note for later use that

$$d_3 = 2 \text{ and } d_4 = 9 \tag{4}$$

and

$$\frac{d_{n+1}}{(n+1)!} - \frac{d_n}{n!} = \frac{(-1)^{n+1}}{(n+1)!},$$
$$\text{or} \quad (n+1)d_n - d_{n+1} = (-1)^n \tag{5}$$

for $n = 1, 2, \ldots$, whence $(n+2)d_{n+1} - d_{n+2} = d_{n+1} - (n+1)d_n$, or

$$d_{n+2} = (n+1)d_{n+1} + (n+1)d_n \tag{6}$$

for $n = 1, 2, \ldots$. Now define for $n = 1, 2, \ldots$:

$$\alpha_n = \frac{d_{n+2}}{n+1}. \tag{7}$$

Then by (4) and (6), $\alpha_1 = d_3/2 = 1$, and $\alpha_2 = d_4/3 = 3$, and for $n \geq 3$ we have

$$(n+1)\alpha_n = (n+1)n\alpha_{n-1} + (n+1)(n-1)\alpha_{n-2},$$

or $\alpha_n = n\alpha_{n-1} + (n-1)\alpha_{n-2}$. Comparison with (2) shows that for $n = 1, 2, \ldots$, we have $\alpha_n = a_n$. Thus by (7):

$$a_n = \frac{d_{n+2}}{n+1} \tag{8}$$

for $n = 1, 2, \ldots$. Now (8) and (5) imply

$$\begin{aligned}(-1)^n \cdot \frac{n!}{a_n a_{n+1}} &= (-1)^{n+2} \cdot \frac{n!(n+1)(n+2)}{d_{n+2}d_{n+3}} \\ &= \big((n+3)d_{n+2} - d_{n+3}\big) \cdot \frac{(n+2)!}{d_{n+2}d_{n+3}},\end{aligned}$$

and so, for $n = 1, 2, \ldots$:

$$(-1)^n \cdot \frac{n!}{a_n a_{n+1}} = \frac{(n+3)!}{d_{n+3}} - \frac{(n+2)!}{d_{n+2}}. \tag{9}$$

Using (9) and (3), and noting that $3 = 3!/d_3$, we get

$$\begin{aligned} & 3 - \frac{1!}{a_1 a_2} + \frac{2!}{a_2 a_3} - \frac{3!}{a_3 a_4} + \cdots + (-1)^n \cdot \frac{n!}{a_n a_{n+1}} \\ = \;& \frac{3!}{d_3} + \left(\frac{4!}{d_4} - \frac{3!}{d_3}\right) + \left(\frac{5!}{d_5} - \frac{4!}{d_4}\right) + \cdots + \left(\frac{(n+3)!}{d_{n+3}} - \frac{(n+2)!}{d_{n+2}}\right) \\ = \;& \frac{(n+3)!}{d_{n+3}} \;=\; \left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{(-1)^{n+3}}{(n+3)!}\right)^{-1} \\ \to \;& (e^{-1})^{-1} \;=\; e \qquad \text{as } n \to \infty\,, \end{aligned}$$

which proves (1).

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.

Herzig used continued fractions to solve the problem.

2331. [1998: 176] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.*

Let p be an odd prime. Show that there is at most one non-degenerate integer triangle with perimeter $4p$ and integer area. Characterize those primes for which such triangles exist.

Composite solution by Jill S. Taylor, student, Mount Allison University, Sackville, New Brunswick; and the proposer.

Consider such a triangle of sides a, b, c, and semi-perimeter s, with area $\sqrt{s(s-a)(s-b)(s-c)}$. Then $1 \le a,\, b,\, c \le 2p - 1$.

Since $s = 2p$ and the area of the triangle is an integer, one of $(s-a)$, $(s-b)$, $(s-c)$ must be divisible by p. Without loss of generality, assume that $p|(s-a)$. Then $s - a = p$ since $1 \le s - a \le 2p - 1$.

Hence $(s-b) + (s-c) = a = p$ and $(s-b)(s-c)$ must be twice a square. Clearly, $(s-b)$ and $(s-c)$ are relatively prime. [*Ed.*: If $d \in \mathbb{N}$ is such that $d|(s-b)$ and $d|(s-c)$, then $d|p$, and thus $d = 1$ or $d = p$. However, if $d = p$, then $p|(2p-b)$, $(2p-c)$ would imply that $b = c = p$, and so, $2s = a + b + c = 3p$, which is a contradiction.]

It follows that $s - b = x^2$ and $x - c = 2y^2$ for relatively prime integers x and y. Hence $p = x^2 + 2y^2$.

Conversely, if $p = x^2 + 2y^2$, then p, $p + x^2$ and $2p - x^2$ would be the sides of a triangle with the described properties, since

$$\begin{aligned} 2s \;&=\; p + (p + x^2) + (2p - x^2) \;=\; 4p \qquad \text{and} \\ s(s-p)(s-(p+x^2))(s-(2p-x^2)) \;&=\; 2p^2(p - x^2)x^2 \\ &=\; 2p^2(2x^2y^2) = (2pxy)^2\,. \end{aligned}$$

By well-known results in elementary number theory (see [3]), the representation $p = x^2 + 2y^2$ is possible and unique if and only if $p \equiv 1, 3 \pmod 8$. Hence such a triangle exists (and is unique) if and only if $p \equiv 1, 3 \pmod 8$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; JOEL SCHLOSBERG, student, Bayside, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; KENNETH M. WILKE, Topeka, Kansas, USA.

The first four triangles with the described proprties are $(3, 4, 5; 6)$, $(11, 13, 20; 66)$, $(17, 25, 26; 204)$, *and* $(19, 20, 37; 114)$, *(where the components denote the side lengths and the areas of the triangles, respectively). These were given by Hess, Leversha and the proposer. Hess and Leversha actually listed the first six and eleven such triangles, respectively.*

The result cited in the solution above can be found in many books on number theory. Besides [3], Konečný quoted [1] and two others. Both Seiffert and Wilke quoted [2].

References:

[1] Ethan D. Bolker, *Elementary Number Theory*, W.A. Benjamin Inc., New York, 1970, pp. 113–116.

[2] T. Nagell, *Introduction to Number Theory*, 2nd Ed., Chelsea, 1964, pp. 188–191.

[3] W. Sierpiński, *A Selection of Problems in the Theory of Numbers*, Pergamon Press Ltd., Oxford, England, 1964, p. 72.

2332. [1998: 177] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Suppose x and y are integers. Solve the equation

$$x^2y^2 - 7x^2y + 12x^2 - 21xy - 4y^2 + 63x + 70y - 174 = 0 .$$

Solution by Michel Bataille, Rouen, France.

The equation reduces to the equivalent form

$$(x-2)(y-3)((x+2)(y-4) - 21) = 0 .$$

The first two factors provide two families of solutions: $(2, a)$ and $(b, 3)$, where a and b are integers. Any other solution is such that $(x+2)(y-4) = 21$; that is, $(x+2, y-4)$ is one of the pairs $(1, 21)$, $(21, 1)$, $(3, 7)$, $(7, 3)$, $(-1, -21)$, $(-21, -1)$, $(-3, -7)$, and $(-7, -3)$. Examining each case separately, we obtain seven new solutions:

$$(-1, 25)\,,\ (19, 5)\,,\ (1, 11)\,,\ (5, 7)\,,\ (-3, -17)\,,\ (-5, -3)\,,\ \text{and}\ (-9, 1)\,.$$

Also solved by HAYO AHLBURG, Benidorm, Spain; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ZAVOSH AMIR-KHOSRAVI, student, North Toronto Collegiate Institute, Toronto; SAM BAETHGE, Nordheim, Texas, USA; EDWARD J. BARBEAU, University of Toronto, Toronto, Ontario; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, Athens, Greece; GORAN CONAR, student, Gymnasium Varaždin,

Varaždin, Croatia; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; ANTHONY FONG, student, Eric Hamber Secondary School, Vancouver; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines and GIOVANNI MAZZARELLO, Ferrovie dello Stato, Florence, Italy; DOUGLASS L. GRANT, University College of Cape Breton, Sydney, Nova Scotia; YEO KENG HEE, Hwa Chong Junior College, Singapore; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; JESSIE LEI, student, Vincent Massey Secondary School, Windsor, Ontario; GERRY LEVERSHA, St. Paul's School, London, England; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; CHRISTO SARGOTIS, Thessaloniki, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; BOB PRIELIPP, University of Wisconsin–Oshkosh, Wisconsin, USA; DIGBY SMITH, Mount Royal College, Calgary, Alberta; PANOS E. TSAOUSSOGLOU, Athens, Greece; ERIC UMEGÅRD, Västerås, Sweden; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer. There was one incomplete solution submitted.

Most of the submitted solutions are similar to the one given above.

2333. [1998: 177] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

You are given that D and E are points on the sides AC and AB respectively of $\triangle ABC$. Also, DE is not parallel to CB. Suppose F and G are points of BC and ED respectively such that

$$\overline{BF} : \overline{FC} = \overline{EG} : \overline{GD} = \overline{BE} : \overline{CD}.$$

Show that GF is parallel to the angle bisector of $\angle BAC$.

I. *Solution by Michael Lambrou, University of Crete, Crete, Greece.*

With respect to position vectors originating from A, write $\overrightarrow{AB} = \mathrm{b}$, $\overrightarrow{AC} = \mathrm{c}$, so that for some p, $q \in (0,1)$ (see remark) we have $\mathrm{e} := \overrightarrow{AE} = p\mathrm{b}$, $\mathrm{d} := \overrightarrow{AD} = q\mathrm{c}$. If the given equal ratios are denoted by $\lambda > 0$ (see remark), we have

$$\mathrm{f} := \overrightarrow{AF} = \frac{\lambda\mathrm{c} + \mathrm{b}}{\lambda + 1} \quad \text{and} \quad \mathrm{g} := \overrightarrow{AG} = \frac{\lambda\mathrm{d} + \mathrm{e}}{\lambda + 1} = \frac{\lambda q\mathrm{c} + p\mathrm{b}}{\lambda + 1}.$$

Moreover, the condition $\overline{BE} : \overline{CD} = \lambda$ becomes $\left|\overrightarrow{BE}\right| = \lambda\left|\overrightarrow{CD}\right|$, and so $(1-p)|\mathrm{b}| = \lambda(1-q)|\mathrm{c}|$. Thus,

$$\begin{aligned}\overrightarrow{GF} &= \mathrm{f} - \mathrm{g} = \frac{\lambda(1-q)}{\lambda+1}\mathrm{c} + \frac{1-p}{\lambda+1}\mathrm{b} \\ &= \frac{\lambda(1-q)|\mathrm{c}|}{\lambda+1}\frac{\mathrm{c}}{|\mathrm{c}|} + \frac{(1-p)|\mathrm{b}|}{\lambda+1}\frac{\mathrm{b}}{|\mathrm{b}|} = \frac{(1-p)|\mathrm{b}|}{\lambda+1}\left(\frac{\mathrm{c}}{|\mathrm{c}|} + \frac{\mathrm{b}}{|\mathrm{b}|}\right),\end{aligned}$$

which is parallel to $\left(\frac{\mathrm{c}}{|\mathrm{c}|} + \frac{\mathrm{b}}{|\mathrm{b}|}\right)$. But this last is a sum of unit vectors along AB, AC and so is along the bisector of A, completing the proof.

Remarks. 1. The condition "DE not parallel to CB" has not been used. In the parallel case it turns out that GF coincides with the bisector.

[*Editor's additional remark*: Most solvers made a similar remark, with different interpretations according to whether or not one's definition of *parallel lines* permits the lines to coincide.]

2. We have taken here D, E, F in the interior of the corresponding sides of $\triangle ABC$. Thus we have $0 < p,\ q < 1$ and $\lambda > 0$. However, this is not necessary and the proof can be adapted to the more general case. For example, if $\lambda > 0$ then p, q must be both greater than 1 or both less than 1.

II. *Solution by Nikolaos Dergiades, Thessaloniki, Greece.*

Let K, L be chosen so that the quadrangles $EBKG$ and $GLCD$ are parallelograms, and therefore KG is equal and parallel to BE while LG is equal and parallel to CD. Consequently, $\angle BAC = \angle KGL$, and since $BK||LC$ and

$$\frac{BK}{LC} = \frac{EG}{GD} = \frac{BF}{FC}$$

[the last equality by assumption], we deduce that $\triangle BKF$ is similar to $\triangle CLF$, so that KL passes through F. Therefore,

$$\frac{KF}{FL} = \frac{BF}{FC} = \frac{BE}{CD} = \frac{KG}{LG}$$

[the middle equality by assumption], which implies that in $\triangle KGL$, we have that GF is the bisector of $\angle KGL$ and (since $KG||BA$ and $LG||CA$) parallel to the bisector of $\angle BAC$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; FLORIAN HERZIG, student, Cambridge, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

2334. [1998: 177] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that ABC is a triangle with incentre I, and that BI, CI meet AC, AB at D, E respectively. Suppose that P is the intersection of AI with DE. Suppose that $PD = PI$.

Find angle ACB.

Solution by Gerry Leversha, St. Paul's School, London, England.

We are given that $PD = PI$; hence $\angle AID = \angle EDI$. But

$$\angle AID = \angle ABI + \angle BAI = \frac{A+B}{2}$$

Therefore
$$\angle EDI = \frac{A+B}{2}.$$

We need to calculate some angles.

$$\begin{aligned}
\angle DEI &= 180^\circ - \angle EDI - \angle EID \\
&= 180^\circ - \frac{A+B}{2} - \left(180^\circ - \frac{B+C}{2}\right) \;=\; \frac{C-A}{2}\,, \\
\angle DEA &= 180^\circ - \angle EAD - \angle ADE \\
&= 180^\circ - A - (180^\circ - \angle EDI - \angle BDC) \\
&= -A + \frac{A+B}{2} + \left(180^\circ - C - \frac{B}{2}\right) \\
&= 180^\circ - C - \frac{A}{2}\,.
\end{aligned}$$

Now let $ED = x$ and use the Sine Rule on triangles AED and CED:

$$\frac{x}{AD} = \frac{\sin A}{\sin\left(C + \frac{A}{2}\right)} \qquad \text{and} \quad \frac{x}{DC} = \frac{\sin\frac{C}{2}}{\sin\left(\frac{C-A}{2}\right)}\,.$$

Now we divide and recall that $\dfrac{DC}{AD} = \dfrac{a}{c}$ by the Angle Bisector Theorem.

Therefore $$\frac{\sin A}{\sin C} = \frac{a}{c} = \frac{\sin A \sin\left(\frac{C-A}{2}\right)}{\sin\frac{C}{2}\sin\left(C+\frac{A}{2}\right)}.$$

Thus,

$$\begin{aligned}
\sin\frac{C}{2}\sin\left(C+\frac{A}{2}\right) &= \sin C \sin\left(\frac{C-A}{2}\right) \\
&= 2\sin\frac{C}{2}\cos\frac{C}{2}\sin\left(\frac{C-A}{2}\right)\,,
\end{aligned}$$

and $$\sin\left(C+\frac{A}{2}\right) = 2\cos\frac{C}{2}\sin\left(\frac{C-A}{2}\right) = \sin\left(C-\frac{A}{2}\right) - \sin\frac{A}{2}.$$

Hence $$\sin\frac{A}{2} = \sin\left(C-\frac{A}{2}\right) - \sin\left(C+\frac{A}{2}\right) = -2\cos C\sin\frac{A}{2},$$

and $$\cos C = -\frac{1}{2}\,, \qquad \text{so that } C = 120^\circ.$$

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer. There was one incorrect solution.

2335. [1998: 177] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Triangle ABC has circumcircle Γ. A circle Γ' is internally tangent to Γ at P, and touches sides AB, AC at D, E respectively. Let X, Y be the feet of the perpendiculars from P to BC, DE respectively.

Prove that $PX = PY \sin\frac{A}{2}$.

Solution by Florian Herzig, student, Cambridge, UK.

Rename the circles Γ and Γ' (respectively) as Γ_1 and Γ_2 to avoid later confusion. Invert the configuration in any circle with centre P, and denote the image of a point or circle X by X'. Then Γ_1' and Γ_2' are parallel lines, and the latter touches the circumcircles of $PC'E'A'$ and $PA'D'B'$ (because the corresponding lines touch Γ_2). Therefore $B'C' = 2D'E'$ and also

$$\angle E'PD' = \angle E'PA' + \angle A'PD' = \frac{1}{2}(\angle C'PA' + \angle A'PB') = \frac{1}{2}\angle C'PB'.$$

Moreover, $\angle C'PB' = \angle BPC = 180^o - A$. Since X is the foot of the perpendicular to BC from P (the centre of inversion), the image of BC is the circle $B'C'P$ having PX' as diameter.

[*Editor's comment*. Herzig provided a simple justification of the claim, but proofs are readily found in any text with a section on inversions, such as Coxeter and Greitzer's *Geometry Revisited*.]

Similarly, PY' is a diameter of circle $E'PD'$. Hence PX' is twice the circumradius of circle $B'PC'$, and PY' of $E'PD'$. By the Sine Law, therefore,

$$PX' = \frac{B'C'}{\sin A}, \quad \text{and} \quad PY' = \frac{D'E'}{\sin\angle E'PD'} = \frac{D'E'}{\sin(90^o - \frac{A}{2})}.$$

Finally,

$$\frac{PX}{PY} = \frac{PY'}{PX'} = \frac{D'E' \cdot \sin A}{B'C' \cdot \cos\frac{A}{2}} = \sin\frac{A}{2},$$

as we wanted to show.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

The configuration investigated here has appeared before in **CRUX** *— in Leon Bankoff's "Mixtilinear Adventure"* [1983: 2-7] *and in S. Shirali's "On a Generalized Ptolemy Theorem"* [1996: 49-53]. *See also Nathan Altshiller Court, College Geometry, p. 239 theorem 537. We thank Bellot and Seimiya for these references. Bellot also found the configuration in a 1991 Bulgarian journal and in the Cambridge Mathematical Tripos of 1929. It is remarkable how he is so often able to find relevant references. This editor wonders if the method can be applied to finding my glasses (which occasionally get misplaced.)*

2336. [1998: 177] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

The bisector of angle A of a triangle ABC meets BC at D. Let Γ and Γ' be the circumcircles of triangles ABD and ACD respectively, and let P, Q be the intersections of AD with the common tangents to Γ, Γ' respectively.

Prove that $PQ^2 = AB \cdot AC$.

Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.

We first will prove that, with the notation of the problem, calling t the length of the segment of common tangent to both circles (AD the common chord),

$$PQ^2 \;=\; AD^2 + t^2\,. \tag{1}$$

It is easy to see that P is the midpoint of the common tangent segment (for example, calculating the power of P with respect to both circles in two forms: if L, N are the points of tangency, then $PL^2 = PA \cdot PD = PN^2 \Longrightarrow PL = PN$.)

Then calling $x = AP = QD$, we have

$$PQ \;=\; AD + 2x \Longrightarrow PQ^2 \;=\; AD^2 + 4x(AD + x)\,,$$

and as

$$\left(\frac{t}{2}\right)^2 \;=\; x(AD + x)\,,$$

the formula (1) follows.

As AD is the internal bisector of angle A, we have the well known formulae

$$AD \;=\; \frac{2bc}{b+c}\cos\frac{A}{2}\,, \tag{2}$$

$$AD^2 \;=\; cb\left[1 - \frac{a^2}{(b+c)^2}\right]\,. \tag{3}$$

Therefore, we must prove that, if AD is the bisector of the angle A,

$$AD^2 + t^2 \;=\; cb \qquad \text{or, using (3),} \quad t \;=\; \frac{a\sqrt{bc}}{b+c}\,. \tag{4}$$

To this end, we first find the radius R_1 of the circle ABD with aid of the Sine Law:

$$\frac{BD}{\sin\frac{A}{2}} \;=\; 2R_1 \Longleftrightarrow R_1 \;=\; \frac{ab}{2(b+c)\sin\frac{A}{2}}\,, \tag{5}$$

and analogously the radius of the circle ADC:

$$R_2 = \frac{ab}{2(b+c)\sin\frac{A}{2}}. \tag{6}$$

If O_1, O_2 are the centres of these circles, we have

$$t^2 = \overline{O_1O_2}^2 - (R_1 - R_2)^2, \tag{7}$$

and, calling $M = O_1O_2 \cap AD$, we obtain:

$$\overline{O_1M}^2 = R_1^2 - \frac{AD^2}{4}; \quad \overline{O_2M}^2 = R_2^2 - \frac{AD^2}{4},$$

from which, using (2), (5) and the Sine Law, we find

$$\overline{O_1M}^2 = \frac{c^2(a^2 - b^2\sin^2 A)}{4(b+c)^2\sin^2\frac{A}{2}} = \frac{c^2a^2\cos^2 B}{4(b+c)^2\sin^2\frac{A}{2}};$$

that is, $O_1M = \dfrac{ca\cos B}{2(b+c)\sin\frac{A}{2}}$, and analogously, $O_2M = \dfrac{ba\cos C}{2(b+c)\sin\frac{A}{2}}$, from which results

$$O_1O_2 = \frac{a^2}{2(b+c)\sin\frac{A}{2}}.$$

Going back to (7), we obtain

$$t^2 = \frac{a^2[a^2 - (c-b)^2]}{4(b+c)^2\sin^2\frac{A}{2}} = \frac{a^2(a+c-b)(a-c+b)}{4(b+c)^2\sin^2\frac{A}{2}} = \frac{a^2bc}{(b+c)^2},$$

which is precisely (4), and the problem is solved.

Also solved by NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer. There was one incorrect solution.